M2 Optimization and AMS

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Calculus of Variations

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Exercises

Exercise 1. Solve the problem

$$\min\{J(f) := \int_0^1 \left[\frac{1}{2}f'(t)^2 + tf(t) + \frac{1}{2}f(t)^2\right] dt\,;\ f \in \mathcal{A}\}, \quad \text{où} \quad \mathcal{A} := \{f \in C^1([0,1])\ :\ f(0) = 0\}.$$

Find the minimal value of J on $\mathcal{A}$ and the function(s) f which attain it, proving that they are actually minimizers

Exercise 2. Consider the problem

$$\min\{J(u) := \int_0^1 \left[\frac{1}{2}u'(t)^2 + u(t)f(t)\right] dt\,;\ u \in W^{1,2}([0,1])\}.$$

Find a necessary and sufficient condition on f so that this problem admits a solution.

Exercise 3. Given $f \in C^2(\mathbb{R})$, consider the problem

$$\min\{F(u) := \int_0^1 \left[(u'(t)^2 - 1)^2 + f(u(t))\right] dt\,;\ u \in C^1([0,1]), u(0) = a,\ u(1) = b\}.$$

Prove that the problem does not admit any solution if $|b - a| \leq \frac{1}{\sqrt{3}}$.

Exercise 4. Consider a minimization problem of the form

$$\min\{F(u) := \int_0^1 L(t, u(t), u'(t))dt\,;\ u \in W^{1,1}([0,1]), u(0) = a,\ u(1) = b\},$$

where $L \in C^2([0,1] \times \mathbb{R} \times \mathbb{R})$. We denote as usual by (t, x, v) the variables of L. Suppose that $\bar{u}$ is a solution to the above problem. Prove that we have

$$\frac{\partial^2 L}{\partial v^2}(t, \bar{u}(t), \bar{u}'(t)) \geq 0\, a.e.$$

Exercise 5. Consider the problem

$$\min\left\{\int_0^1 \left[|u'(t)|^2 + \arctan(u(t))\right]\, dt\ :\ u \in C^1([0,1])\right\},$$

and prove that it has no solutions. Prove the existence of a solution if we add the boundary condition $u(0) = 0$, write the optimality conditions and discuss the regularity of the solution.

Exercise 6. Prove existence and uniqueness of the solution of

$$\min\left\{\int_\Omega \left(f(x)|u(x)| + |\nabla u(x)|^2\right) dx\,;\, u \in H^1(\Omega),\ \int_\Omega u = 1\right\},$$

when Ω is an open, connected and bounded subset of $\mathbb{R}^n$ and $f \in L^2(\Omega)$, $f \geq 0$ (the sign of f is not important for existence). Where do we use connectedness ? Also prove that, if Ω is not connected (but has a finite number of connected components and we keep the assumption $f \geq 0$), then we have existence but maybe not uniqueness, and that if we withdraw both connectedness and positivity of f, then maybe we don't even have existence.

Exercise 7. Fully solve

$$\min\left\{\int_Q \left(|\nabla u(x,y)|^2 + u(x,y)^2\right)\,dx\,dy \;:\; u\in C^1(Q),\; u=\phi \text{ sur } \partial Q\right\},$$

where $Q=[-1,1]^2\subset\mathbb{R}^2$ and $\phi:\partial Q\to\mathbb{R}$ is given by

$$\phi(x,y)=\begin{cases}0 & \text{si } x=-1,\ y\in[-1,1]\\ 2(e^y+e^{-y}) & \text{if } x=1,\ y\in[-1,1]\\ (x+1)(e+e^{-1}) & \text{if } x\in[-1,1],\ y=\pm1.\end{cases}$$

Find the minimizer and the value of the minimum. Writing the Euler-Lagrange equation is not compulsory, but could help.

Exercise 8. Show that for every function $f:\mathbb{R}\to\mathbb{R}_+$ l.s.c. there exists a sequence of functions $f_k:\mathbb{R}\to\mathbb{R}_+$, each $k-$Lipschitz, such that for every $x\in\mathbb{R}$ the sequence $(f_k(x))_k$ increasingly converges to $f(x)$.

Use this fact and the theorems we saw in class to prove semicontinuity, wrt to weak convergence in $H^1(\Omega)$, of the functional

$$J(u)=\int_\Omega f(u(x))|\nabla u(x)|^p\,dx,$$

where $p\geq 1$ and $f:\mathbb{R}\to\mathbb{R}_+$ is l.s.c.

Exercise 9. Let $\Omega\subset\mathbb{R}^n$ be bounded and open, and $\phi:\partial\Omega\to\mathbb{R}$ be Lipschitz continuous. Prove that there exists at least a function $\bar u$ which is Lipschitz on $\mathbb{R}^n$ and such that $\bar u=\phi$ on $\partial\Omega$.

Consider the problem

$$\min\left\{\int_\Omega \left(|\nabla u|^2-\varepsilon_0 u^2\right)\,dx \;:\; u\in H^1(\Omega),\; u-\bar u\in H^1_0(\Omega)\right\},$$

where the condition $u-\bar u\in H^1_0(\Omega)$ is a way of saying $u=\phi$ on $\partial\Omega$.

Prove that, at least for small $\varepsilon_0>0$ the above problem admits a solution, and give an example with large ε_0 where the solution does not exist. Also prove that, for small $\varepsilon_0>0$, the solution is unique. What does the smallness of ε_0 depend on ? Write the PDE satisfied by the minimizer (if you attended Elliptic equations, also prove $u\in C^\infty(\Omega)$).

Exercise 10. Find the Poincaré constant of the interval $(-A,A)$, i.e. the smallest constant C such that

$$\int_{-A}^{A} u^2(x)dx\leq C\int_{-A}^{A}(u')^2(x)dx$$

for every function in $H^1_0((-A,A))$.

What is the largest value of A such that $H^1_0((-A,A))\ni u\mapsto\int_{-A}^{A}[(u')^2(x)-u^2(x)]dx$ is a convex functional ? What about strict convexity ?

Exercise 11. If $f:\mathbb{R}^n\to\mathbb{R}$ is given by $f(x)=|x|\log|x|$, compute f^* and f^{**}.

Exercise 12. Let $f:\mathbb{R}^n\to\mathbb{R}$ be convex. Prove that f is strictly convex if and only if f^* is C^1 and that f is $C^{1,1}$ if and only if f^* is elliptic (meaning that there exists $c>0$ such that $f(x)-c|x|^2$ is convex).

Exercise 13. Consider the problem

$$\min\left\{\int_\Omega \frac{1}{p}|v|^p dx+\langle\bar u_0,\pi_0\rangle+\langle\bar u_1,\pi_1\rangle \quad:\quad \nabla v=f+\pi_0-\pi_1\right\},$$

where the minimization is done on the triplets (v,π_0,π_1) with $v\in L^2(\Omega;\mathbb{R}^d)$, $\pi_i\in(H^1(\Omega))'$ satisfying $\langle\pi_i,\phi\rangle=0$ for every $\phi\in H^1_0(\Omega)$ and $\langle\pi_i,\phi\rangle\geq 0$ for every $\phi\geq 0$. Here $f\in(H^1(\Omega))'$ and $\bar u_i\in H^1(\Omega)$ are given, and $\bar u_0+\bar u_1\geq 0$. Find its dual.

Exercise 14. Let $H : \mathbb{R}^n \to \mathbb{R}$ be given by

$$H(v) = \frac{(4|v|+1)^{3/2} - 6|v| - 1}{12}.$$

a) Prove that H is C^1 and strictly convex. Is it $C^{1,1}$? Is it elliptic ?

b) Compute H^*. Is it C^1, strictly convex, $C^{1,1}$ and/or elliptic ?

c) Consider the problem $\min\{\int H(v) \ : \ \nabla \cdot v = f\}$ (on the d-dimensional torus, for simplicity) and find its dual.

d) Supposing $f \in L^2$, prove that the optimal u in the dual problem is H^2.

e) Under the same assumption, prove that the optimal v in the primal problem belongs to $W^{1,p}$ for every $p < 2$ if $d = 2$, for $p = d/(d-1)$ if $3 \le d \le 5$, and for $p = 6/5$ if $d \ge 3$.

Exercise 15. Consider the problem

$$\min\{A(v) := \int_{\mathbb{T}^d} H(v(x))dx \ : \ v \in L^2, \nabla \cdot v = f\}$$

for a function H which is elliptic. Prove that the problem has a solution, provided there exists at least an admissible v with $A(v) < +\infty$. Prove that, if f is an H^1 function with zero mean, then the optimal v is also H^1.

Exercise 16. Consider

$$\min\left\{\int_{\mathbb{T}^d} e^{c(x)}\left[\frac{|\nabla u(x)|^2}{2} + \frac{u^2(x)}{2} + f(x)u(x)\right]dx, \ u \in H^1(\mathbb{T}^d)\right\}$$

where $c, f : \mathbb{T}^d \to \mathbb{R}$ are given C^∞ functions.

a) Prove that the problem admits a unique solution.

b) Write the Euler-Lagrange equation of the problem.

c) Prove, using this PDE, that the solution is a C^∞ function.

Exercise 17. Given $u_0 \in C^1([0,1])$ consider the problem

$$\min\left\{\int_0^1 \frac{1}{2}|u - u_0|^2 dx \quad : \quad u' \ge 0\right\},$$

which consists in the projection of u_0 onto the set of monotone increasing functions (where the condition $u' \ge 0$ is intended in the weak sense).

a) Prove that this problem admits a unique solution.

b) Write the dual problem

c) Prove that the solution is actually the following : define U_0 through $U_0' = u_0$, set $U_1 := (U_0)^{**}$ to be the largest convex and l.s.c. function smaller than U_0, take $u = U_1'$.

Exercise 18. Given $u \in W^{1,p}_{loc}(\mathbb{R}^d)$, suppose that we have

$$\int_{B(x_0,r)} |\nabla u|^p dx \le C r^d f(r),$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is defined as $f(r) = -\log r$ for $r \le e^{-1}$ and $f(r) = 1$ for $r \ge e^{-1}$. Prove that u is continuous and that we have

$$|u(x) - u(y)| \le C|x - y| f(|x - y|)$$

(possibly for a different constant C).

Exercise 19. Let Ω be an open set in $\mathbb{R}^d$ and $F : H^1(\Omega) \to \mathbb{R}$ be defined through

$$F(u) := \int_\Omega \frac{1}{2}(|\nabla u| - 1)_+^2 + fu.$$

Suppose $f \in L^d(\Omega)$. Prove that minimizers of F are perturbative quasi-minimizers for $\int |\nabla u|^2$. Which regularity can we deduce for the minimizers of F ?

Exercise 20. Let $u \in H^1_{loc}(\mathbb{R}^d)$ be a solution of $\Delta u = f$, where $f = \nabla \cdot v$ and $v \in C^{0,\alpha}_{loc}(\mathbb{R}^d; \mathbb{R}^d)$. Prove $u \in C^{1,\alpha}_{loc}$. Deduce the result $\Delta u = f$, $f \in C^{0,\alpha}_{loc} \Rightarrow u \in C^{2,\alpha}_{loc}$.

Exercise 21. Given a bounded open set Ω and a Lipschitz function $g : \partial\Omega \to \mathbb{R}$, prove that the following problem admits at least a solution

$$\min \left\{ ||\frac{\pi}{2} + \arctan(u) + |\nabla u|^2||_{L^\infty(\Omega)} \,:\, u \in \mathrm{Lip}(\overline{\Omega}),\, u = g \text{ on } \partial\Omega \right\}.$$

Also prove that the problem

$$\min \left\{ || \arctan(u) + |\nabla u|^2||_{L^\infty(\Omega)} \,:\, u \in \mathrm{Lip}(\overline{\Omega}),\, u = g \text{ on } \partial\Omega \right\}$$

admits a solution, if we suppose $g \geq 0$.

Exercise 22. Set $F(u) := ||u + |u'|||_{L^\infty([0,1])}$. Prove that

$$\inf \{F(u) \,:\, u \in \mathrm{Lip}([0, 1]),\, u(0) = u(1) = -1\} < \inf \left\{ F(u) \,:\, u \in C^1([0, 1]),\, u(0) = u(1) = -1 \right\}.$$

Exercise 23. Prove that the function $u : \mathbb{R}^2 \to \mathbb{R}$ defined through $u(x_1, x_2) = x_1^{4/3} - x_2^{4/3}$ satisfies $\Delta_\infty u = 0$ in the viscosity sense. Deduce that, differently from harmonic function, we cannot get C^∞ regularity for solutions of $\Delta_\infty u = 0$.

Exercise 24. For given $f \in L^1(\Omega)$ with $\int_\Omega f(x)dx = 0$ and $p > d$ consider the functions u_p which solve

$$\min \left\{ \frac{1}{p} \int |\nabla u|^p dx + \int fu \,:\, u \in W^{1,p}(\Omega) \right\}.$$

Prove that the sequence u_p is compact in $C^0(\Omega)$. What can we say about the possible limits as $p \to \infty$? (in particular, do they minimize something ?).

Exercise 25. Let $A \subset \mathbb{R}^2$ be a bounded measurable set. Let $\pi(A) \subset \mathbb{R}$ be defined via $\pi(A) = \{x \in \mathbb{R} : \mathcal{L}^1(\{y \in \mathbb{R} : (x, y) \in A\} > 0\}$. Prove that we have $Per(A) \geq 2\mathcal{L}^1(\pi(A))$. Appy this result to the isoperimetric problem in $\mathbb{R}^2$, proving the existence of a solution to

$$\min \left\{ Per(A) \,:\, A \subset \mathbb{R}^2,\, A \text{ bounded } \mathcal{L}^2(A) = 1 \right\}.$$

Exercise 26. Let $\Omega = B(0, 1) \subset \mathbb{R}^N$ be the unit ball in dimension N. Consider the following variational problem :

$$\min \left\{ \int_\Omega |\nabla u|^2 + \int_\Omega (u - 1)^2 + Per(A) \,:\, u \in H^1_0(\Omega), A \subset \Omega \text{ of finite perimeter, such that } u = 0 \text{ a.e. on } A^c \right\}.$$

where $Per(A)$ stands for the perimeter of A in the BV sense.

a) Prove that the problem admits a solution (u, A).
b) Prove that for every solution (u, A) we have $0 \leq u \leq 1$.
c) Prove that the problem admits at least a radial solution (i.e. $A = B(0, r)$ and $u = u^*$).
d) Find or characterize the radial solution.

Exercise 27. In the disk $B(0,1) \subset \mathbb{R}^2$ we need to place a disk $B(x_0, r)$ centered at $x_0 \in B(0,1)$ and of radius $r \le 1 - |x_0|$ so that we minimize

$$J(x_0, r) := \frac{1}{r} + \frac{1}{\sqrt{1-|x_0|^2}} + \int_{B(0,1)} g(x, u_{x_0,r}(x))dx,$$

where $g : B(0,1) \times \mathbb{R} \to \mathbb{R}$ is a given bounded continuous function and $u_{x_0,r}$ denotes the solution of

$$\begin{cases} \Delta u = 1 & \text{dans } B(0,1) \setminus \overline{B(x_0,r)}, \\ u = 0 & \text{sur } \partial B(0,1) \cup \partial B(x_0,r), \end{cases}$$

that we extend to 0 on $B(x_0, r)$. Prove that there exists a solution.

Exercise 28. Let Ω be an open connected and smooth subset of $\mathbb{R}^d$ such that $\lambda_1(\Omega) > 1$ and fix $\bar{u} \in H^1(\Omega)$. Prove that the following minimization problem admits a unique solution

$$\min \left\{ \int \left(\frac{1}{2}|\nabla u|^2 + \sin(u) \right) dx \;:\; u - \bar{u} \in H_0^1(\Omega) \right\}$$

and prove that u is a solution of the above problem if and only if it solves

$$\begin{cases} \Delta u = \cos(u) & \text{weakly in } \Omega, \\ u = \bar{u} & \text{on } \partial\Omega. \end{cases}$$

Exercise 29. Let $\Omega = B(0,1) \subset \mathbb{R}^d$ be the d-dimensional ball, with $d > 1$, $g : \partial\Omega \to \mathbb{R}$ a given Lipschitz function and $f \in L^\infty(\Omega)$. Consider the problem

$$\inf \left\{ J(u) := \int_\Omega \left(|x||\nabla u(x)|^2 + f(x)u(x) \right) dx \;:\; u = g \text{ on } \partial\Omega \right\}.$$

Find a suitable functional space where J is well-defined (valued in $\mathbb{R} \cup \{+\infty\}$) as well as the boundary condition, and where the minimization problem has a solution. Can we choose $W^{1,p}$? H^1 ?

Also say whether this solution is unique, and write the Euler-Lagrange equation of the problem. Find the solution when g and f are constant.

Exercise 30. Let $u \ge 0$ be a Lipschitz continuous function with compact support in $\mathbb{R}^d$, and u^* its simmetrization. Prove that u^* is also Lipschitz, and $\mathrm{Lip}(u^*) \le \mathrm{Lip}(u)$.

Exercise 31. By using the isoperimetric inequality and the coarea formula, prove the inequality

$$||u||_{L^{d/(d-1)}} \le C||\nabla u||_{L^1},$$

for $u \in C_c^\infty(\mathbb{R}^d)$ and find the optimal constant C. Deduce the injection $BV(\mathbb{R}^d) \subset L^{d/(d-1)}(\mathbb{R}^d)$.

Exercise 32. Let $(F_n)_n$ be a sequence of functionals defined on a common metric space X, such that $F_n \le F_{n+1}$. Suppose that each F_n is l.s.c. and set $F := \sup_n F_n$. Prove $F_n \xrightarrow{\Gamma} F$.

Exercise 33. Let us define the following functionals on $X = L^2([-1,1])$

$$F_n(u) := \begin{cases} \frac{1}{2n}\int_{-1}^1 |u'(t)|^2 dt + \frac{1}{2}\int_{-1}^1 |u(t) - t|^2 dt & \text{if } u \in H_0^1([-1,1]), \\ +\infty & \text{otherwise}; \end{cases}$$

together with

$$H(u) = \begin{cases} \frac{1}{2}\int_{-1}^1 |u(t) - t|^2 dt & \text{if } u \in H_0^1([-1,1]), \\ +\infty & \text{otherwise}; \end{cases}$$

and $F(u) := \frac{1}{2}\int_{-1}^1 |u(t) - t|^2 dt$ for every $u \in X$.

a) Prove that, for each n, the functional F_n is l.s.c. for the L^2 (strong) convergence ;

b) Prove that also F is l.s.c. for the same convergence, but not H ;

c) Find the minimizer u_n of F_n over X ;

d) Find the limit as $n \to \infty$ of u_n. Is it a strong L^2 limit ? is it a uniform limit ? a pointwise a.e. limit ?

e) Find the Γ-limit of F_n (which, without surprise, is one of the functionals F or H), proving Γ-convergence ;

f) Does the functional H admit a minimizer in X ?

Exercise 34. Consider the functions $a_n : [0,1] \to$ given by $a_n(x) = a(nx)$ where $a = 2\sum_{k\in\mathbb{Z}} \mathbb{1}_{[2k,2k+1]} + \sum_{k\in\mathbb{Z}} \mathbb{1}_{[2k-1,2k]}$. Given $f \in L^1([0,1])$ with $\int_0^1 f(t)dt = 0$ and $p \in]1, +\infty[$, compute

$$\lim_{n\to\infty} \min \left\{ \int_0^1 \left(\frac{1}{p} a_n |u'(t)|^p dt + f(t)u(t) \right) dt \, : \, u \in W^{1,p}([0,1]) \right\}.$$