ENS Lyon and Université Claude Bernard Lyon 1, M2A
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Calculus of Variations and Elliptic PDEs

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Mid-Term Examination

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All kind of documents (notes, books...) are authorized. The total number of points is much larger than 20, which means that attacking only some exercises could be a reasonable option. The exercises are not necessarily ordered by difficulty.

Exercice 1 (6 points)**.** Find the solution of the problem

$$\min\left\{\int_0^\pi e^{\cos(t)}\left(\frac{u'(t)^2}{2}+u(t)(1-\cos(t)-\cos^2(t))\right)dt \quad : \quad u\in C^1([0,\pi]),\ u(0)=0\right\},$$

properly justifying its minimality and its uniqueness.

Answer

Let us write the Euler-Lagrange equation of the problem. Using

$$L(t,x,v)=e^{\cos(t)}\left(\frac{v^2}{2}+x(1-\cos(t)-\cos^2(t))\right)$$

we see that the equation, together with its boundary conditions, is

$$\begin{cases}(e^{\cos(t)}u'(t))'=e^{\cos(t)}(1-\cos(t)-\cos^2(t)),\\ u(0)=0,\\ e^{\cos(t)}u'(\pi)=0.\end{cases}$$

Expanding the first equation we obtain

$$u''(t)-\sin(t)u'(t)=1-\cos(t)-\cos^2(t)).$$

We look for a solution in the form $u(t)=A\sin(t)+B\cos(t)+C$ (no guaranee that the solution will actually be of this form, but we try). The boundary conditions impose $B+C=0$ and $A=0$. Hence we look for a solution in the form $u(t)=B(\cos(t)-1)$. In order for this function to solve the equation we need $-B\cos(t)+B\sin^2(t)=1-\cos(t)-\cos^2(t))$, and we can observe that $B=1$ provides (by chance) a solution. This solution is a C^1 function.

Hence the function $u(t)=\cos(t)-1$ solves the Euler-Lagrange equation of the problem. Since L is convex in both v and x, solving the Euler-Lagrange equation is both necessary and sufficient for being a minimizer. Thus, this function u is a minimizer. Since we found a solution to the equation by guessing it, we have noidea if other solutions exist. Yet, we observe that L is strictly convex in v, which implies that if two minimizers u_1 and u_2 exist we should have $u_1'=u_2'$ a.e. The difference u_1-u_2 is then constant, and using the intiial boundary condition we obtain $u_1-u_2=0$, so that the minimizers is finally unique.

Exercice 2 (4 points)**.** Let Ω be a bounded open subset of $\mathbb{R}^d$. Consider the minimization problem

$$\min\left\{\int_\Omega\left(\sqrt{u(x)^4+|\nabla u(x)|^4}+\cos(u(x)-g(x))+\sqrt{1+u(x)^2|\nabla u(x)|^2}\right)dx \ : \ u\in H_0^1(\Omega)\right\},$$

where g is a given measurable function defined on Ω. Prove that the problem has a solution.

Answer

We consider the function L given by

$$L(x,u,v) = \sqrt{u^4+|v|^4} + \cos(u-g(x)) + \sqrt{1+u^2|v)|^2}$$

and we see that the functional to minimize has the form

$$J(u) = \tilde{J}(u,\nabla u), \quad \text{where } \tilde{J}(u,v) := \int_\Omega L(x,u(x),v(x))dx.$$

The function L is such that $L \geq 0$ (since $\sqrt{u^4+|v|^4} \geq 0$, $\cos(u-g(x)) \geq -1$ and $\sqrt{1+u^2|v|^2} \geq 1$) and is continuous in u and convex in v (just use the convexity of $v \mapsto \sqrt{1+|v|^2}$ possibly composed with powers larger than one or multiplications by a positive constant). Hence, the lower semicontinuity theorems we proved in class show that $\tilde{J}$ is lower semicontinuous for the strong L^2 convergence of u and the weak L^2 convergence of v. In particular, J is lower semicontinuous for the weak H^1 convergence. We now take a minizing sequence u_n such that $J(u_n) \to \inf J$ and we want to extract a weakly converging subsequence. We use $L(x,u,v) \geq |u|^2$ (just keep only the first term and remove $|v|^4$ in the square root) as well as $L(x,u,v) \geq |v|^2$ (now, remove $|u|^4$ in the square root). This implies $L(x,u,v) \geq \frac{1}{2}(|u|^2+|v|^2)$ and $J(u) \geq \frac{1}{2}||u||^2_{H^1}$. Hence, every minimizing sequence is bounded in H^1 and we can extract a weakly converging subsequence. The limit still belongs to H^1_0, which is a colsed subspace, hence also weakly closed because it is convex.

Exercice 3 (5 points)**.** Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{x^2}{2} + \cos(x)$. Prove that f is strictly convex and that f^* is a C^1 function of the form $f^*(y) = \frac{y^2}{2} + h(y)$, where h satisfies $|h| \leq 1$, $h(0) = -1$ and $h'(x-\sin(x)) = \sin(x)$. Find the value of f^* at all the points $y = k\pi$ for $k \in \mathbb{Z}$.

Answer

We compute f' and f'', obtaining $f'(x) = x - \sin(x)$ and $f''(x) = 1-\cos(x)$. The function f is convex since $f'' \geq 0$. Moreover, f'' only vanishes at isolated points, so there is no interval on which f is affine, and hence f is strictly convex. for the same reason f' is strictly increasing. We also not that we have $f'(0) = 0$, so that 0 is a minimizer of f and hence $\min f = f(0) = 1$.

Of course f^* is convex; for every $x \in \partial f^*(y)$ we also have $y \in \partial f^*(x)$ i.e. $y = f'(x)$. Since f' is strictly increasing for every y there is at most one such an x, so that the subdifferential of f^* at each point only contains one point, and f^* is hence C^1.

From $-1+g \leq f \leq 1+g$, where $g(x) = \frac{x^2}{2}$, we deduce $1+g^* \geq f^* \geq -1+g^*$ and using $g^*(y) = \frac{y^2}{2}$ we obtain $f^*(y) = \frac{y^2}{2} + h(y)$ where h satisfies $|h| \leq 1$.

We have $f^*(0) = \sup_x -f(x) = -\min f = -1$, so that we have $h(0) = -1$.

From the relation $(f^*)' \circ f' = id$ we obtain $f'(x) + h'(f'(x)) = x$, which can be rewritten as $h'(x - \sin(x)) = \sin(x)$. We can then consider the function $x \mapsto H(x) := h(x-\sin(x))$. We have $H(0) = h(0) = -1$ and $H'(x) = h'(x-\sin(x))(1-\cos(x)) = \sin(x)(1-\cos(x))$. Hence we have

$$h(k\pi) = H(k\pi) = -1 + \int_0^{k\pi} \sin(t)(1-\cos(t))dt = -1 + \int_0^{k\pi} \sin(t)dt,$$

where we removed the integral of $\sin(t)\cos(t)$ since this function is $\pi-$periodic with zero average on each period. Hence we obtain

$$h(k\pi) = H(k\pi) = -1 + [-\cos(t)]_0^{k\pi} = -\cos(k\pi) = -(-1)^k.$$

We then have $f^*(k\pi) = \frac{k^2}{2}\pi^2 - (-1)^k$.

Exercice 4 (12 points)**.** Let $\mathbb{T}^d$ be the d-dimensional torus. Consider the following minimization problem

$$\inf \left\{ J_f(u) := \int_\Omega \left(\frac{1}{3}|\nabla u(x)|^3 + f(x)u(x)^2 \right) dx \ : \ u \in W^{1,3}(\mathbb{T}^d) \right\},$$

where f is a given Lipschitz continuous function on $\mathbb{T}^d$.

1. Find all the solutions of the problem when f is the zero function.
2. Prove that when $\int f(x)dx < 0$ there is no solution.
3. Prove that when $\int f(x)dx = 0$ but f is not the zero function there is no solution.
4. Assume $\int f(x)dx > 0$: prove that there exists a minimizing sequence $(u_n)_n$ with $\int f(x)u_n(x)dx = 0$.
5. Assume $\int f(x)dx \neq 0$: prove the following Poincaré-type inequality: there exists a constant C such that $||u||_{L^3} \leq C||\nabla u||_{L^3}$ for all functions $u \in W^{1,3}(\mathbb{T}^d)$ such that $\int f(x)u(x)dx = 0$.
6. Assuming $\int f(x)dx > 0$, prove that the problem admits a solution.
7. Prove that the functional J_f is convex if and only if $f \geq 0$ and prove, when f is not everywhere nonnegative but $\int f(x)dx > 0$, that the solution is not unique.
8. Write the PDE satisfied by the solutions (Euler-Lagrange equation).
9. Prove that we have $|\nabla u|^{1/2}\nabla u \in H^1(\mathbb{T}^d)$. Can we weaken the assumption on f in order to obtain the same result (replacing $f \in \text{Lip}$ with $f \in W^{1,p}$, and for which p)?

Answer

1. When f is the zero function we have $J_f \geq 0$ and $J_f(0) = 0$. The minimum is thus 0 and it is realized by all the constant functions, and only by them.
2. Set $a := \int f(x)dx < 0$. We can take $u = c$ constant and we have $J_f(u) = ac^2$. Sending $c \to \infty$ we obtain $\inf J_f = -\infty$, so that the minimum is not attained.
3. We expand $J_f(u + c)$, where c is a constant: we obtain $J_f(u + c) = J_f(u) + 2c\int f(x)u(x)dx + c^2 \int f(x)dx$. Using $\int f(x)dx = 0$ we have $J_f(u + c) = J_f(u) + 2ca$ where $a = \int f(x)u(x)dx$. If f is not the zero function we can choose u such that $a \neq 0$. In this case $J_f(u+c)$ is an unbounded affine function of c, which proves that we have again $\inf J_f = -\infty$, and the minimum is not attained.
4. We use again the expansion of $J_f(u + c) = J_f(u) + 2c\int f(x)u(x)dx + c^2 \int f(x)dx$. We see that this quantity is a second-order polynomial in c which admits a minimum because we supposed $intf(x)dx > 0$. If u_n is a minimizing sequence, we can produce a new minimizing sequence by taking $\tilde{u}_n = u_n + c_n$ and choosing c_n so that $J_f(u_n + c_n) = \min_c J_f(u_n + c)$. Such a value c_n exists. Renaming the functions, this means that we can assume that our new minimizing sequence, that we now call u_n is such that for every n the minimum $\min_c J_f(u_n + c)$ is attained for $c = 0$. Taking the derivative of the above expansion this means $\int f(x)u_n(x)dx = 0$.
5. The desired Poincaré-type inequality can be proven by contradiction. It it does not hold, then for every k we find a function u_k with $||u_k||_{L^3} > k||\nabla u_k||_{L^3}$ and $\int f(x)u_k(x)dx = 0$. We can normalize u_k so that $||u_k||_{L^3} = 1$. In this way, the sequence u_k is bounded in $W^{1,3}$ and weakly converges, up to a subsequence, to a limit u. This limit satisfies $||\nabla u||_{L^3} = 0$ and hence u is a constant. By compact embedding of $W^{1,3}$ into L^3 we also have $1 = |u_k||_{L^3} \to |u||_{L^3}$, so that u is not the zero constant. On the other hand, u also satisfies $\int f(x)u(x)dx = 0$ but this is impossible because u is a non-zero constant and $\int f(x)dx \neq 0$. Hence the desired inequality holds.
6. We assume $\int f(x)dx > 0$ and take a minimizing sequence u_n with the extra property $\int f(x)u_n(x)dx = 0$. We then observe that, applying first a Holder inequality with exponents 3/2 and 3 to the second term, and then the Poincaré-type inequality that we just proved, we have for this minimizing sequence
$$J_f(u_n) \geq \frac{1}{3}||\nabla u_n||^3_{L^3} - ||f||_{L^3}||u_n||^2_{L^3} \geq \frac{1}{3}||\nabla u_n||^3_{L^3} - C||\nabla u_n||^2_{L^3}.$$
This shows that $||\nabla u_n||_{L^3}$ has to stay bounded because the positive term explodes faster than the negative one in the right hand side. Hence u_n is bounded in $W^{1,3}$ (using again the Poincaré-type

inequality) and admits subsequence weakly converging to some u. We then have $||\nabla u||_{L^3} \leq \liminf_n ||\nabla u_n||_{L^3}$ and $\int f u_n^2 \to \int f u^2$ since the weak convergence in $W^{1,3}$ implies strong convergence in L^2. We then obtain $J_f(u) \leq \liminf_n J_f(u_n)$ and u is a minimizer.

7. If $f \geq 0$, the functional J_f is of course convex. On the other hand when f is not everywhere nonnegative, we can consider a function $\varphi \in C^\infty$ supported on an open set where $f < 0$ (remember that f is supposed to be continuous). We then consider $t \mapsto J_f(t\varphi) = c_1 t^3 + c_2 t^2$ and we observe that we have $c_2 = \int f \varphi^2 < 0$. This quantity cannot be convex in a neighborhood of $t = 0$ because the second-order term has a negative sign. Hence J_f is not convex. We observe moreover that this also shows that 0 is never a solution of the minimization problem unless $f \geq 0$. In the case where f is not everywhere nonnegative but $\int f(x)dx > 0$, we consider a minimizer u and its opposite $-u$. Thanks to $J_f(u) = J_f(-u)$, we observe that we have two minimizers (since u is not the zero function), so that the minimizer is not unique.

8. The PDE satisfied by the solutions (Euler-Lagrange equation) is

$$\Delta_3(u) := \nabla \cdot (|\nabla u| \nabla u) = 2fu.$$

9. The results based on regularity via duality that we proved in class show that whenver we have $\Delta_p u = g$ and $g \in W^{1,q}$ (the exponent q being the dual of p, we obtain (interpreting u as thesolution of a convex variaitonal problem and using its dual as well) $|\nabla u|^{p/2-1}\nabla u \in H^1$. Here we just need to guarantee $2fu \in W^{1,3/2}$. In this case this is easy because we have $f \in W^{1,\infty}$ and $u \in W^{1,3}$, so that the product fu is $W^{1,3}$ and hence $W^{1,3/2}$ (indeed, we have $\nabla(fu) = f\nabla u + \nabla f u$ and we use the boundedness of both f and ∇f and the L^3 integrability of u and ∇u). The assumption on f can be weakened, as we only need $f\nabla u, \nabla f u \in L^{3/2}$. Using $\nabla u \in L^3$ we have $|\nabla u|^{3/2} \in L^2$, so that the condition on $f \nabla u$ is satisfied whenever $f^{3/2} \in L^2$, i.e. $f \in L^3$. For the conditon on $\nabla f u$ we distinguish according to the dimension. If the dimension is 1 or 2 then $u \in W^{1,3}$ implies $u \in L^\infty$, so that we just need $\nabla f \in L^{3/2}$. In this case $f \in W^{1,3/2}$ is thus sufficient because it would imply, both in the 1D and in the 2D case, $f \in L^3$. If the dimension is 3 we have $u \in L^p$ for every p so that in order to obtain $\nabla f u \in L^{3/2}$ we just need $\nabla f \in L^r$ for some $r > 3/2$. In this case this also implies $f \in L^3$. Finally, if the dimension d is at least 4, we use $u \in L^{3^*}$, where $3^* = 3d/(d-3)$ and we need $\nabla f \in L^{3d/(d+3)}$. Note that $W^{1,3d/(d+3)}$ exactly embeds into L^3, so that in this case we also guarantee $f \in L^3$. Summarizing, we can replace the assumption $f \in W^{1,\infty}$ with

   - $f \in W^{1,3/2}$ for $d = 1, 2$,
   - $f \in W^{1,r}$ for some $r > 3/2$ if $d = 3$,
   - $f \in W^{1,3d/(d+3)}$ for $d \geq 4$.