ENS Lyon and Université Claude Bernard Lyon 1, M2A
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Calculus of Variations and Elliptic PDEs
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Final Exam
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3h duration. All kind of documents (notes, books...) are authorized. The total number of points is much larger than 20, which means that attacking only some exercises could be a reasonable option. The exercises are not necessarily ordered by difficulty.

Exercice 1 (4 points)**.** Consider the two optimization problems

$$(\mathrm{P_D}) \quad \min\left\{\int_0^1 (|u'(t)|^2+|u(t)-t|^2)\,dt \quad : \quad u\in C^1([0,1]),\ u(0)=0, u(1)=1\right\},$$
$$(\mathrm{P_N}) \quad \min\left\{\int_0^1 (|u'(t)|^2+|u(t)-t|^2)\,dt \quad : \quad u\in C^1([0,1])\right\}.$$

Prove that u_D given by $u_D(t)=t$ is a solution of $(\mathrm{P_D})$, and that it is its unique solution.
Then, find the solution u_N of $(\mathrm{P_N})$.

Solution: For $(\mathrm{P_D})$, consider that we have $\int_0^1|u'(t)|^2dt \geq \left(\int_0^1 u'(t)dt\right)^2 = 1$ and $\int_0^1|u(t)-t|^2dt\geq 0$ for every u, and that both ineualities are equalities for $u=u_D$. Moreover, if we want equality we need equality in the second inequality, which only occurs for $u=u_D$.

For $(\mathrm{P_N})$, we write the Euler-Lagrange equation, which reads

$$\begin{cases} 2u''=2(u(t)-t),\\ u'(0)=0,\\ u'(1)=0.\end{cases}$$

The only solution of this system is $u(t)=\frac{e}{e+1}e^{-t}-\frac{1}{e+1}e^{t}+t$, and it is a solution of the minimization problem since the problem is convex.

Exercice 2 (5 points)**.** Given a bounded and smooth domain $\Omega\subset\mathbb{R}^d$ and a function $u_0\in H^1(\Omega)$, prove that we have

$$\min\left\{\int_\Omega\left(\frac{|\nabla u|^2}{2}+|u-u_0|\right)dx \ : \ u\in H^1(\Omega)\right\}$$
$$=\max\left\{\int_\Omega\left(-\frac{|v|^2}{2}+v\cdot\nabla u_0\right)dx \ : v\in L^2(\Omega;\mathbb{R}^d), \nabla\cdot v\in L^\infty, ||\nabla\cdot v||_{L^\infty}\leq 1\right\}.$$

Proe that both problems admit a unique solution, and deduce that the optimizer of the first problem is a function u such that $|\Delta u|\leq 1$, $\Delta u=1$ on $\{u>u_0\}$, and $\Delta u=-1$ on $\{u<u_0\}$.

Solution: We can use the Fenchel-Rockafellar theorem with $X=H^1$, $Y=L^2$, $A=\nabla$, $A^t=-\nabla\cdot$, and write the first problem as $\min F(u)+G(Au)$ with $F(u)=\int|u-u_0|$ and $G(w)=\frac{1}{2}\int|w|^2$. The transform of G is easy to compute and we have $G^*(v)=\frac{1}{2}\int|w|^2$. As for the transform of F we obtain

$$F^*(p)=\sup_u <p,u> -||u-u_0||_{L^1}=\sup_{\tilde u}<p,\tilde u>+<p,u_0>-||\tilde u||_{L^1}=<p,u_0>+I_A(p),$$

where I_A is the function taking value 0 on A and $+\infty$ outisde A, the set A being the set of elements $p\in(H^1)'$ which belong to L^∞ and have L^∞ norm smaller than 1. The Fenchel-Rockafellar theorem can be applied because all spaces are Hilbert spaces, G is continuous everywhere and F is finite everywhere. Then, we just have to re-write $-<\nabla\cdot v, u_0>$ as $\int v\cdot\nabla u_0$.

Both problems admit a solution since a minimizing sequence u_n for the problem on the left (the primal problem) will be such that $||\nabla u_n||_{L^2}$ and $||u_n||_{L^1}$ are bounded. As a consequence, $⨍ u_n$ is bounded, and $||u_n - ⨍ u_n||_{L^2}$ is bounded by Poincaré-Wirtinger. We deduce that u_n is bounded in H^1 and can conclude by semicontinuity. For the dual problem, any maximizing sequence will be bounded in L^2 and we conclude again by semicontinuity.

If we call u and v the two optimizers we have

$$\begin{aligned} 0 &= \int\left(\frac{|\nabla u|^2}{2}+|u-u_0|\right)+\int\left(\frac{|v|^2}{2}-v\cdot\nabla u_0\right) \\ &\geq \int \nabla u\cdot v+(\nabla\cdot v)(u-u_0)-v\cdot\nabla u_0 \\ &= \int \nabla u\cdot v - v\cdot\nabla(u-u_0)-v\cdot\nabla u_0 = 0, \end{aligned}$$

where in the second line we used the Young ineuality $\frac{1}{2}|a|^2+\frac{1}{2}|b|^2\geq a\cdot b$ on $a=v$ and $b=\nabla u$ (the only euality case being $a=b$) and the inequality $|u-u_0\geq(\nabla\cdot v)(u-u_0)$ coming from $|\nabla\cdot v|\leq 1$ (where we only have equality only equality if ∇v takes value 1 on $\{u>u_0\}$, and -1 on $\{u<u_0\}$). Since all these inequalities must be eualities we obtain $v=\nabla u$ and the desired condition on $\Delta u=\nabla\cdot\nabla u$.

Exercice 3 (10 points)**.** We consider the following optimization problem

$$\text{(P)}\quad \min\left\{M(u):=\int_{\mathbb{R}^3} u^3(x)dx\ ,\ u\in\mathcal{S}(\mathbb{R}^3)\right\}$$

where $\mathcal{S}(\mathbb{R}^3)$ is the set of non-trivial nonnegative subsolutions of $\Delta u+u^3=0$, i.e.

$$\mathcal{S}(\mathbb{R}^3)=\left\{u\in L^3(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)\cap H^1_{loc}(\mathbb{R}^3), u\geq 0, \Delta u+u^3\geq 0\right\}\setminus\{0\},$$

(the inequality $\Delta u+u^3\geq 0$ has to be considered in its weak form $-\int\nabla u\cdot\nabla\psi+\int u^3\psi\geq 0$ for everynon-negative test function $\psi\in H^1$ with compact support).

1. Prove that $\mathcal{S}(\mathbb{R}^3)$ is non-empty and that, if g denotes the Gaussian function $g(x)=e^{-|x|^2/2}$ and $A>0$ is a constant, the function Ag belongs to $\mathcal{S}(\mathbb{R}^3)$ if and only if $A\geq e$.

2. Prove that for every $\lambda>0$ if $u\in\mathcal{S}(\mathbb{R}^3)$ the function u_λ defined via $u_\lambda(x)=\lambda u(\lambda x)$ also belongs to $\mathcal{S}(\mathbb{R}^3)$ and $M(u_\lambda)=M(u)$ and deduce that one can construct a minimizing sequence $(u_n)_n$ for (P) such that $||u_n||_{L^\infty}=1$ for every n.

3. In arbitrary dimension d (not necessarily $d=3$ in this question), prove that there exists a constant $C=C(d)$ such that whenever u satisfies $\Delta u\geq -1$ then $R\mapsto ⨍_{B(x_0,R)} u(x)dx+CR^2$ is nondecreasing for every x_0 and admits a limit as $R\to 0$. Prove that such a limit, as a function of x_0, is a representative of u and that for this representative we have $⨍_{B(x_0,R)} u(x)dx\geq u(x_0)-CR^2$. Also prove that the same representative is upper-semicontinuous.

4. Again in arbitrary dimension, prove that for every $p\in[1,\infty)$ and every function $u\in L^p(\mathbb{R}^d)$ satisfying $\Delta u\geq -1$, if we choose the representative described in the previous question, then we have $\lim_{|x|\to\infty} u(x)=0$.

5. Deduce that (P) admits a minimizing sequence $(u_n)_n$ made of upper-semicontinuous functions functions u_n tending to 0 at infinity and such that $u_n(0)=\max u_n$ and $⨍_{B(x_0,R_0)} u_n(x)dx>\frac{1}{2}$ for a certain radius R_0 independent of n.

6. Prove that such a minimizing sequence is bounded in $H^1_{loc}(\mathbb{R}^3)$ and deduce that it admits a subsequence which locally weakly converges in H^1 to a function $\bar{u}\in\mathcal{S}$.

7. Prove that $\bar{u}$ is a solution of (P).
8. Suppose that, in an open set $\Omega \subset \mathbb{R}^3$, we have $\Delta\bar{u} + \bar{u}^3 = f \in C^0(\Omega)$. Prove that the optimality of $\bar{u}$ implies $f = 0$ in Ω and deduce that the solution of (P) is not of the form $u = Ag$.

Solution:

1. From $\nabla g = -xg$ we get $\Delta g = (-3 + |x|^2)g$ (using that the divergence of the vector field x is 3 in dimension 3) and hence $\Delta(Ag) + (Ag)^3 = (Ag)(-3 + |x|^3 + A^2g^2)$. The function Ag hence belongs to $\mathcal{S}(\mathbb{R}^3)$ if and only if A is such that $-3 + |x|^3 + A^2g^2 \geq 0$. Using $g^2 = e^{-|x|^2}$ and writing in terms of $t = |x|^2$ we need $A^2 \geq \max_{t\geq 0}(3 - t)e^t$. We can compute that this maximum is attained for $t = 2$ and its value is e^2. We then have $Ag \in \mathcal{S}(\mathbb{R}^3)$ if and only if $A \geq 3$. In particular, $\mathcal{S}(\mathbb{R}^3) \neq \emptyset$.

2. We can see that we have $\Delta(u_\lambda)(x) + u_\lambda^3(x) = \lambda^3(\Delta u + u^3)(\lambda x) \geq 0$. Moreover we have $\int u_\lambda^3 = \int \lambda^3 u(\lambda x)^3 = \int u^3$ by change of variables. Hence, when taking a minimizing sequence $\tilde{u}_n$ for (P), we can replace it by $u_n = (\tilde{u}_n)_{\lambda_n}$ where $\lambda_n = ||\tilde{u}_n||_{L^\infty}^{-1}$, thus guaranteeing at the same time that we have $||u_n||_{L^\infty} = 1$ and that we still havea minimizing sequence.

3. For functions v such that $\Delta v \geq 0$ we have that $R \mapsto ⨍_{B(x_0,R)} v(x)dx$ is nondecreasing. Here we can apply it to $v(x) = u(x) + \frac{|x-x_0|^2}{2d}$. The value of the constant C to be chosen is then $C = \frac{1}{2(d+2)}$. Once we know that $R \mapsto ⨍_{B(x_0,R)} u(x)dx + CR^2$ is nondecreasing it is clear that it admits a limit as $R \to 0$, and this limit is also the limit of $R \mapsto ⨍_{B(x_0,R)} u(x)dx$ since the other term tends to 0. This limit coincides with $u(x_0)$ whenever x_0 is a Lebesgue point of u, so it provides a representative of u. Since we obtained $u(x_0)$ as a limit of a nondrecreasign quantity, we also have the inequality $u(x_0) \leq ⨍_{B(x_0,R)} u(x)dx + CR^2$ for every x_0 and R.

   In order to prove that u is usc, take a sequence $x_n \to x_0$ and consider the inequality $u(x_n) \leq ⨍_{B(x_n,R)} u(x)dx + CR^2$. We then pass to the limit, and the integral for fixed R converges so that we obtain $\limsup_n u(x_n) \leq ⨍_{B(x_0,R)} u(x)dx + CR^2$. Taking then the limit as $R \to 0$ we obtain the desired semicontinuity.

4. Suppose that there exists $\varepsilon > 0$ and a sequence x_n with $|x_n| \to \infty$ and $u(x_n) \geq \varepsilon$. For R small enough and only depending on ε but not on x_n (say, $CR^2 < \varepsilon/2$) we also have $⨍_{B(x_n,R)} u(x)dx > \varepsilon/2$ and hence $\int_{B(x_n,R)} u^p(x)dx \geq C = C(\varepsilon, R) > 0$. We can choose our sequence x_n such that all the balls $B(x_n, R)$ are disjoint (it is enough to impose $|x_{n+1}| > |x_n| + 2R$) and this gives $\int u^p(x)dx \geq \sum_n \int_{B(x_n,R)} u^p(x)dx = +\infty$, a contradiction.

5. Using the preivous results we can choose a minimizing sequence such that $||u_n||_{L^\infty} = 1$. In particular, $\Delta u_n \geq -1$. This implies that u_n is usc and tends to 0 at infinity, and hence admits a maximum point. We can translate this function so that the maximum is attained at 0. Moreover $⨍_{B(0,R)} u(x)dx \geq 1 - CR^2$ and we just need to choose R_0 small enough so that $1 - CR_0^2 > 1/2$.

6. Testing the ineuality $\Delta u_n \geq -1$ against $u_n\eta^2$, where η is a cut-off function, we obtain $\int |\nabla u_n|^2\eta^2 \leq \int u_n\eta^2 + \int 2\nabla u_n \cdot \nabla\eta u_n\eta \leq C(\eta)(1 + (\int |\nabla u_n|^2\eta^2)^{1/2})$. This provides a bound on $\int |\nabla u_n|^2\eta^2$, i.e. an H^1_{loc} bound. We can then extract, for every ball, a subsequence which is weakly converging in H^1 on such a ball, but with a diagonal extraction we can guarantee that the same subsequence satisfies wak convergence on each ball to a limit $\bar{u}$.

7. First of all, let us prove $\bar{u} \in \mathcal{S}(\mathbb{R}^3)$. Taking a test function ψ with compact support we have $-\int \nabla u_n \cdot \nabla\psi + \int u_n^3\psi \geq 0$. This inequality passes to the limit thanks to the weak convergence $\nabla u_n \rightharpoonup \nabla\bar{u}$ in L^2 and the strong convergence $u_n \to \bar{u}$ in L^3 (because 3 is below the critical Sobolev exponent, equal to 6 in dimension $d = 3$). This proves that $\bar{u}$ satisfies the differential condition. Of course we have $0 \leq \bar{u} \leq 1$ as a consequence of $0 \leq u_n \leq 1$. Moreover the condition $⨍_{B(x_0,R_0)} u(x)dx > \frac{1}{2}$ passes to the limit and guarantees that $\bar{u}$ is not the zero function. Moreover, the strong convergence on every bounded set implies a.e. convergence and, by Fatou's lemma, we obtain $\int \bar{u}^3 \leq \liminf_n \int u_n^3$, which proves that $\bar{u}$ is a minimizer.

8. Suppose that f is not the zero function and take an open set Ω' where f is bounded from below by a strictly positive constant. Let us take a smooth function $\varphi \geq 0$ supported in Ω' and define $u_\varepsilon = \bar{u} - \varepsilon\varphi$. We have $\Delta u_\varepsilon + u_\varepsilon^3 \geq f - \varepsilon\Delta\varphi - 3\varepsilon\bar{u}^2\varphi - C\varepsilon^3\varphi^3$. For small ε, this quantity is still nonnegative, but $M(u_\varepsilon) < M(\bar{u})$. Of course, since for the Gaussians $\Delta u + u^3$ ia continuous but non-identically-zero function, they cannot be optimizers.

Exercice 4 (4 points)**.** Given a function $u \in L^1(\mathbb{R}^d)$ and a number $R > 0$, define a function $a_R[u]$ as follows: $a_R[u](x) := ⨍_{B(x,R)} u(y)dy$. Suppose that a function $u \in H^1_{loc}(\mathbb{R}^d)$ satisfies $u > 0$ a.e. and

$$\nabla \cdot (a_R[u]\nabla u) = \frac{u}{1+u}.$$

Prove that we have $u \in C^\infty(\mathbb{R}^d)$.

Solution: The condition $u \in H^1_{loc}$ is only needed to give a meaning to the term in the divergence, all the regularity will only need weaker assumptions.

For any $u \in L^1$ the function $a_R[u]$ is continuous since when $x_n \to x$ we have $|a_R[u](x_n) - a_R[u](x)| \leq C\int_{A_n} u$ where A_n is the symmetric difference of $B(x_n, R)$ and $B(x, R)$, whose measure tends to 0. Moreover, using $u > 0$ a.e., we have $a_R[u] > 0$ everywhere. As a consequence, being a continuous and strictly positive function, $a_R[u]$ is locally bounded both from below and from above by positive constants. We can then apply DeGiorgi-Nash-Moser's results in order to obtain $u \in C^{0,\alpha}$ as soon that we guarantee that $u \in L^2_{loc}$ (this is a consequence of $u \in H^1 loc$) and that the right-hand side is the divergence of an L^p_{loc} function for $p > d$. Here the right-hand side $f = u/(1 + u)$ itself is L^∞, which is better. Indeed, setting $F(x_1, \dots, x_n) = e_1 \int_0^{x_1} f(t, x_2, \dots, x_n)dt$ we have $f = \nabla \cdot F$ and $F \in L^\infty_{loc}$. This proves that we have $u \in C^{0,\alpha}$. Then, we can look again at the equation: we have now $\nabla\cdot(a\nabla u) = \nabla\cdot F$ with $a, F \in C^{0,\alpha}$. We deduce $u \in C^{1,\alpha}$ and we can go on obtaining $C^{k,\alpha}$ for every k.

Exercice 5 (7 points)**.** Take $X = \{u \in L^1([-1, 1]) : 0 \leq u \leq 1\, a.e.\}$ and define a family of functionals on X, indexed by $\varepsilon > 0$, as follows:

$$F_\varepsilon(u) = \begin{cases} \frac{\varepsilon}{2}\int_{-1}^1 |u'(t)|^2dt + \frac{1}{2\varepsilon}\int_{-1}^1 u^2(t)(1-u(t))^2dt & \text{if } u \in H^1_0([-1,1]) \text{ and } u(0) = 0, \\ +\infty & \text{if not.} \end{cases}$$

Also define the functional F as follows: if u is the indicator function of a union of intervals whose closures are disjoint and which do not contain 0 in their interior, then $F(u)$ is 1/6 times the number of endpoints of these intervals (i.e. 1/3 times the number of intervals); if u is the indicator function of a union of intervals whose closures are disjoint and such that one of them contains 0 in its interior, then $F(u)$ is 1/6 times the number of endoints of these intervals increased by 2 (i.e. 1/3 plus 1/3 times the number of intervals); otherwise $F(u) = +\infty$.

Prove that we have $F_\varepsilon \xrightarrow{\Gamma} F$ for the L^1 strong convergence on X.

Solution: The proof is almost the same as in the approximation of the perimeter functional. The only difference is that since we impose $u(0) = 0$ we need to count a possible jump down to 0 and then a possible jump up at 0. For the $\Gamma - \liminf$ part, we have as usual $F_\varepsilon(u_\varepsilon) \geq ||\nabla\Phi(u_\varepsilon)||_{\mathcal{M}}$. Using $u_\varepsilon(-1) = u_\varepsilon(0) = u_\varepsilon(1) = 0$ we decompose this into two parts, since $(u_\varepsilon)_{|[-1,0]} \in H^1_0([-1, 0])$ and $(u_\varepsilon)_{|[0,1]} \in H^1_0([0, 1])$. Summing the two results we obtain that the $\Gamma - \liminf$ is bounded below by a functional which is only finite on indicator functions and it is the sum of $c_0 = \int_0^1 \sqrt{W}$ times the perimeter functionals on $[-1, 0]$ plus the perimeter functional on $[0, 1]$. This sum is only finite on finite disjoint unions of intervals and adds two artificial jumps at 0 if $u = 1$ in a neighborhood of 0. Finally, note that here $W(u) = u^2(1 - u^2)$ so that $c_0 = \int_0^1 t(1 - t)dt = \int_0^1 tdt - \int_0^1 t^2dt = 1/2 - 1/3 = 1/6$. This proves the $\Gamma - \liminf$ part of the proof.

For the $\Gamma - \limsup$ part the construction is the same as in the standard proof, even easier in dimension 1, but if $u = \sum_{i=1}^N \mathbb{1}_{[a_i,b_i]}$ with $a_{i_0} < 0 < b_{i_0}$ then one has to replace $[a_{i_0}, b_{i_0}]$ with $[a_{i_0}, -C\varepsilon] \cup [C\varepsilon, b_{i_0}]$. More precisely, fixing $\delta > 0$ one can find a number $L > 0$ and function $\phi : [-L, L] \to [0, 1]$ such that

$\int_{-L}^{L} \frac{1}{2}|\phi'|^2 + \frac{1}{2}W(\phi) < c_0(1+\delta)$ and $\phi(-L) = 0, \phi(L) = 1$. We then define $u` - \varepsilon$ using a scaled copy of ϕ, on each interval $[a_i - L\varepsilon, a_i + L\varepsilon]$, and also on $[0, 2L\varepsilon]$, as well as a reversed copy of it on each interval of the form $[b_i - L\varepsilon, b_i + L\varepsilon]$, and also on $[-2L\varepsilon, 0]$. This only works if $a_0 > -1$ and $b_N < 1$, but this condition can be fixed by density exactly as in the standard proof. Then,on $[a_i + L\varepsilon, b_i - L\varepsilon]$ we set $u_\varepsilon = 1$ and elsewhere $u_\varepsilon = 0$. The obtained seuence of functions u_ε will converge pointwisely to u a.e. and is dominated by a constant, so it converges L^1, satisfies the constraint $u_\varepsilon(-1) = u_\varepsilon(0) = u_\varepsilon(1) = 0$, and has $2N+2$ transitions, each one costing at most $c_0(1+\delta)$. This shows that on the class of functions which are indicators of disjoint unions of intervals far from the boundary the $\Gamma - \limsup$ is bounded from above by $(1+\delta)F$ and the result follows by letting $\delta \to 0$ and proving that this class is dense in energy in the set of functions u such that $F(u) < +\infty$.