ENS Lyon and Université Claude Bernard Lyon 1, M2A
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Calculus of Variations and Elliptic PDEs
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Mid-Term Examination
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All kind of documents (notes, books...) are authorized. The total number of points is much larger than 20, which means that attacking only some exercises could be a reasonable option. The exercises are not necessarily ordered by difficulty.

Exercice 1 (5 points)**.** Consider the problem

$$\min\left\{\int_0^1\left(e^t\frac{u'(t)^2}{2}+e^{2t}u(t)\right)dt\quad:\quad u\in C^1([0,1]),\ u(1)=0\right\}.$$

Find its minimizer, proving that it is unique and justifying its minimality.

Solution:

The Euler-Lagrange system for this problem is given by

$$\begin{cases}(e^tu')'=e^{2t},\\ u(1)=0,\\ u'(0)=0.\end{cases}$$

Using the last condition we obtain

$$e^tu'(t)=\int_0^t e^{2s}ds=\frac{1}{2}(e^{2t}-1),$$

hence

$$u'(t)=\frac{1}{2}(e^t-e^{-t}),\qquad u(t)=\frac{1}{2}(e^t+e^{-t})+c.$$

In order to impose the Dirichlet condition at $t=1$ we need $c=-\frac{1}{2}(e+e^{-1})$.

The function u given by $u(t)=\frac{1}{2}(e^t+e^{-t}-(e+e^{-1}))$ satisfies all the conditions of the Euler-Lagrange system. The problem being convex, this function is a minimizer. It is the unique minimizer since the integrand is strictly convex in u' and convex in u: as a consequence, two minimizers could only differ by a constant, bbut the final value being fixed they should coincide.

Exercice 2 (10 points)**.** Let Ω be a bounded open subset of $\mathbb{R}^d$. Consider the minimization problem

$$\min\left\{\int_\Omega(e^{|\nabla u(x)|}-|\nabla u(x)|-u(x)^2-f(x)u(x))dx\ :\ u\in H_0^1(\Omega)\cap C^0(\Omega)\right\},$$

where $f\in L^1(\Omega)$ is a given function.

1. Prove that the problem has a solution.

2. Find the Euler-Lagrange equation of the problem, and in which sense do solutions solve it.

3. If $d=1$, prove that minimizers are Lipschitz functions, and that they are C^∞ if $f\in C^\infty$.

4. In case a minimizer $\bar u$ is Lipschitz continuous, prove $\int_\Omega(e^{|\nabla\bar u|}-1)|\nabla\bar u|dx=\int_\Omega(2\bar u^2+f\bar u)dx$.

5. *(Much more difficult: the goalhere is to prove a similar relation without assuming $\bar u \in W^{1,\infty}$).*

   (a) Prove that at least a minimizer $\bar u$ satisfies $\int_\Omega (e^{|\nabla \bar u|}-1)|\nabla \bar u| dx \le \int_\Omega (2\bar u^2 + f\bar u)dx$.

   (b) Prove that the same inequality is satisfied by any minimizer $\bar u$.

Solution:

1. Let us take a minimizing sequence u_n. From $e^{|\nabla u|} - |\nabla u| = 1 + \sum_{k\ge 2}\frac{1}{k!}|\nabla u|^k$, we see that all competitor for which the functional is finite actually belongs to $W^{1,p}_0$ for every p. Choosing an integer p with $p > \max\{2,d\}$, we deduce a $W^{1,p}$ bound on u_n. Indeed, we have

$$\frac{1}{p!}||\nabla u||^p_{L^p} - ||u_n||^2_{L^2} - ||f||_{L^1}||u_n||_{L^\infty} \le C.$$

   Using the injection of $W^{1,p}$ into both L^2 and L^∞ (because p is large) and the equivalence between the $W^{1,p}$ norm and the L^p norm of the gradient (because of the boundary condition), we see that this provides a bound on $\frac{1}{p!}||\nabla u||^p_{L^p}$. Up to subsequences, we can suppose that we have $u_n \rightharpoonup u$ in $W^{1,p}$, with uniform convergence $u_n \to u$.

   Since $z \mapsto e^{|z|} - |z|$ is convex, the functional is lower semicontinuous for the weak $W^{1,p}$ convergence. The function u is an admissible competitor since it belogns to H^1_0 as a weak limit (in $W^{1,p}$ and hence also in H^1) of H^1_0 functions and since it is continuous as it belogs to $W^{1,p}$ for $p > d$. Hence, u is a minimizer.

2. The Euler-Lagrange equation is given by

$$\nabla \cdot \left((e^{|\nabla u|} - 1)\frac{\nabla u}{|\nabla u|}\right) = -2u - f.$$

   Yet, if we want to guarantee that we can differentiate under the integral sign in order to obtain its weak version, i.e.

$$\int_\Omega (e^{|\nabla u|} - 1)\frac{\nabla u}{|\nabla u|}\cdot \nabla\varphi = \int_\Omega (2u+f)\varphi,$$

   we need to bound the integral of $e^{|\nabla u + \varepsilon\nabla\varphi|}$, which is possible if $|\nabla\varphi|$ is bounded. This is why we can state the weak version of the Euler-Lgrange equation is satsfied if restricted to Lipschitz test functions φ with $\varphi = 0$ on $\partial\Omega$.

3. If $d = 1$, the weak deriviative of the L^1 function $(e^{|\bar u'|} - 1)\frac{\bar u'}{|\bar u'|}$ equals $-2\bar u - f$, which is L^1. Primitevs of L^1 functions are bounded, which provides a bound on $|\bar u'|$, and $\bar u$ is Lipschitz continuous. If $f \in C^\infty$ we can iterate the regularity argument: we now know that $(e^{|\bar u'|} - 1)\frac{\bar u'}{|\bar u'|}$ has a Lipschitz derivative, hence it is $C^{1,1}$. We deduce $\bar u \in C^{2,1}$ and by induction $u \in C^{k,1}$ for every k.

4. In case a minimizer $\bar u$ is Lipschitz continuous we can use it as a test function in the Euler-Lgrange equation, and the resulting condition exactly provides $\int_\Omega (e^{|\nabla \bar u|}-1)|\nabla \bar u| dx = \int_\Omega (2\bar u^2 - f\bar u)dx$.

5. The difficult point here is that we cannot in general use the minimizer itself as a test function, while it is the case when the integrand has a polynomial growth. We therefore define $F_N(u) = \int_\Omega (1 + \sum_{k=2}^N \frac{|\nabla u|^k}{k!})$ and $F(u) = \int_\Omega e^{|\nabla u|} - |\nabla u| = \lim_{N\to\infty} F_N(u)$, and consider the problem where F is replaced with F_N.

   (a) If we call u_N a minimizer of $F_N(u) - \int_\Omega u^2 + fu$ (which exists for $N > \max\{2,d\}$), we can prove now that we have

$$\int_\Omega \sum_{k\ge 2}^{N} \frac{1}{(k-1)!}|\nabla u_N|^k = \int_\Omega 2u_N^2 + fu_N.$$

If we fix again an integer p with $p > \max\{2, d\}$ and we restrict to $N \geq p$ we can obtain a uniform bound (not depending on N) on $||\nabla u_N||_{L^p}$. We can then extract a subsequence (that we still denote by u_N) such that $u_N \rightharpoonup u_\infty$ in $W^{1,p}$. This weak convergence implies the uniform convergence of u_N to u_∞. We then consider

$$\int_\Omega \sum_{k\geq 2}^{p} \frac{1}{(k-1)!}|\nabla u_N|^k \leq \int_\Omega 2u_N^2 + fu_N$$

and pass to the limit, using convexity and hence lower semicontinuity on the left hand side. We then obtain

$$\int_\Omega \sum_{k\geq 2}^{p} \frac{1}{(k-1)!}|\nabla u_\infty|^k \leq \int_\Omega 2u_\infty^2 + fu_\infty$$

and, taking the limit $p \to \infty$, we obtain the desired result. We just need now to show that u_∞ is a minimizer of the original problem. This comes from the minimality of u_N, since we have, for an arbitrary competitor u and $N \geq p$,

$$\int_\Omega 1+\sum_{k\geq 2}^{p} \frac{1}{k!}|\nabla u_N|^k - \int_\Omega (u_N^2+fu_N) \leq \int_\Omega 1+\sum_{k\geq 2}^{N} \frac{1}{k!}|\nabla u|^k - \int_\Omega (u^2+fu) \leq \int_\Omega e^{|\nabla u|} - |\nabla u| - \int_\Omega (u^2+fu).$$

From the semicontinuity on the left hand side we obtain

$$\int_\Omega 1 + \sum_{k\geq 2}^{p} \frac{1}{k!}|\nabla u_\infty|^k - \int_\Omega (u_\infty^2 + fu_\infty) \leq \int_\Omega e^{|\nabla u|} - |\nabla u| - \int_\Omega (u^2 + fu).$$

Sending $p \to \infty$ we obtain the desired minimality for u_∞.

(b) Proving that the same inequality is satisfied by any minimizer $\bar{u}$ is more delicate since we do not have uniqueness of the minimizer. We then act in the following way. Let $\bar{u}$ be a minimizer and set $g = 2\bar{u} + f$. We now consider u_N a minimizer of $F_N(u) - \int gu$. The same arguments as above prove that we have

$$\int_\Omega \sum_{k\geq 2}^{N} \frac{1}{(k-1)!}|\nabla u_N|^k = \int_\Omega gu_N$$

and that u_N is bounded in $W^{1,p}$ for $p > \max\{2, d\}$. We then call, again, u_∞ its weak the limit and obtain (via the same computations as before) that u_∞ minimizes the functional $u \mapsto \int_\Omega e^{|\nabla u|} - |\nabla u| - \int_\Omega gu$ and that we have, indeed

$$\int_\Omega (e^{|\nabla u_\infty|} - 1)|\nabla u_\infty| dx \leq \int_\Omega (2\bar{u} + f)u_\infty.$$

We are left to prove $u_\infty = \bar{u}$.
To do this, we exploit both the minimality of u_∞ and $\bar{u}$:

$$\begin{aligned} F(u_\infty) - \int gu_\infty &\leq F(\bar{u}) - \int g\bar{u} \\ F(\bar{u}) - \int \bar{u}^2 + f\bar{u} &\leq F(u_\infty) - \int u_\infty^2 + fu_\infty. \end{aligned}$$

If we sum the two ineqalities and simplify the expressions using $g = 2\bar{u} + f$, we obtain $\int |u_\infty - \bar{u}|^2 \leq 0$, i.e. $u_\infty = \bar{u}$.

Exercice 3 (5 points)**.** Consider the two functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ given by

$$f_1(x) = \begin{cases} x(\log x)^2 & \text{if } x \geq 1 \\ 0 & \text{if } x < 1, \end{cases} \qquad f_2(x) = \begin{cases} x(\log x)^2 & \text{if } x \geq 1 \\ +\infty & \text{if } x < 1, \end{cases}.$$

1. Prove that f_1 and f_2 are convex.

2. Find f_1^* and f_2^*.

Solution: Let us compute the first derivative of $x \mapsto x(\log x)^2$: it is given by $(\log x)^2 + 2\log x$, which is positive and increasing on $x \geq 1$. Hence, both f_1 and f_2 are convex and increasing on $[1, +\infty)$, and they are also convex on $\mathbb{R}$ since in the case of f_1 the function has be extended to be equal to its tangent at $x = 1$, in the case of f_2 to $+\infty$ our of a convex set.

We now compute f_1^*. We have $f_1^*(y) = \sup_x xy - f_1(x)$. Since $f_1 = 0$ before 1, if $y > 0$ the minimization can be restricted to $\{x \geq 0\}$ since for $x < 1$ we obtain results smaller than what is obtained for $x = 1$. On the other hand, for $y < 0$ the sup can be attained using a sequence of points diverging to $-\infty$, and we have $f_1^*(y) = +\infty$. For $y = 0$ we have $f_1^*(0) = \sup -f_1 = 0$. We ust have to compute the value of $f_1^*(y)$ for $y > 0$ and we look for the maximizer by computing the derivative. The derivative of $x \mapsto xy - f_1(x)$ is equal for $x \geq 0$ to $y - (\log x)^2 + 2\log x$. Setting $z = \log x \geq 0$, this vanishes whenever $(z+1)^2 = y + 1 > 1$, hence $z = \pm\sqrt{y+1} - 1$. Since the choice of the negative sign would give $z < 0$, the maximum is attained for a point x such that $\log x = \sqrt{y+1} - 1 > 1 - 1 = 0$, so that $x > 1$. We now compute this maximum:

$$f_1^*(y) = xy - x(\log x)^2 = x(y - z^2) = x((z+1)^2 - 1 - z^2) = 2xz = 2e^{\sqrt{y+1}-1}(\sqrt{y+1} - 1).$$

This same expression is also valid for $y = 0$, as it gives 0 as a value.

Summarizing, we have

$$f_1^*(y) = \begin{cases} 2e^{\sqrt{y+1}-1}(\sqrt{y+1} - 1) & \text{if } y \geq 0, \\ +\infty & \text{if } y < 0. \end{cases}$$

The computations for f_2^* are exactly the same for $y > 0$ since in this case it is obvious that the maximum should be restricted to $x \geq 1$ and the maximizer obtained by canceling the derivative satisfies $x > 1$. For $y = 0$ we have again $f_2^*(0) = \sup -f_2 = 0$. For $y < 0$ the function $x \mapsto xy - f_2(x)$ is decreasing, so that $x = 1$ is optimal, and we have $f_2^*(y) = y$. In this case we have

$$f_2^*(y) = \begin{cases} 2e^{\sqrt{y+1}-1}(\sqrt{y+1} - 1) & \text{if } y \geq 0, \\ y & \text{if } y < 0. \end{cases}$$

Exercice 4 (8 points)**.** Let Ω be a bounded open subset of $\mathbb{R}^d$. Consider the following minimization problem

$$\inf \left\{ \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - 2\sqrt{u} \right) dx \ : \ u \geq 0, u - 1 \in H_0^1(\Omega) \right\}.$$

1. Prove that it admits a unique solution.

2. Prove that the solution $\bar{u}$ satisfies $\bar{u} \geq 1$.

3. Prove that the solution is C^∞ on the interior of Ω.

4. In the cases where Ω is a cube, prove that we have $\bar{u} \in W^{2,p}(\Omega)$ for every $p < \infty$.

5. In the cases where Ω is a ball, prove that the solution is radially decreasing and C^∞ up to the boundary.

Solution:

1. Any minimizing sequence u_n satisfies $||\nabla u_n||_{L^2}^2 \leq C(1 + \int \sqrt{u_n}) \leq C(1 + ||u_n||_{L^1}^{1/2})$ (the last inequality comes from Jensen's or Hölder's inequalities). Using $||u_n||_{L^1} \leq ||u_n - 1||_{L^1} + C$ together with Poincaré's inequality we see that this imples a bound on $||\nabla u_n||_{L^2}$. We can then extract a weakly converging subsequence in H^1. The limit u is non-negative and is a minimizer since the functional is l.s.c. as the integrand is convex in both u and ∇u. The minimizer is unique since the functional is even strictly convex.

2. If the solution $\bar{u}$ does not satisfy $\bar{u} \geq 1$ then $\tilde{u} := \max\{1, \bar{u}\}$ would provide a better value than $\bar{u}$.

3. the Euler-Lagrange equation of the problem is $\Delta u = u^{-1/2}$ and is solved by $u = \bar{u} \geq 1$. This implies $\Delta \bar{u} \in L^\infty \subset L^p$ for every p, so that $\bar{u} \in W^{2,p}_{loc}$. The function $s \mapsto s^{-1/2}$ is a C^∞ diffeomorphism with bounded derivatives on $[1, \infty)$, so that $\bar{u}^{-1/2}$ has the same regularity as $\bar{u}$. Iterating the regulairty argument we obtain $\bar{u} \in W^{2k,p}_{loc}$ for every k, hence $\bar{u} \in C^\infty$ far from the boundary.

4. In the cases where Ω is a cube the standard reflection arguments allow to see $\bar{u}$ as the restriction of a solution u to $\Delta u = f$ with $f \in L^\infty$ (but no extra regularity since the reflection breaks the continuity of f). this proves that the $W^{2,p}$ regularity is global.

5. In the cases where Ω is a ball, the uniqueness of the solution implies that $\bar{u}$ is radial. We then write $\bar{u}(x) = \bar{f}(|x|)$ and look at the variational problem solved by $\bar{f}$, which minimizes

$$\int_0^R r^{d-1} \left(\frac{|f'(r)|^2}{2} - \sqrt{f(r)} \right) dr$$

among functions with $f(R) = 1$. The correspondin Euler-Lagrange equation is $(r^{d-1} f'(r))' = -r^{d-1} f(r)^{-1/2}$. Since we know that $\bar{u}$ is smooth inside the ball, then we have $f'(0) = 0$, which implies $f'(r) < 0$ for every r using the equation, and hence $\bar{u}$ is radially decreasing. Moreover, the above differential equation shows that, at least far from $r = 0$, the regularity of f' is one extra derivative better than that of f, which iteratively implies $f \in C^\infty$. This can be applied on (ε, R), while the general result can be applied on $[0, R - \varepsilon)$ and, together, they imply $\bar{u} \in C^\infty$ up to the boundary.