**ENS Lyon and Université Claude Bernard Lyon 1, M2A**
**November 8th, 2024**

# Calculus of Variations and Elliptic PDEs

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# Mid-Term Examination

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All kind of documents (notes, books...) are authorized. The total number of points is much larger than 20, which means that attacking only some exercises could be a reasonable option. The exercises are not necessarily ordered by difficulty.

**Exercice 1** (6 points)**.** Find the solution of the problem

$$\min\left\{\int_0^\pi e^{\cos(t)}\left(\frac{u'(t)^2}{2}+u(t)(1-\cos(t)-\cos^2(t))\right)dt \quad : \quad u\in C^1([0,\pi]),\ u(0)=0\right\},$$

properly justifying its minimality and its uniqueness.

**Exercice 2** (4 points)**.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^d$. Consider the minimization problem

$$\min\left\{\int_\Omega\left(\sqrt{u(x)^4+|\nabla u(x)|^4}+\cos(u(x)-g(x))+\sqrt{1+u(x)^2|\nabla u(x)|^2}\right)dx \ : \ u\in H^1_0(\Omega)\right\},$$

where $g$ is a given measurable function defined on $\Omega$. Prove that the problem has a solution.

**Exercice 3** (5 points)**.** Consider the function $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=\frac{x^2}{2}+\cos(x)$. Prove that $f$ is strictly convex and that $f^*$ is a $C^1$ function of the form $f^*(y)=\frac{y^2}{2}+h(y)$, where $h$ satisfies $|h|\le 1$, $h(0)=-1$ and $h'(x-\sin(x))=\sin(x)$. Find the value of $f^*$ at all the points $y=k\pi$ for $k\in\mathbb{Z}$.

**Exercice 4** (12 points)**.** Let $\mathbb{T}^d$ be the $d$-dimensional torus. Consider the following minimization problem

$$\inf\left\{J_f(u):=\int_{\mathbb{T}^d}\left(\frac{1}{3}|\nabla u(x)|^3+f(x)u(x)^2\right)dx \ : \ u\in W^{1,3}(\mathbb{T}^d)\right\},$$

where $f$ is a given Lipschitz continuous function on $\mathbb{T}^d$.

1. Find all the solutions of the problem when $f$ is the zero function.
2. Prove that when $\int f(x)dx<0$ there is no solution.
3. Prove that when $\int f(x)dx=0$ but $f$ is not the zero function there is no solution.
4. Assume $\int f(x)dx>0$: prove that there exists a minimizing sequence $(u_n)_n$ with $\int f(x)u_n(x)dx=0$.
5. Assume $\int f(x)dx\neq 0$: prove the following Poincaré-type inequality: there exists a constant $C$ such that $||u||_{L^3}\le C||\nabla u||_{L^3}$ for all functions $u\in W^{1,3}(\mathbb{T}^d)$ such that $\int f(x)u(x)dx=0$.
6. Assuming $\int f(x)dx>0$, prove that the problem admits a solution.
7. Prove that the functional $J_f$ is convex if and only if $f\ge 0$ and prove, when $f$ is not everywhere nonnegative but $\int f(x)dx>0$, that the solution is not unique.
8. Write the PDE satisfied by the solutions (Euler-Lagrange equation).
9. Prove that we have $|\nabla u|^{1/2}\nabla u\in H^1(\mathbb{T}^d)$. Can we weaken the assumption on $f$ in order to obtain the same result (replacing $f\in \mathrm{Lip}$ with $f\in W^{1,p}$, and for which $p$)?