**ENS Lyon and Université Claude Bernard Lyon 1, M2A**
**January 11th, 2020**

# Calculus of Variations and Elliptic PDEs

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# Final Exam

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Duration: 3h. All kind of documents (notes, books...) are authorized, but you cannot collaborate with anyone else. The total number of points is much larger than 20, which means that attacking two or three exercises could be a reasonable option.

**Exercice 1** (6 points)**.** Consider the problem

$$\min\left\{\int_0^{2\pi}\frac{|u'(t)|^2}{2+\sin t}dt \quad : \quad u\in C^1([0,2\pi]),\ u(0)=0, u(2\pi)=1\right\}.$$

Find the minimizer as well as its minimal value.

More generally, given a function $h\in C^0([a,b])$, $h>0$, characterize the solution and the find the minimal value of

$$\min\left\{\int_a^b\frac{|u'(t)|^2}{h(t)}dt \quad : \quad u\in C^1([a,b]),\ u(a)=A, u(b)=B\right\}$$

in terms of $h$, $A$ and $B$.

**Exercice 2** (8 points)**.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^d$ and $a,f:\Omega\to\mathbb{R}$ two Lipschitz functions. Suppose $\inf a>0$. Consider the following minimization problem

$$\min\left\{\int_\Omega a\left(\frac{|\nabla u|^2}{2}+\frac{u^2}{2}+fu\right)dx\ :\ u\in C^1(\Omega)\cap L^2(\Omega)\right\},$$

where the condition $u\in C^1(\Omega)$ is to be intended locally in the interior of $\Omega$ (no behavior or regularity on the boundary is imposed). Prove that the problem admits a solution. It can be useful to treat separately the cases of dimension $d=1$, $d=2,3$, and $d>3$ and to prove the following lemmas

1. If we have $d\leq 3$ and a function $v$ satisfying $v\in H^1(\Omega)$, then $v$ can be written as $v=\nabla\cdot F$ where $F$ is locally Hölder continuous.
2. If $u$ solves an equation of the form $\nabla\cdot(a\nabla u)=g$, where $a$ is Lipschitz continuous and bounded from below by a strictly positive constant and $g\in L^1_{loc}$, then we can also deduce, in the weak sense, the equation $\Delta u=\frac{g}{a}-\nabla(\log a)\cdot\nabla u\in L^1_{loc}$.

**Exercice 3** (8 points)**.** Let $H:\mathbb{R}^d\to\mathbb{R}$ be given by $H(z)=\sqrt{1+|z|^2+|z|^4}$ and $\Omega$ be the $d$-dimensional torus. Consider the equation

$$\nabla\cdot\left(\frac{(1+2|\nabla u|^2)\nabla u}{H(\nabla u)}\right)=f.$$

1. Given $f\in H^{-1}(\Omega)$ such that $\langle f,1\rangle=0$ (i.e. $f$ has zero average), prove that there exists a solution $u\in H^1(\Omega)$ to this equation, which is unique up to additive constants.
2. If $f\in C^\infty$, prove that the solution $u$ is also $C^\infty$.

**Look at the back of the paper for the last exercise**

**Exercice 4** (8 points)**.** The goal is to find the limit as $\varepsilon \to 0$ of the following minimal value

$$\min \left\{ \frac{\varepsilon}{2} \int_{B(0,R_0)} |\nabla u|^2 + \frac{1}{2\varepsilon} \int_{B(0,R_0)} u^2(1-u)^2 + \left( \int_{B(0,R_0)} u \right)^{-1} \; : \; u \in H_0^1(B(0,R_0)),\, 0 \le u \le 1 \right\}.$$

In order to find and justify this limit one can go through the following steps.

1. Prove that the functionals $F_\varepsilon$ defined on $L^1(B(0,R_0))$ through

$$F_\varepsilon(u) := \begin{cases} \frac{\varepsilon}{2} \int_{B(0,R_0)} |\nabla u|^2 + \frac{1}{2\varepsilon} \int_{B(0,R_0)} u^2(1-u)^2 & \text{if} u \in H_0^1(B(0,R_0)),\, 0 \le u \le 1 \\ +\infty & \text{if not} \end{cases}$$

   $\Gamma$-converge for the strong $L^1$ topology to $F$ given by

$$F(u) := \begin{cases} c\mathrm{Per}(A) & \text{if} u = I_A,\, A \subset B(0,R_0) \\ +\infty & \text{if not} \end{cases}$$

   where $c$ is a suitable constant (to be found) and $\mathrm{Per}(A)$ is the perimeter in the BV sense of the set $A \subset \mathbb{R}^d$ viewed as a subset of the whole space. For this proof, it is enough to explain which arguments have to be modified compared to the result seen in class.

2. Find and characterize the minimizers of $u \mapsto F(u) + (\int u)^{-1}$ making use, if needed, of the isoperimetric inequality.

3. Conclude and compute the limit value, which can depend on the dimension $d$ and on $R_0$.