Université Paris-Saclay
M2 AMS and Optimization
January 2016

Calculus of Variations
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Mock Exam
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This is just an example of possible exam classwork. Try to do it in 3h
All kind of documents (notes, books...) will be authorized.

Exercice 1 (6 points)**.** Consider the problem

$$\min\left\{\int_0^1 \left(u'(t)^2 + 16e^t u(t) + 9u(t)^2\right) dt + 3u(1)^2 \quad : \quad u \in H^1([0,1]),\ u(0) = -1\right\}.$$

Prove that it admits a minimizer, that it is unique, and find it.

Exercice 2 (6 points)**.** Let Ω be an open connected and smooth subset of $\mathbb{R}^d$. Prove that the following minimization problem admits a solution

$$\min\left\{\int \left(\frac{1}{2}|\nabla u|^2 - (1+v^2)\sin\left(\frac{u}{1+v^2}\right) + (2+\arctan(u))|\nabla v|^2\right) dx \ :\ u, v \in H_0^1(\Omega)\right\}$$

and write the necessary optimality conditions as a system of elliptic PDEs in u, v.

Exercice 3 (4 points)**.** Let $f \in L^1([0,1])$ and $F \in W^{1,1}([0,1])$ be such that $F' = f$ and $F(0) = F(1) = 0$. Let $1 < p < \infty$ be a given exponent, and q be its conjugate exponent. Prove

$$\min\left\{\int_0^1 \frac{1}{p}|u'(t)|^p dt + \int_0^1 f(t)u(t)dt \ :\ u \in W^{1,p}([0,1])\right\} = -\frac{1}{q}||F||^q_{L^q([0,1])}.$$

Exercice 4 (6 points)**.** Let Ω be an open connected and smooth subset of $\mathbb{R}^d$ and $p \in]1,\infty[$. Prove that the following minimization problem admits a solution

$$\min\left\{\frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} \ :\ u \in W_0^{1,p}(\Omega) \setminus \{0\}\right\},$$

that this solution may be taken non-negative, and that it satisfies

$$\Delta_p u = c u^{p-1}$$

for a suitable constant $c < 0$. Suppose $p > 2$ and consider $G := |\nabla u|^{p-2}\nabla u$: prove $G \in H^1_{loc}(\Omega)$.

Exercice 5 (8 points)**.** Let u_ε be solutions of the minimization problems P_ε given by

$$P_\varepsilon \ :=\ \min\left\{\int_0^\pi \left(\frac{\varepsilon}{2}|u'(t)|^2 + \frac{1}{2\varepsilon}\sin^2(u(t)) + 10^3|u(t) - t|\right) dt \ :\ u \in H^1([0,\pi])\right\}.$$

Prove that u_ε converges strongly in L^1 to a function u_0 as $\varepsilon \to 0$, find this function, and prove that the convergence is actually strong in all the L^p spaces with $p < \infty$. Is it a strong L^∞ convergence ?