ENS Lyon and Université Claude Bernard Lyon 1, M2A
December 1st, 2020

Calculus of Variations and Elliptic PDEs
–
Mock Exam
–

Exercice 1 (6 points)**.** Consider the problem

$$\min\left\{\int_0^{\pi/2}\Big(u'(t)^2+u(t)^2+2\sin(t)u(t)\Big)\,dt\quad:\quad u\in C^1([0,\pi/2]),\ u(0)=0\right\}.$$

Prove that it admits a minimizer, that it is unique, and find it.

Exercice 2 (8 points)**.** Let Ω be a bounded open subset of $\mathbb{R}^d$. Consider the following minimization problem

$$\min\left\{\int_\Omega(1+e^u)(1+|\nabla u|^2)dx\ :\ u\in X\right\}.$$

1. If $X=H^1(\Omega)$, prove that the problem has no solution.
2. If $X=H^1_0(\Omega)$, prove that the problem admits at least a solution $\bar u$, and prove $\bar u\le 0$.
3. Via a suitable change of variable $v=g(u)$ prove that the minimizer $\bar u$ is unique, and that we have $\bar u\in C^\infty(\Omega)$ (interior regularity only).

Exercice 3 (6 points)**.** Given a continuous function $L:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, positive and convex in the second variable, and a bounded open domain $\Omega\subset\mathbb{R}^n$, prove that the following minimization problem admits a solution

$$\min\left\{\operatorname{Per}(A)+\int_\Omega(|\nabla u|^p+|u|^p)+\int_A L(u,\nabla u)\ :\ u\in W^{1,p}(\Omega),A\subset\Omega,\,|A|=\frac{|\Omega|}{2}\right\},$$

where $\operatorname{Per}(A)$ stands for the perimeter - in the BV theory - of A, and the minimization is performed over u and A.

Exercice 4 (7 points)**.** For given $f\in L^1(\Omega)$ with $\int_\Omega f(x)dx=0$ and $p>d$ consider the functions u_p which solve

$$\min\left\{\frac{1}{p}\int|\nabla u|^p dx+\int fu\ :\ u\in W^{1,p}(\Omega)\right\}.$$

Prove that the sequence u_p is compact in $C^0(\Omega)$ and that we have, up to extracting subsequences, $u_p\to u_\infty$ uniformly, where u_∞ is a solution of the following problem

$$\min\left\{\int fu\ :\ u\in \operatorname{Lip}_1(\Omega)\right\},$$

where Lip_1 is the space of Lipschitz functions with Lipschitz constant at most 1.