ENS Lyon and Université Claude Bernard Lyon 1, M2A
October 25th, 2021

Calculus of Variations and Elliptic PDEs
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Mid-Term Examination
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All kind of documents (notes, books...) are authorized. The total number of points is much larger than 20, which means that attacking only some exercises could be a reasonable option. The exercises are not necessarily ordered by difficulty.

Exercice 1 (5 points)**.** Consider the problem

$$\min\left\{\int_0^1 \left(e^t\frac{u'(t)^2}{2} + e^{2t}u(t)\right)dt \quad : \quad u \in C^1([0,1]),\ u(1) = 0\right\}.$$

Find its minimizer, proving that it is unique and justifying its minimality.

Exercice 2 (10 points)**.** Let Ω be a bounded open subset of $\mathbb{R}^d$. Consider the minimization problem

$$\min\left\{\int_\Omega (e^{|\nabla u(x)|} - |\nabla u(x)| - u(x)^2 - f(x)u(x))dx \ : \ u \in H^1_0(\Omega) \cap C^0(\Omega)\right\},$$

where $f \in L^1(\Omega)$ is a given function.

1. Prove that the problem has a solution.
2. Find the Euler-Lagrange equation of the problem, and in which sense do solutions solve it.
3. If $d = 1$, prove that minimizers are Lipschitz functions, and that they are C^∞ if $f \in C^\infty$.
4. In case a minimizer $\bar u$ is Lipschitz continuous, prove $\int_\Omega (e^{|\nabla \bar u|} - 1)|\nabla \bar u|dx = \int_\Omega (2\bar u^2 + f\bar u)dx$.
5. *(Much more difficult: the goal here is to prove a similar relation without assuming $\bar u \in W^{1,\infty}$).*
   (a) Prove that at least a minimizer $\bar u$ satisfies $\int_\Omega (e^{|\nabla \bar u|} - 1)|\nabla \bar u|dx \le \int_\Omega (2\bar u^2 + f\bar u)dx$.
   (b) Prove that the same inequality is satisfied by any minimizer $\bar u$.

Exercice 3 (5 points)**.** Consider the two functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ given by

$$f_1(x) = \begin{cases} x(\log x)^2 & \text{if } x \ge 1, \\ 0 & \text{if } x < 1, \end{cases} \qquad f_2(x) = \begin{cases} x(\log x)^2 & \text{if } x \ge 1, \\ +\infty & \text{if } x < 1. \end{cases}$$

1. Prove that f_1 and f_2 are convex.
2. Find f_1^* and f_2^*.

Exercice 4 (8 points)**.** Let Ω be a bounded open subset of $\mathbb{R}^d$. Consider the following minimization problem

$$\inf\left\{\int_\Omega \left(\frac{1}{2}|\nabla u|^2 - 2\sqrt{u}\right)dx \ : \ u \ge 0,\ u - 1 \in H^1_0(\Omega)\right\}.$$

1. Prove that it admits a unique solution.
2. Prove that the solution $\bar u$ satisfies $\bar u \ge 1$.
3. Prove that the solution is C^∞ on the interior of Ω.
4. In the cases where Ω is a cube, prove that we have $\bar u \in W^{2,p}(\Omega)$ for every $p < \infty$.
5. In the cases where Ω is a ball, prove that the solution is radially decreasing and C^∞ up to the boundary.