ENS Lyon and Université Claude Bernard Lyon 1, M2A
December 13th, 2021

Calculus of Variations and Elliptic PDEs – Final Exam

Duration: 3h. All kind of documents (notes, books...) are authorized. The total number of points is much larger than 20, which means that attacking two or three exercises could be a reasonable option.

Exercice 1 (7 points)**.** Consider the minimization problem

$$\min\left\{\int_0^T \frac{e^{-2t}}{2}\left(|u'(t)|^2 - |u(t)|^2\right) dt \ : \ u \in H^1([0,T]),\ u(0) = 1\right\}.$$

1. Prove that the minimization problem admits a solution at least if T is small enough.
2. Write and solve the Euler-Lagrange equation of this minimization problem, with its boundary conditions.
3. Prove the inequality $\int_0^T e^{-2t}|h(t)|^2 dt \le \int_0^T e^{-2t}|h'(t)|^2 dt$ for any $h \in H^1([0,T])$ with $h(0) = 0$.
4. Prove that the minimization problem admits a unique solution for any $T > 0$.

Exercice 2 (5 points)**.** Consider a Lipschitz domain $\Omega \subset \mathbb{R}^d$, a measurable map $\tau : \Omega \to \Omega$ and a scalar function $u \in H^1_{loc}(\Omega)$ which is a weak solution in Ω of

$$\nabla \cdot \left(\frac{2 + u(\tau(x))^2}{1 + u(\tau(x))^2}\nabla u\right) = 0.$$

1. If τ is the identity, prove that $u + \arctan u$ is a harmonic function.
2. If $\tau \in C^\infty$, prove $u \in C^\infty(\Omega)$.

Exercice 3 (7 points)**.** Consider a smooth bounded and connected domain $\Omega \subset \mathbb{R}^d$, a function $f \in L^2(\Omega)$ with $⨍_\Omega f = 0$, a number $\alpha \in \mathbb{R}$, and the minimization problem

$$\min\left\{\int_\Omega (1 + |\nabla u|)^2 + \cos(\alpha u + |\nabla u|) + fu \ : \ u \in H^1(\Omega)\right\}.$$

1. For $\alpha = 0$ prove that the minimization can be restricted to the functions u with $⨍_\Omega u = 0$ and for $\alpha \neq 0$ to the functions u with $|⨍_\Omega u| \le \frac{\pi}{|\alpha|}$.
2. Prove that the problem admits a solution.
3. Prove that any solution satisfies $\alpha \int_\Omega \sin(\alpha u + |\nabla u|) = 0$.
4. Prove that the solution is unique up to additive constants if $\alpha = 0$.
5. Find all the minimizers in the case $f = 0$ and $\alpha \neq 0$.

Exercice 4 (9 points)**.** Given a curve $\omega \in C^0(\mathbb{R};\mathbb{R}^n)$ which is 2π-periodic, consider the minimization problem

$$\min\left\{\int_0^{2\pi}\left(\sqrt{\varepsilon^2 + |u'(t)|^2} + \frac{\varepsilon}{2}|u'(t)|^2 + \frac{1}{2}|u(t) - \omega(t)|^2\right) dt \ : \ u \in H^1_{per}(\mathbb{R};\mathbb{R}^n)\right\},$$

where $H^1_{per}(\mathbb{R})$ denotes the set of functions $u \in H^1_{loc}(\mathbb{R})$ which are 2π-periodic.

1. Prove that this minimization problem admits a unique solution u_ε.
2. Prove that the sequence of minimizers is bounded, when $\varepsilon \to 0$, in $BV(I)$ for any interval $I \subset \mathbb{R}$.
3. Prove that u_ε converges (in which sense?) to the unique solution $\bar{u}$ of

   $$\min\left\{|u'|([0,2\pi[) + \int_0^{2\pi} \frac{1}{2}|u(t) - \omega(t)|^2 dt \ : \ u \in BV_{per}(\mathbb{R};\mathbb{R}^n)\right\},$$

   where $BV_{per}(\mathbb{R})$ denotes the set of functions $u \in BV_{loc}(\mathbb{R})$ which are 2π-periodic and u' is their distributional derivative, which is a measure, and $|u'|([0,2\pi[)$ is its total variation on one period.
4. Find $\bar{u}$ in the case $n = 2$ and $\omega(t) = (R\cos t, R\sin t)$ for $R > 1$.