ENS Lyon and Université Claude Bernard Lyon 1, M2A
December 13th, 2021

Calculus of Variations and Elliptic PDEs – Final Exam

Duration: 3h. All kind of documents (notes, books...) are authorized. The total number of points is much larger than 20, which means that attacking two or three exercises could be a reasonable option.

Exercice 1 (7 points)**.** Consider the minimization problem

$$\min\left\{\int_0^T \frac{e^{-2t}}{2}\left(|u'(t)|^2 - |u(t)|^2\right)dt \ :\ u \in H^1([0,T]),\ u(0) = 1\right\}.$$

1. Prove that the minimization problem admits a solution at least if T is small enough.
2. Write and solve the Euler-Lagrange equation of this minimization problem with its boundary conditions.
3. Prove the inequality $\int_0^T e^{-2t}|h(t)|^2 dt \leq \int_0^T e^{-2t}|h'(t)|^2 dt$ for any $h \in H^1([0,T])$ with $h(0) = 0$.
4. Prove that the minimization problem admits a unique solution for any $T > 0$.

Solution:

1. We know the inequality $|u(t)| = |\int_0^t u'| \leq (\int_0^T |u'|^2)^{1/2} T^{1/2}$ which is valid whenever $u(0) = 0$. Hence we deduce $\int_0^T |u|^2 \leq T^2 \int_0^T |u'|^2$. As a consequence, it T is such that $e^{-2T} > T^2$ we have $\int_0^T e^{-2t}|u'|^2 \geq c\int_0^T e^{-2t}|u|^2$ for $c > 1$. This means that any minimizing sequence will be bounded in H^1 and then standard semicontinuity results (since the integrand is convex in u' but not in u) allow to prove the existence of a minimizer.

2. The Euler-Lagrange equation is $(e^{-2t}u')' = -e^{-2t}u$, which becomes $u'' - 2u' = -u$, and the boundary conditions are $u(0) = 1$ and $u'(T) = 0$. The family of general solutions of the equation is given by the functions of the form $u(t) = (A + Bt)e^t$ and we need $A = 1$ to satisfy $u(0) = 1$. If we compute $u'(t) = (A + B + Bt)e^t$ imposing $u'(T) = 0$ means taking $B(1 + T) = -A = -1$, so the solution si given by $u(t) = (1 - \frac{t}{T+1})e^t$.

3. The easiest way to prove the inequality $\int_0^T e^{-2t}|h(t)|^2 dt \leq \int_0^T e^{-2t}|h'(t)|^2 dt$ for $h(0) = 0$ is to write $h(t) = e^t v(t)$, with $v(0) = 0$. Then we have $e^{-2t}(|h'(t)|^2 - |h(t)|^2) = |v'(t)|^2 + 2v(t)'(t)$, so that

$$\int_0^T e^{-2t}|h'(t)|^2 dt - \int_0^T e^{-2t}|h(t)|^2 dt = \int_0^T [|v'(t)|^2 + 2v(t)v'(t)]dt = \int_0^T |v'(t)|^2 dt + v(T)^2 \geq 0.$$

   Another possibility is to solve

$$M := \min \frac{\int_0^T e^{-2t}|h'(t)|^2 dt}{\int_0^T e^{-2t}|h't)|^2 dt}$$

   and prove that the optimal h (which exists) satisfies $(e^{-2t}h')' = -Me^{-2t}h$ together with $h(0) = 0$ and $h'(T) = 0$. Suppose then $M < 1$ and prove that it is not possible to find a solution.

4. There is a unique solution $u(t) = (1 - \frac{t}{T+1})e^t$ to the Euler-Lagrange equation. Hence there can be at most one minimizer. We then take an arbitrary competitor of the form $u + h$ and observe that we have (if we call J the functional we minimize)

$$J(u + \varphi) = J(u) + \int_0^T e^{-2t}[u'(t)\varphi'(t) - u(t)\varphi(t)]dt + J(\varphi).$$

   We then have $\int_0^T e^{-2t}[u'(t)\varphi'(t) - u(t)\varphi(t)]dt = 0$ because of the condition on u, and $J(\varphi) \geq 0$ because of the previous estimate. This proves that the solution of the Euler-Lagrange equation is indeed a minimizer.

Exercice 2 (5 points)**.** Consider a Lipschitz domain $\Omega \subset \mathbb{R}^d$, a measurable map $\tau : \Omega \to \Omega$ and a scalar function $u \in H^1_{loc}(\Omega)$ which is a weak solution in Ω of

$$\nabla \cdot \left(\frac{2 + u(\tau(x))^2}{1 + u(\tau(x))^2} \nabla u \right) = 0.$$

1. If τ is the identity, prove that $u + \arctan u$ is a harmonic function.
2. If $\tau \in C^\infty$, prove $u \in C^\infty(\Omega)$.

Solution:

1. In the case where τ is the identity, we use $\frac{2+u^2}{1+u^2} = 1 + \frac{1}{1+u^2}$ and $(1 + \frac{1}{1+u^2})\nabla u = \nabla(u + \arctan u)$, so that the equation is equivalent to $\Delta(u + \arctan u) = 0$.
2. First note that $a := \frac{2+u^2\circ\tau}{1+u^2\circ\tau}$ is bounded from below and above (it takes values in $(1, 2]$). Then, DeGiorgi-Nash-Moser's theorem implies $u \in C^{0,\alpha}$ (locally) for some α. We now see that whenever we have some regularity for u, then a has the same regularity since we compose with C^∞ functions. Hence $a \in C^{0,\alpha}$ and $u \in C^{1,\alpha}$. But then, by induction, $u \in C^{k,\alpha}_{loc}$ provides $u \in C^{k+1,\alpha}_{loc}$ and finally $u \in C^\infty$.

Exercice 3 (7 points)**.** Consider a smooth bounded and connected domain $\Omega \subset \mathbb{R}^d$, a function $f \in L^2(\Omega)$ with $\fint f = 0$, a number $\alpha \in \mathbb{R}$, and the minimization problem

$$\min \left\{ \int_\Omega (1 + |\nabla u|)^2 + \cos(\alpha u + |\nabla u|) + fu \ : \ u \in H^1(\Omega) \right\}.$$

1. For $\alpha = 0$ prove that the minimization can be restricted to the functions u with $\fint u = 0$ and for $\alpha \neq 0$ to the functions u with $|\fint u| \leq \frac{\pi}{|\alpha|}$.
2. Prove that the problem admits a solution.
3. Prove that any solution satisfies $\alpha \int \sin(\alpha u + |\nabla u|) = 0$.
4. Prove that the solution is unique up to additive constants if $\alpha = 0$.
5. Find all the minimizers in the case $f = 0$ and $\alpha \neq 0$.

Solution:

1. Thanks to the condition $\fint f = 0$, if $\alpha = 0$ the functional is invariant when adding a constant to u, so we can restrict to those functions which have zero-mean (or fix any other mean). When $\alpha \neq 0$, the functional is only invariant if adding a constant of the form $2k\pi\alpha^{-1}$. So, we cannot change the mean of u into an arbitrary mean, but we can bring it into any interval whose length is $2\pi/|\alpha|$, in particular into $[-\frac{\pi}{|\alpha|}, \frac{\pi}{|\alpha|}]$.
2. We take a minimizing sequence u_n and, thanks to the previous considerations, we suppose that $\fint u_n$ is bounded. If we call J the functional to be minimized we have $J(u) \geq \int |\nabla u|^2 - |\Omega| - ||f||_{L^2}||u||_{L^2}$. We have $||u - \fint u||_{L^2} \leq C||\nabla u||_{L^2}$ for any u, so that we have in our case $||u_n||_{L^2} \leq C(1 + ||\nabla u_n||_{L^2})$ and $J(u_n) \geq ||\nabla u_n||^2_{L^2} - C - C||\nabla u_n||_{L^2}$. This proves, using $J(u_n) \leq C$, that$||\nabla u_n||_{L^2}$ is bounded, and hence u_n is bounded in H^1 and we can extract a weakly converging subsequence. We then note that the integrand $L(u, v) = (1+|v|^2)+\cos(\alpha u+|v|)$ is continuous in (u, v) and convex in v (for this last property, we first look at the function $s \mapsto (1 + s^2) + \cos(\alpha u + s)$ which is increasing and convex on $s \in \mathbb{R}_+$ (just compute two derivatives), and then compose with $s = |v|$, which is a convex function. This implies the semicontinuity of the functional wrt the weak convergence in H^1, and hence the existence of a minimizer.
3. We take u an optimizer and then consider $\frac{d}{ds}J(u+s)$, where s is a constant, and impose that this derivative vanishes for $s = 0$.
4. The computation of the second derivative of $L(u, v) = (1+|v|^2)+\cos(\alpha u+|v|)$ wrt v shows that L is strictly convex in v. If $\alpha =$, then L does not depend on u, so that we have strict convexity in terms of the gradient of u. The other term is linear, so it does not affect the convexity. The strict convexity in ∇u implies that two different minimizers must have the same gradient, hence differ by a constant since Ω is connected.

5. For $f = 0$ and $\alpha \neq 0$ take $u = \frac{(\pi+2k\pi)}{\alpha}$. Since u is a constant, its gradient vanishes and we have $J(u) = |\Omega|(1-1) = 0$. On the other hand, we have for any u the inequalities $(1+|\nabla u|)^2 \geq 1$ and $\cos(\alpha u+|\nabla u|) \geq -1$, which show $J(u) \geq 0$ for every u. If u realizes $J(u) = 0$ then we necessarily have $|\nabla u| = 0$ a.e. and $\cos(\alpha u) = -1$, which shows that $u = \frac{(\pi+2k\pi)}{\alpha}$ covers all the possible solutions.

Exercice 4 (9 points). Given a curve $\omega \in C^0(\mathbb{R};\mathbb{R}^n)$ which is 2π-periodic, consider the minimization problem

$$\min\left\{\int_0^{2\pi}\left(\sqrt{\varepsilon^2+|u'(t)|^2}+\frac{\varepsilon}{2}|u'(t)|^2+\frac{1}{2}|u(t)-\omega(t)|^2\right)dt \;:\; u \in H^1_{per}(\mathbb{R};\mathbb{R}^n)\right\},$$

where $H^1_{per}(\mathbb{R})$ denotes the set of functions $u \in H^1_{loc}(\mathbb{R})$ which are 2π-periodic.

1. Prove that this minimization problem admits a unique solution u_ε.
2. Prove that the sequence of minimizers is bounded, when $\varepsilon \to 0$, in $BV(I)$ for any interval $I \subset \mathbb{R}$.
3. Prove that u_ε converges (in which sense?) to the unique solution $\bar{u}$ of

$$\min\left\{|u'|([0,2\pi[)+\int_0^{2\pi}\frac{1}{2}|u(t)-\omega(t)|^2dt \;:\; u \in BV_{per}(\mathbb{R};\mathbb{R}^n)\right\},$$

   where $BV_{per}(\mathbb{R})$ denotes the set of functions $u \in BV_{loc}(\mathbb{R})$ which are 2π-periodic and u' is their distributional derivative, which is a measure, and $|u'|([0,2\pi[)$ is its total variation on one period.
4. Find $\bar{u}$ in the case $n = 2$ and $\omega(t) = (R\cos t, R\sin t)$ for $R > 1$.

Solution: Let us set $J_\varepsilon := L_\varepsilon + H_\omega$, where $L_\varepsilon(u) := \int_0^{2\pi}\sqrt{\varepsilon^2+|u'(t)|^2}+\frac{\varepsilon}{2}|u'(t)|^2$ and $H_\omega(u) = \int_0^{2\pi}\frac{1}{2}|u(t)-\omega(t)|^2$. We also set $L(u) := |u'|([0,2\pi[)$ (which is equal to $\int_0^{2\pi}|u'|$ for smooth functions u) and $J = L + H_\omega$.

1. For fixed $\varepsilon > 0$ the two terms L_ε and H_ω are bounded from below in terms of the L^2 norm of u' and of u respectively (we use the fact that ω is bounded). Hence, any minimizing sequence is bounded in $H^1(I)$ for any interval I (here we use periodicity) and we can extract a weakly converging subsequence. The terms depending on u' are convex in u', so the functional is lsc for this convergence, which proves the existence of a minimizer. Uniqueness comes from the strict convexity of H_ω.

2. We have $L_\varepsilon(u) \geq L(u)$ so that $J_\varepsilon(u) \geq L(u) + ||u||^2_{L^2([0,2\pi])} - C$. Comparing the optimal $u\varepsilon$ to the zero function we have $J_\varepsilon(u_\varepsilon) \leq 2\pi(\varepsilon + ||\omega||^2_{L^2}$ which bounds both the L^2 norm of u_ε and the norm of u'_ε in the space of measures when $\varepsilon \to 0$. This can be done on a periodicity interval, or on any bounded interval, of course.

3. We want to prove the Γ-convergence of J_ε to J in the space of 2π-periodic L^2 functions endowed with the $L^2([0,2\pi])$ norm. The bound in BV on the minimizers together with the compact embedding $BV \subset L^2$ which is true in dimension one allows to obtain the compactness assumption which is needed to guarantee that Γ-convergence implies the convergence of the minimizers. We would obtain in this case strong L^2 convergence but actually the compact embedding $BV \subset L^p$ for any $p < +\infty$ also provides strong convergence in all the L^p spaces. We can't conclude about L^∞ convergence, even if we have weak-* L^∞ convergence of u_ε to $\bar{u}$ and weak-* convergence of u'_ε to $\bar{u}'$. Extra assumptions on ω could imply better convergence of the minimizers.

   The Γ-convergence is not difficult to establish in this case. First we observe that H_ω is continuous for the L^2 convergence so that it is enough to prove the Γ-convergence of L_ε to L. We have $L_\varepsilon \geq L$ and L is lsc for the L^2 convergence (indeed, if we take a sequence $u_k \to u$ with $L(u_k) \leq C$ we automatically obtain a BV nound and we can extract a subsequence such that u'_k weakly-* converges as measures to u', so that the mass of u' is smaller than the liminf of the masses of u'_k). This provides the Γ-liminf inequality at no cost.

   The Γ-liminf inequality when $\varepsilon \to 0$ is straightforward on smooth functions. We just need to prove that this is a class dense in energy. Given any u with $L(u) < +\infty$ we can define $u_n := \eta_n * u$ where η_n is a standard smooth convolution kernel. We have $u_n \to u$ in L^2 and $|u'_n|([0,2\pi[) \leq |u'|([0,2\pi[)$ since the norm of the derivative is a convex functional invariant by translations, and hence it decreases by convolution. This shows that this sequence can be used to prove the density in energy of the class f smooth functions and concludes the proof.

4. We look for a solution of the form $\bar{u}(t) = (r\cos t, r\sin t)$. For this function we have $|\bar{u}'| = r$ and $\bar{u}'' = -\bar{u}$. Let us take an arbitrary competitor $u = \bar{u} + \varphi$ and suppose that it is smooth. We then write the inequality which can be easily obtained by convexity of both terms:

$$J(\bar{u} + \varphi) \geq J(u) + \int_0^{2\pi} \frac{\bar{u}'}{|\bar{u}'|} \cdot \varphi' + (\bar{u} - \omega) \cdot \varphi.$$

Integrating by parts and using $|\bar{u}'| = r$ we have

$$J(\bar{u} + \varphi) \geq J(u) + \int_0^{2\pi} \left(-\frac{\bar{u}''}{r} + (\bar{u} - \omega) \right) \cdot \varphi.$$

It is then enough to impose $-\frac{\bar{u}''}{r} + (\bar{u} - \omega) = 0$ to obtain that $\bar{u}$ minimizes among smooth functions. Any other competitor u can be approximated by convolution by smooth functions u_k, with $J(u_k) \to J(u)$ as we saw in the previous question, so the optimality among smooth functions is equialent to the global optimality. The desired differential condition (which is nothing but the Euler-Lagrange equation, but we discuss this into detals because of the non-smoothness of both competitors and $u'/|u'|$) is equivalent to $\bar{u}(1 + \frac{1}{r}) = \omega$ and hence $r = R - 1$ is sufficient for optimality. This optimizer is the unique one, because of strict convexity of H_ω.