ENS Lyon and Université Claude Bernard Lyon 1, M2A
December 20th, 2024

Calculus of Variations and Elliptic PDEs

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Final Exam

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3h duration. All kind of documents (notes, books...) are authorized, but not communication devices. The total number of points is much larger than 20, so attacking only some exercises could be a reasonable option. The exercises are not necessarily ordered by difficulty.

Exercice 1 (4 points)**.** Consider the two optimization problems

$$(\mathrm{P_D})\quad \min\left\{\int_0^1 \left(|u'(t)|^2+|u(t)-t|^2\right)dt \quad : \quad u\in C^1([0,1]),\ u(0)=0, u(1)=1\right\},$$
$$(\mathrm{P_N})\quad \min\left\{\int_0^1 \left(|u'(t)|^2+|u(t)-t|^2\right)dt \quad : \quad u\in C^1([0,1])\right\}.$$

Prove that u_D given by $u_D(t)=t$ is a solution of $(\mathrm{P_D})$, and that it is its unique solution.
Then, find the solution u_N of $(\mathrm{P_N})$.

Exercice 2 (5 points)**.** Given a bounded and smooth domain $\Omega\subset\mathbb{R}^d$ and a function $u_0\in H^1(\Omega)$, prove that we have

$$\min\left\{\int_\Omega\left(\frac{|\nabla u|^2}{2}+|u-u_0|\right)dx\ :\ u\in H^1(\Omega)\right\}$$
$$=\max\left\{\int_\Omega\left(-\frac{|v|^2}{2}+v\cdot\nabla u_0\right)dx\ : v\in L^2(\Omega;\mathbb{R}^d), \nabla\cdot v\in L^\infty, ||\nabla\cdot v||_{L^\infty}\leq 1\right\}.$$

Proe that both problems admit a unique solution, and deduce that the optimizer of the first problem is a function u such that $|\Delta u|\leq 1$, $\Delta u=1$ on $\{u>u_0\}$, and $\Delta u=-1$ on $\{u<u_0\}$.

Exercice 3 (10 points)**.** We consider the following optimization problem

$$(\mathrm{P})\quad \min\left\{M(u):=\int_{\mathbb{R}^3}u^3(x)dx\ ,\ u\in\mathcal{S}(\mathbb{R}^3)\right\}$$

where $\mathcal{S}(\mathbb{R}^3)$ is the set of non-trivial nonnegative subsolutions of $\Delta u+u^3=0$, i.e.

$$\mathcal{S}(\mathbb{R}^3)=\left\{u\in L^3(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)\cap H^1_{loc}(\mathbb{R}^3), u\geq 0, \Delta u+u^3\geq 0\right\}\setminus\{0\},$$

(the inequality $\Delta u+u^3\geq 0$ has to be considered in its weak form $-\int\nabla u\cdot\nabla\psi+\int u^3\psi\geq 0$ for everynon-negative test function $\psi\in H^1$ with compact support).

1. Prove that $\mathcal{S}(\mathbb{R}^3)$ is non-empty and that, if g denotes the Gaussian function $g(x)=e^{-|x|^2/2}$ and $A>0$ is a constant, the function Ag belongs to $\mathcal{S}(\mathbb{R}^3)$ if and only if $A\geq e$.

2. Prove that for every $\lambda>0$ if $u\in\mathcal{S}(\mathbb{R}^3)$ the function u_λ defined via $u_\lambda(x)=\lambda u(\lambda x)$ also belongs to $\mathcal{S}(\mathbb{R}^3)$ and $M(u_\lambda)=M(u)$ and deduce that one can construct a minimizing sequence $(u_n)_n$ for (P) such that $||u_n||_{L^\infty}=1$ for every n.

3. In arbitrary dimension d (not necessarily $d=3$ in this question), prove that there exists a constant $C=C(d)$ such that whenever u satisfies $\Delta u\geq -1$ then $R\mapsto \fint_{B(x_0,R)}u(x)dx+CR^2$ is nondecreasing for every x_0 and admits a limit as $R\to 0$. Prove that such a limit, as a function of x_0, is a representative of u and that for this representative we have $\fint_{B(x_0,R)}u(x)dx\geq u(x_0)-CR^2$. Also prove that the same representative is upper-semicontinuous.

4. Again in arbitrary dimension, prove that for every $p \in [1, \infty)$ and every function $u \in L^p(\mathbb{R}^d)$ satisfying $\Delta u \geq -1$, if we choose the representative described in the previous question, then we have $\lim_{|x| \to \infty} u(x) = 0$.

5. Deduce that (P) admits a minimizing sequence $(u_n)_n$ made of upper-semicontinuous functions functions u_n tending to 0 at infinity and such that $u_n(0) = \max u_n$ and $⨍_{B(x_0,R_0)} u_n(x)dx > \frac{1}{2}$ for a certain radius R_0 independent of n.

6. Prove that such a minimizing sequence is bounded in $H^1_{loc}(\mathbb{R}^3)$ and deduce that it admits a subsequence which locally weakly converges in H^1 to a function $\bar{u} \in \mathcal{S}$.

7. Prove that $\bar{u}$ is a solution of (P).

8. Suppose that, in an open set $\Omega \subset \mathbb{R}^3$, we have $\Delta \bar{u} + \bar{u}^3 = f \in C^0(\Omega)$. Prove that the optimality of $\bar{u}$ implies $f = 0$ in Ω and deduce that the solution of (P) is not of the form $u = Ag$.

Exercice 4 (4 points)**.** Given a function $u \in L^1(\mathbb{R}^d)$ and a number $R > 0$, define a function $a_R[u]$ as follows: $a_R[u](x) := ⨍_{B(x,R)} u(y)dy$. Suppose that a function $u \in H^1_{loc}(\mathbb{R}^d)$ satisfies $u > 0$ a.e. and

$$\nabla \cdot (a_R[u] \nabla u) = \frac{u}{1+u}.$$

Prove that we have $u \in C^\infty(\mathbb{R}^d)$.

Exercice 5 (7 points)**.** Take $X = \{u \in L^1([-1,1]) \,:\, 0 \leq u \leq 1\, a.e.\}$ and define a family of functionals on X, indexed by $\varepsilon > 0$, as follows:

$$F_\varepsilon(u) = \begin{cases} \frac{\varepsilon}{2} \int_{-1}^1 |u'(t)|^2 dt + \frac{1}{2\varepsilon} \int_{-1}^1 u^2(t)(1-u(t))^2 dt & \text{if } u \in H^1_0([-1,1]) \text{ and } u(0) = 0, \\ +\infty & \text{if not.} \end{cases}$$

Also define the functional F as follows: if u is the indicator function of a union of intervals whose closures are disjoint and which do not contain 0 in their interior, then $F(u)$ is $1/6$ times the number of endpoints of these intervals (i.e. $1/3$ times the number of intervals); if u is the indicator function of a union of intervals whose closures are disjoint and such that one of them contains 0 in its interior, then $F(u)$ is $1/6$ times the number of endoints of these intervals increased by 2 (i.e. $1/3$ plus $1/3$ times the number of intervals); otherwise $F(u) = +\infty$.

Prove that we have $F_\varepsilon \xrightarrow{\Gamma} F$ for the L^1 strong convergence on X.