Calculus of Variations

Mock Exam

This is just an example of possible exam classwork. Try to do it in 3h
All kind of documents (notes, books...) will be authorized.

Exercice 1 (6 points). Consider the problem

\[
\min \left\{ \int_0^1 \left( u'(t)^2 + 16e^t u(t) + 9u(t)^2 \right) dt + 3u(1)^2 : \ u \in H^1([0,1]), \ u(0) = -1 \right\}.
\]

Prove that it admits a minimizer, that it is unique, and find it.

Exercice 2 (6 points). Let \( \Omega \) be an open connected and smooth subset of \( \mathbb{R}^d \). Prove that the following minimization problem admits a solution

\[
\min \left\{ \int \left( \frac{1}{2} |\nabla u|^2 - (1 + v^2) \sin \left( \frac{u}{1 + v^2} \right) + (2 + \arctan(u)) |\nabla v|^2 \right) dx : \ u, v \in H^1_0(\Omega) \right\}
\]

and write the necessary optimality conditions as a system of elliptic PDEs in \( u, v \).

Exercice 3 (4 points). Let \( f \in L^1([0,1]) \) and \( F \in W^{1,1}([0,1]) \) be such that \( F' = f \) and \( F(0) = F(1) = 0 \). Let \( 1 < p < \infty \) be a given exponent, and \( q \) be its conjugate exponent. Prove

\[
\min \left\{ \int_0^1 \frac{1}{p} |u'(t)|^p dt + \int_0^1 f(t)u(t) dt : \ u \in W^{1,p}([0,1]) \right\} = -\frac{1}{q} \|F\|_{L^q([0,1])}.
\]

Exercice 4 (6 points). Let \( \Omega \) be an open connected and smooth subset of \( \mathbb{R}^d \) and \( p \in ]1, \infty[ \). Prove that the following minimization problem admits a solution

\[
\min \left\{ \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} : \ u \in W^{1,p}_0(\Omega) \setminus \{0\} \right\},
\]

that this solution may be taken non-negative, and that it satisfies

\[
\Delta_p u = cu^{p-1}
\]

for a suitable constant \( c < 0 \). Suppose \( p > 2 \) and consider \( G := |\nabla u|^{p-2} \nabla u \): prove \( G \in H^1_{loc}(\Omega) \).

Exercice 5 (8 points). Let \( u_\varepsilon \) be solutions of the minimization problems \( P_\varepsilon \) given by

\[
P_\varepsilon := \min \left\{ \int_0^\pi \left( \frac{\varepsilon}{2} |u'(t)|^2 + \frac{1}{2\varepsilon} \sin^2(u(t)) + 10^3 |u(t) - t| \right) dt : \ u \in H^1([0,\pi]) \right\}.
\]

Prove that \( u_\varepsilon \) converges strongly in \( L^1 \) to a function \( u_0 \) as \( \varepsilon \to 0 \), find this function, and prove that the convergence is actually strong in all the \( L^p \) spaces with \( p < \infty \). Is it a strong \( L^\infty \) convergence?