ENS Lyon and Université Claude Bernard Lyon 1, M2A
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Calculus of Variations and Elliptic PDEs
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Mock Exam
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Exercice 1 (6 points)**.** Consider the problem

$$\min \left\{ \int_0^{\pi/2} \Big(u'(t)^2 + u(t)^2 + 2\sin(t)u(t) \Big)\, dt \quad : \quad u \in C^1([0,\pi/2]),\ u(0) = 0 \right\}.$$

Prove that it admits a minimizer, that it is unique, and find it.

Solution

The function L given by $L(t,x,v) = |v|^2 + |x|^2 + 2\sin(t)x$ is convex in (x,v). Hence, any solution of the Euler-Lagrange equation of the problem coupled with the suitable boundary conditions, is also a minimizer. considering that we have $\partial_v L(t,x,v) = 2v$ and $partial_x L(t,x,v) = 2x + 2\sin(t)$ the Euler-Lagrange system is

$$\begin{cases} 2u'' = 2u + 2\sin(t), \\ u(0) = 0, \\ u'(\pi/2) = 0. \end{cases}$$

It is easy to see that the solution is of the form $u(t) = Ae^t + Be^{-t} - \frac{1}{2}\sin(t)$ and find $A + B = 0$ and $Ae^{\pi/2} = Be^{-\pi/2}$, hence $A = B = 0$. The solution is then given by $u(t) = -\frac{1}{2}\sin(t)$.

This function is C^1 and solves the Euler-Lagrange system, it then minimizes the functional among any admissible competitor. Uniqueness can be justified either using the strict convexity of L in (x,v), or just be noting that this is the only solution of the Euler-Lagrange system.

Exercice 2 (8 points)**.** Let Ω be a bounded open subset of $\mathbb{R}^d$. Consider the following minimization problem

$$\min \left\{ \int_\Omega (1+e^u)(1+|\nabla u|^2)dx \ : \ u \in X \right\}.$$

1. If $X = H^1(\Omega)$, prove that the problem has no solution.
2. If $X = H_0^1(\Omega)$, prove that the problem admits at least a solution $\bar{u}$, and prove $\bar{u} \le 0$.
3. Via a suitable change of variable $v = g(u)$ prove that the minimizer $\bar{u}$ is unique, and that we have $\bar{u} \in C^\infty(\Omega)$ (interior regularity only).

Solution

1. If $X = H^1(\Omega)$, we can easily see that the inf of the problem is $|\Omega|$. Indeed, for every function $u \in X$ we have $F(u) > \int 1 = |\Omega$ (let F be the functional we minimize). Moreover, $F(-n) = |\Omega|(1+e^{-n}) \to |\Omega$ (where $-n$ is the constant function $-n$). Yet, no function u can satisfy the equality $F(u) = |\Omega|$ since this would imply $e^u = 0$.

2. If $X = H_0^1(\Omega)$, we take a minimizing sequence u_n. From the bound from above on $F(u_n$à) we deduce that $\int |\nabla u_n|^2 \le \int (1+|\nabla u_n|^2) \le F(u_n)$ is also bounded. Hence, u_n is bounded in H_0^1 and admits a subsequence which weakly converges in H^1 and strongly in L^2. Since the integrand $L(x,v) = (1+e^x)(1+|v|^2)$ is continuous in (x,v) and convex in v, the functional F is l.s.c. for this convergence, and the limit $\bar{u}$ is a minimizer.

   Let us compare $\bar{u}$ and $-|\bar{u}|$. The modulus of their gradient is the same, so that we have

$$(1+e^{-|\bar{u}|})(1+|\nabla|\bar{u}||^2) \le (1+e^{\bar{u}})(1+|\nabla \bar{u}|^2),$$

   with strict inequality where $-|\bar{u}| < \bar{u}$, i.e. where $\bar{u} > 0$. This proves that we have $F(-|\bar{u}|) < F(\bar{u})$ unless $\bar{u} \le 0$ a.e. Since $\bar{u}$ is a minimizer, we deduce $\bar{u} \le 0$.

3. We can write $F(u)$ in the form $F(u) = \int |h(u)|^2 |\nabla u|^2 + |h(u)|^2$, where $h(s) = \sqrt{1+e^s}$. Take g the anti-derivative of h, i.e. $g(0) = 0$ and $g' = h$. Do not look for an explicit expression of g. Set $v = g(u)$ and $\tilde{F}(v) = F(u)$. The problem becomes then

$$\min \tilde{F}(v) = \int |\nabla v|^2 + \phi(v)$$

   where $\phi = |h^2| \circ g^{-1}$. Note that g is a C^∞ diffeomorphism, since $g \in C^\infty$ and $g' \ge 1$. The function ϕ is also C^∞.

   The Euler-Lagrange equation of the problem is

$$2\Delta v = \phi'(v)$$

   and a simple bootstrap procedure proves $v \in C^\infty$. Indeed, we start from $v \in H^1$ and we can prove by induction $v \in H^k$ since $v \in H^k \Rightarrow v \in H^{k+2}$ by elliptic regularity. Hence, the minimizer $\bar{u}$ is also C^∞.

   Uniqueness can be proven once we check that ϕ is convex, and this makes $\tilde{F}$ strictly convex on X. Let us compute ϕ'. We have

$$\phi(s) = 1 + e^{g^{-1}(s)}, \quad \phi'(s) = \frac{e^{g^{-1}(s)}}{g'(g^{-1}(s))} \quad \phi'(g(s)) = \frac{e^s}{h(s)} = \frac{e^s}{\sqrt{1+e^s}}.$$

   The function ϕ is convex if and only if ϕ' is increasing, which is also equivalent to $\phi' \circ g$ being increasing. A simple computation proves that $s \mapsto \frac{e^s}{\sqrt{1+e^s}}$ is increasing since its derivative is $\frac{e^s + \frac{1}{2}e^{2s}}{(1+e^s)^{3/2}}$ and is positive, hence ϕ is convex.

Exercice 3 (6 points)**.** Given a continuous function $L : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, positive and convex in the second variable, and a bounded open domain $\Omega \subset \mathbb{R}^n$, prove that the following minimization problem admits a solution

$$\min \left\{ \mathrm{Per}(A) + \int_\Omega (|\nabla u|^p + |u|^p) + \int_A L(u, \nabla u) \ : \ u \in W^{1,p}(\Omega), A \subset \Omega, \, |A| = \frac{|\Omega|}{2} \right\},$$

where $\mathrm{Per}(A)$ stands for the perimeter - in the BV theory - of A, and the minimization is performed over u and A.

Solution

Sorry, we should have written $p > 1$... Also, it is maybe not clear that, even if we impose $A \subset \Omega$, we compute the perimeter of A as a subset of of $\mathbb{R}^n$ (i.e., we count the part of the boundary which is on $\partial\Omega$).

Take a minimizing sequence (A_n, u_n). From the bound of the functional we obtain that u_n is bounded in $W^{1,p}$ and that I_{A_n} is bounded in BV. We can extract a subsequence such that $u_n \rightharpoonup u$ in $W^{1,p}$ and $I_{A_n} \to v$ strongly in L^1 and a.e. Moreover, by pointwise a.e. convergence, we see that v is also an indicator function $v = I_A$. The strong convergence in L^1 impies $\int v = |\Omega|/2$, so that A also satisfies the constraint. We can also assume $\nabla I_{A_n} \rightharpoonup \nabla v = \nabla I_A$ as measures. We then have $\mathrm{Per}(A) \leq \liminf_n \mathrm{Per}(A_n)$ and $\int_\Omega (|\nabla u|^p + |u|^p) \leq \liminf_n \int_\Omega (|\nabla u_n|^p + |u_n|^p)$. In order to obtain the lower semicontinuity of the functional and prove existence we just need to prove that the term $\int_A L(u, \nabla u)$ is also l.s.c.

This term can be written as $\int I_A L(u, \nabla u)$. Consider the function $\tilde{L}(t, x, w) := t_+ L(x, w)$. This function is continuous in (t, x, w) and convex in w. Since (I_{A_n}, u_n) converges strongly to (I_A, u) and ∇u_n weakly to ∇u, we obtain the desired lower semicontinuity, and hence the existence of a minimizer.

Exercice 4 (7 points)**.** For given $f \in L^1(\Omega)$ with $\int_\Omega f(x)dx = 0$ and $p > d$ consider the functions u_p which solve

$$\min\left\{\frac{1}{p}\int |\nabla u|^p dx + \int fu \,:\, u \in W^{1,p}(\Omega)\right\}.$$

Prove that the sequence u_p is compact in $C^0(\Omega)$ and that we have, up to extracting subsequences, $u_p \to u_\infty$ uniformly, where u_∞ is a solution of the following problem

$$\min\left\{\int fu \,:\, u \in \mathrm{Lip}_1(\Omega)\right\},$$

where Lip_1 is the space of Lipschitz functions with Lipschitz constant at most 1.

Solution

It is clear that the minimizer u_p is not unique, and that we can add constants to it. Let us choose the minimizers u_p with 0 average. This should have been clarified in the statement, sorry. Also, We should add an assumption on Ω, which should be supposed to be bounded and sufficiently smooth.

Let us fix an exponent $p_0 > d$. For any $p > p_0$, we have (by comparing u_p to the constant function 0)

$$\frac{1}{p}||\nabla u_p||^p_{L^p} \leq ||f||_{L^1}||u_p||_{L^\infty}.$$

We now use the Jensen inequality in order to obtain $\int_\Omega h^{p_0} \leq (\int_\Omega h^p)^{p_0/p}|\Omega|^{1-p_0/p}$ for any $h \geq 0$, i.e. $||h||_{L^{p_0}} \leq |\Omega|^{1/p_0 - 1/p}||h||_{L^p}$. Hence we have

$$||\nabla u_p||_{L^{p_0}} \leq |\Omega|^{1/p_0-1/p}||\nabla u||_{L^p} \leq |\Omega|^{1/p_0-1/p}p^{1/p}||f||^{1/p}_{L^1}||u_p||^{1/p}_{L^\infty}.$$

We then use $||u_p||_{L^\infty} \leq C||\nabla u_p||_{L^{p_0}}$, which is a consequence of the injection of W^{1,p_0} into L^∞ and of the choice of u_p as a zero-average minimizer, and obtain

$$||\nabla u_p||^{1-1/p}_{L^{p_0}} \leq |\Omega|^{1/p_0-1/p}p^{1/p}|(C|f||_{L^1})^{1/p}.$$

Note that the constant C here depends on p_0.

By raising both sides to the power $p' = p/(p-1)$ we obtain

$$||\nabla u_p||_{L^{p_0}} \leq |\Omega|^{p'/p_0-1/(p-1)}p^{1/(p-1)}|(C|f||_{L^1})^{1/(p-1)}.$$

The r.h.s is bounded, hence the norm $||\nabla u_p||_{L^{p_0}}$ is also bounded independently of p. The compact injection of W^{1,p_0} into C^0 provides the desired compactness.

We can then assume, up to subsequences, that we have uniform convergence $u_p \to u$. The function u also belongs to W^{1,p_0} and satisfies

$$||\nabla u||_{L^{p_0}} \leq |\Omega|^{1/p_0},$$

which can be obtained by passing to the limit $p \to \infty$ (and hence $p' \to 1$) the previous inequality. We then use the fact that $p_0 > d$ is arbitrary and let $p_0 \to \infty$. we then obtain

$$||\nabla u||_{L^\infty} \leq 1$$

and we have $u \in \mathrm{Lip}_1$. We must now prove that the function u minimizes $\int fu$ among functions in Lip_1. Take $v \in \mathrm{Lip}_1$. We can write

$$\int fu_p \leq \frac{1}{p}\int |\nabla u_p|^p dx + \int fu_p \leq \frac{1}{p}\int |\nabla v|^p dx + \int fv \leq \frac{|\Omega|}{p} + \int fv.$$

We take the limit $p \to \infty$ and we obtain

$$\int fu \leq \int fv,$$

which is the desired inequality.