Calculus of Variations and Elliptic PDEs
–
Exercises

Exercise 1. Solve the problem

$$\min\{J(f) := \int_0^1 \left[\frac{1}{2}f'(t)^2 + tf(t) + \frac{1}{2}f(t)^2\right] dt\,;\ f \in \mathcal{A}\}, \quad \text{où} \quad \mathcal{A} := \{f \in C^1([0,1])\ :\ f(0) = 0\}.$$

Find the minimal value of J on $\mathcal{A}$ and the function(s) f which attain it, proving that they are actually minimizers

Exercise 2. Consider the problem

$$\min\left\{\int_0^T e^{-t}\left(u'(t)^2 + 5u(t)^2\right) dt \quad : \quad u \in C^1([0,T]),\ u(0) = 1\right\}.$$

Prove that it admits a minimizer, that it is unique, find it, compute the value of the minimum, and the limit of the minimizer (in which sense ?) and of the minimal value as $T \to +\infty$.

Exercise 3. Consider the problem

$$\min\{J(u) := \int_0^1 \left[\frac{1}{2}u'(t)^2 + u(t)f(t)\right] dt\,;\ u \in W^{1,2}([0,1])\}.$$

Find a necessary and sufficient condition on f so that this problem admits a solution.

Exercise 4. Let $L : \mathbb{R} \to \mathbb{R}$ be a strictly convex C^1 function, and consider

$$\min\{\int_0^1 L(u'(t))dt\,;\ u \in C^1([0,1]), u(0) = a, u(1) = b\}.$$

Prove that the solution is $u(t) = (1-t)a + tb$, whatever is L. What happens if L is not C^1 ? and if L is convex but not strictly convex ?

Exercise 5. Prove that we have

$$\inf\{\int_0^1 t|u'(t)|^2 dt\,;\ u \in C^1([0,1]), u(0) = 1, u(1) = 0\} = 0.$$

Is the infimum above attained ?

What about, instead

$$\inf\{\int_0^1 \sqrt{t}|u'(t)|^2 dt\,;\ u \in C^1([0,1]), u(0) = 1, u(1) = 0\} \quad ?$$

Exercise 6. Consider a minimization problem of the form

$$\min\{F(u) := \int_0^1 L(t, u(t), u'(t))dt\,;\ u \in W^{1,1}([0,1]), u(0) = a,\ u(1) = b\},$$

where $L \in C^2([0,1] \times \mathbb{R} \times \mathbb{R})$. We denote as usual by (t, x, v) the variables of L. Suppose that $\bar{u}$ is a solution to the above problem. Prove that we have

$$\frac{\partial^2 L}{\partial v^2}(t, \bar{u}(t), \bar{u}'(t)) \geq 0\, a.e.$$

Exercise 7. Given $f \in C^2(\mathbb{R})$, consider the problem

$$\min\{F(u) := \int_0^1 \left[(u'(t)^2 - 1)^2 + f(u(t))\right] dt\,;\ u \in C^1([0,1]), u(0) = a,\ u(1) = b\}.$$

Prove that the problem does not admit any solution if $|b - a| \leq \frac{1}{\sqrt{3}}$.

Exercise 8. Consider the problem

$$\min\left\{\int_0^1 \left[|u'(t)|^2+\arctan(u(t))\right]\,dt \;:\; u\in C^1([0,1])\right\},$$

and prove that it has no solutions. Prove the existence of a solution if we add the boundary condition $u(0)=0$, write the optimality conditions and discuss the regularity of the solution.

Exercise 9. Consider the functional $F : H^1([0,T]) \to \mathbb{R}$ defined through

$$F(u)=\int_0^T \Big(u'(t)^2+\arctan(u(t)-t)\Big)\,dt.$$

Prove that

a) the problem $(P) := \min\{F(u) \;:\; u\in H^1([0,T])\}$ has no solution ;

b) the problem $(P_a) := \min\{F(u) \;:\; u\in H^1([0,T]), u(0)=a\}$ admits a solution for every $a\in\mathbb{R}$;

c) we have $F(-|u|)\le F(u)$;

d) the solution of (P_a) is unique as soon as $a\le 0$;

e) there exists $L_0<+\infty$ such that for every $T\le L_0$ the solution of (P_a) is unique for every $a\in\mathbb{R}$

f) the minimizers of (P) and (P_a) are C^∞ functions.

Exercise 10. Prove existence and uniqueness of the solution of

$$\min\left\{\int_\Omega \Big(f(x)|u(x)|+|\nabla u(x)|^2\Big)\,dx\,;\, u\in H^1(\Omega),\ \int_\Omega u=1\right\},$$

when Ω is an open, connected and bounded subset of $\mathbb{R}^n$ and $f\in L^2(\Omega)$, $f\ge 0$ (the sign of f is not important for existence). Where do we use connectedness ? Also prove that, if Ω is not connected (but has a finite number of connected components and we keep the assumption $f\ge 0$), then we have existence but maybe not uniqueness, and that if we withdraw both connectedness and positivity of f, then maybe we don't even have existence.

Exercise 11. Fully solve

$$\min\left\{\int_Q \Big(|\nabla u(x,y)|^2+u(x,y)^2\Big)\,dx\,dy \;:\; u\in C^1(Q),\ u=\phi \text{ sur } \partial Q\right\},$$

where $Q=[-1,1]^2\subset\mathbb{R}^2$ and $\phi:\partial Q\to\mathbb{R}$ is given by

$$\phi(x,y)=\begin{cases}0 & \text{si } x=-1,\ y\in[-1,1]\\ 2(e^y+e^{-y}) & \text{if } x=1,\ y\in[-1,1]\\ (x+1)(e+e^{-1}) & \text{if } x\in[-1,1],\ y=\pm 1.\end{cases}$$

Find the minimizer and the value of the minimum. Writing the Euler-Lagrange equation is not compulsory, but could help.

Exercise 12. Show that for every function $f:\mathbb{R}\to\mathbb{R}_+$ l.s.c. there exists a sequence of functions $f_k:\mathbb{R}\to\mathbb{R}_+$, each $k-$Lipschitz, such that for every $x\in\mathbb{R}$ the sequence $(f_k(x))_k$ increasingly converges to $f(x)$.

Use this fact and the theorems we saw in class to prove semicontinuity, wrt to weak convergence in $H^1(\Omega)$, of the functional

$$J(u)=\int_\Omega f(u(x))|\nabla u(x)|^p\,dx,$$

where $p\ge 1$ and $f:\mathbb{R}\to\mathbb{R}_+$ is l.s.c.

Exercise 13. Find the Poincaré constant of the interval $(-A, A)$, i.e. the smallest constant C such that

$$\int_{-A}^{A} u^2(x)dx \le C \int_{-A}^{A} (u')^2(x)dx$$

for every function in $H_0^1((-A, A))$.

What is the largest value of A such that $H_0^1((-A, A)) \ni u \mapsto \int_{-A}^{A}[(u')^2(x) - u^2(x)]dx$ is a convex functional? What about strict convexity?

Exercise 14. Let $\Omega \subset \mathbb{R}^n$ be bounded and open, and $\phi : \partial\Omega \to \mathbb{R}$ be Lipschitz continuous. Prove that there exists at least a function $\bar{u}$ which is Lipschitz on $\mathbb{R}^n$ and such that $\bar{u} = \phi$ on $\partial\Omega$.

Consider the problem

$$\min \left\{ \int_\Omega \left(|\nabla u|^2 - \varepsilon_0 u^2 \right) dx \ : \ u \in H^1(\Omega), \ u - \bar{u} \in H_0^1(\Omega) \right\},$$

where the condition $u - \bar{u} \in H_0^1(\Omega)$ is a way of saying $u = \phi$ on $\partial\Omega$.

Prove that, at least for small $\varepsilon_0 > 0$ the above problem admits a solution, and give an example with large ε_0 where the solution does not exist. Also prove that, for small $\varepsilon_0 > 0$, the solution is unique. What does the smallness of ε_0 depend on? Write the PDE satisfied by the minimizer.

Exercise 15. Let Ω be an open connected subset in $\mathbb{R}^d$, $a \in L^\infty(\Omega)$ be a function with $a \ge a_0$ where $a_0 > 0$ is a positive constant, and $b \in L^2(\Omega)$ be another function, which is not identically zero. Prove that the following minimization problem admits a solution

$$\min \left\{ \frac{\int_\Omega a|\nabla u|^2 dx}{|\int_\Omega bu\, dx|^2} \ : \ u \in H_0^1(\Omega) \ : \ \int_\Omega bu\, dx \neq 0 \right\},$$

and write the PDE that such a solution satisfies. Finally, compute the value of the above minimum in the case $\Omega = B(0, 1) \subset \mathbb{R}^2$, $a(x) = 1$ and $b(x) = |x|$.

Exercise 16. If $f : \mathbb{R}^n \to \mathbb{R}$ is given by $f(x) = |x| \log |x|$, compute f^* and f^{**}.

Exercise 17. Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Prove that f is strictly convex if and only if f^* is C^1 and that f is $C^{1,1}$ if and only if f^* is elliptic (meaning that there exists $c > 0$ such that $f(x) - c|x|^2$ is convex).

Exercise 18. Given a bounded, smooth and connected domain $\Omega \subset \mathbb{R}^d$, and $f \in L^2(\Omega)$, set $X(\Omega) = \{v \in L^2(\Omega; \mathbb{R}^d) \ : \ \nabla \cdot v \in L^2(\Omega)\}$ and consider the minimization problems

$$\begin{aligned}
(P) \quad &:= \quad \min \left\{ F(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 + f(x)u \right) dx \ : \ u \in H^1(\Omega) \right\} \\
(D) \quad &:= \quad \min \left\{ G(v) := \int_\Omega \left(\frac{1}{2}|v|^2 + \frac{1}{2}|\nabla \cdot v - f|^2 \right) dx \ : \ v \in X(\Omega) \right\},
\end{aligned}$$

a) Prove that (P) admits a unique solution;

b) Prove $\min(P) + \inf(D) \ge 0$;

c) Prove that there exist $v \in X(\Omega)$ and $u \in H^1(\Omega)$ such that $F(u) + G(v) = 0$;

d) Deduce that $\min(D)$ is attained and $\min(P) + \inf(D) = 0$;

e) Justify by a formal inf-sup exchange the duality $\min F(u) = \sup -G(v)$;

f) Prove $\min F(u) = \sup -G(v)$ via a duality proof based on convex analysis.

Exercise 19. Consider the problem

$$\min\left\{\int_\Omega \frac{1}{2}|v|^2 dx + \langle \bar{u}_0, \pi_0\rangle + \langle \bar{u}_1, \pi_1\rangle \quad : \quad \nabla\cdot v = f + \pi_0 - \pi_1\right\},$$

where the minimization is done on the triplets (v, π_0, π_1) with $v \in L^2(\Omega; \mathbb{R}^d)$, $\pi_i \in (H^1(\Omega))'$ satisfying $\langle \pi_i, \phi\rangle = 0$ for every $\phi \in H^1_0(\Omega)$ and $\langle \pi_i, \phi\rangle \geq 0$ for every $\phi \geq 0$. Here $f \in (H^1(\Omega))'$ and $\bar{u}_i \in H^1(\Omega)$ are given.

Find its dual, distinguishing the case $\bar{u}_0 + \bar{u}_1 \geq 0$ or not.

Exercise 20. Let Ω be the $d-$dimensional flat torus (just to avoid boundary conditions, think at a cube), $p, q > 1$ two given exponents, $a > 0$ and $f : \Omega \to \mathbb{R}$ a given Lipschitz continuous function. Consider the following minimization problem

$$\inf\left\{\int_\Omega \left(\frac{1}{p}|\nabla u|^p - \frac{a}{q}|u|^q + fu\right) dx \; : \; u \in W^{1,p}(\Omega) \cap L^q(\Omega), \; \int_\Omega u = 0\right\}.$$

a) Prove that, if $q > p$, the inf is $-\infty$ and the minimization problem has no solution.

b) Prove that, if $q < p$, the infimum is attained.

c) Prove that, if $q = p$, the infimum is attained, provided a is small enough.

d) In the cases where the infimum is attained, write the Euler-Lagrange equation solved by the minimizers.

e) Recall the condition on f which guarantee that solutions of $\Delta_p u = f$, satisfy $(\nabla u)^{p/2} \in H^1$ (remember that, for a vector v, the expression v^α is to be intended as equal to a vector w with $|w| = |v|^\alpha$ and $w \in \mathbb{R}_+ v$).

f) For $p \geq 2$ and $2 \leq q \leq p$, prove that the solution $\bar{u}$ satisfies $(\nabla \bar{u})^{p/2} \in H^1$.

Exercise 21. Let $H : \mathbb{R}^n \to \mathbb{R}$ be given by

$$H(v) = \frac{(4|v|+1)^{3/2} - 6|v| - 1}{12}.$$

a) Prove that H is C^1 and strictly convex. Is it $C^{1,1}$? Is it elliptic?

b) Compute H^*. Is it C^1, strictly convex, $C^{1,1}$ and/or elliptic?

c) Consider the problem $\min\{\int H(v) \; : \; \nabla\cdot v = f\}$ (on the d-dimensional torus, for simplicity) and find its dual.

d) Supposing $f \in L^2$, prove that the optimal u in the dual problem is H^2.

e) Under the same assumption, prove that the optimal v in the primal problem belongs to $W^{1,p}$ for every $p < 2$ if $d = 2$, for $p = d/(d-1)$ if $3 \leq d \leq 5$, and for $p = 6/5$ if $d \geq 3$.

Exercise 22. Consider the problem

$$\min\{A(v) := \int_{\mathbb{T}^d} H(v(x))dx \; : \; v \in L^2, \nabla\cdot v = f\}$$

for a function H which is elliptic. Prove that the problem has a solution, provided there exists at least an admissible v with $A(v) < +\infty$. Prove that, if f is an H^1 function with zero mean, then the optimal v is also H^1.

Exercise 23. Given a function $g \in L^2([0, L])$, consider the problem

$$\min\left\{\int_0^L \frac{1}{2}|u(t) - g(t)|^2 dt \; : \; u(0) = u(L) = 0, u \in \mathrm{Lip}([0, L]), |u'| \leq 1 \text{ a.e.}\right\}.$$

a) Prove that this problem admits a solution.

b) Prove that the solution is unique.

c) Find the optimal solution in the case where g is the constant function $g = 1$ in the terms of the value of L, distinguishing $L > 2$ and $L \leq 2$.

d) Computing the value of

$$\sup \left\{ -\int_0^L (u(t)z'(t) + |z(t)|)dt \ : \ z \in H^1([0, L]) \right\}$$

find the dual of the previous problem by means of a formal inf-sup exchange.

e) Assuming that the equality $\inf \sup = \sup \inf$ in the duality is satisfied, write the necessary and sufficient optimality conditions for the solutions of the primal and dual problem. Check that these conditions are satisfied by the solution found in the case $g = 1$.

f) Prove the the equality $\inf \sup = \sup \inf$ *(more difficult)*.

Exercise 24. Given $u_0 \in C^1([0, 1])$ consider the problem

$$\min \left\{ \int_0^1 \frac{1}{2}|u - u_0|^2 dx \quad : \quad u' \geq 0 \right\},$$

which consists in the projection of u_0 onto the set of monotone increasing functions (where the condition $u' \geq 0$ is intended in the weak sense).

a) Prove that this problem admits a unique solution.

b) Write the dual problem

c) Prove that the solution is actually the following : define U_0 through $U_0' = u_0$, set $U_1 := (U_0)^{**}$ to be the largest convex and l.s.c. function smaller than U_0, take $u = U_1'$.

Exercise 25. Write and prove a Caccioppoli-type inequality between the $L^p(B_r)$ norm of ∇u and the $L^p(B_R)$ norm of u for solutions of $\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$.

Exercise 26. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ be a distributional solution of $\Delta u = b(x) \cdot \nabla u + f(x)u$, where $f : \mathbb{R}^n \to \mathbb{R}$ and $b : \mathbb{R}^n \to \mathbb{R}^n$ are given C^∞ functions. Prove $u \in C^\infty(\mathbb{R}^n)$.

Exercise 27. Suppose that $\Omega = B(0, 1) \subset \mathbb{R}^2$ represents a circular membrane, attached at its boundary $\partial\Omega$ to a given profile ϕ. It must follow the profile on the boundary but is free inside. The shape of the membrane inside will follow that of the graph of a harmonic function. A given initial profile ϕ_0 on the boundary is given, but we can act on it so as to change the shape of the membrane. The goal is to have it as flat as possible, in the sense of minimizing the norm of the hessian matrix on a given region $A = B(0, R) \subset \Omega$, but we pay a price for the effort we do on the boundary.

Mathematically, we consider this problem : for every $\phi \in H^1(\partial\Omega)$ (attention : $\partial\Omega$ is a circle, so when we say H^1 we mean H^1 functions of one variable) we define u_ϕ as the unique solution of $\Delta u = 0$ in Ω, with $u = \phi$ on $\partial\Omega$; then, we solve

$$\min \quad ||\phi - \phi_0||^2_{H^1(\partial\Omega)} + \int_A |D^2 u_\phi|^2 dx.$$

Prove the existence of the minimum, both in the case $A = B(0, R)$ with $R < 1$ and $A = \Omega = B(0, 1)$.

also prove existence of a solution to the problem

$$\min \quad ||\phi - \phi_0||^2_{H^1(\partial\Omega)} + |\nabla u_\phi(0)|,$$

where we want the membrane to be as horizontal as possible at the middle point.

Exercise 28. Let Ω be an open and bounded subset of $\mathbb{R}^d$, and $p > 1$. Consider the following minimization problem

$$\min \left\{ \int_\Omega \left(\frac{1}{p}|\nabla u(x)|^p - \sqrt{1 + \frac{1}{p}|u(x)|^p} \right) dx \ : \ u \in W_0^{1,p}(\Omega) \right\}.$$

a) Prove that it admits a solution.
b) Prove that there exists Ω such that 0 is not a solution, and hence that the solution is not always unique.
c) Write the PDE solved by any solution.
d) In the case $p = 2$, prove that any solution is C^∞ in the interior of Ω.

Exercise 29. Let Ω be an open connected and smooth subset of $\mathbb{R}^d$ such that $\lambda_1(\Omega) > 1$ (we define $\lambda_1(\Omega) := \inf_{v \in H^1_0(\Omega)\setminus\{0\}} ||\nabla v||^2_{L^2}/||v||^2_{L^2}$) and fix $\bar{u} \in H^1(\Omega)$. Prove that the following minimization problem admits a unique solution

$$\min\left\{\int\left(\frac{1}{2}|\nabla u|^2 + \sin(u)\right)dx \ : \ u - \bar{u} \in H^1_0(\Omega)\right\}$$

and prove that u is a solution of the above problem if and only if it solves

$$\begin{cases}\Delta u = \cos(u) & \text{weakly in } \Omega,\\ u = \bar{u} & \text{on } \partial\Omega.\end{cases}$$

Also prove that the optimal u is a C^∞ function inside Ω.

Exercise 30. Which among the functions u which can be written in polar coordinates as $u(\rho, \theta) = \rho^\alpha \sin(k\theta)$ are harmonic on the unit ball ? (i.e., for which values of $\alpha, k \in \mathbb{R}$).

Exercise 31. Consider the equation $\Delta u = f(u)$, where $f : \mathbb{R} \to \mathbb{R}$ is a given C^∞ function. Suppose that $u \in H^1(\Omega)$ is a weak solution of this equation and f has polynomial growth of order m (i.e. $|f(t)| \le C(1+|t|)^m$. In dimension $d = 1, 2$, prove $u \in C^\infty$. In higher dimension, prove it under the restriction $m < (d+2)/(d-2)$.

Exercise 32. Consider the equation $\Delta u = f(|\nabla u|^2)$, where $f : \mathbb{R} \to \mathbb{R}$ is a given C^∞ function. Prove that, if $u \in C^1(\bar{\Omega}) \subset H^1(\Omega)$ is a weak solution of this equation, then we have $u \in C^\infty$. What is the difficulty in removing the assumption $u \in C^1(\bar{\Omega})$? Which assumption to add on f so as to prove the result for H^1 weak solutions ?

Exercise 33. Let $Q = (a_1, b_1) \times (a_2, b_2) \times \dots (a_n, b_n) \subset \mathbb{R}^n$ be a rectangle and $f \in C^\infty_c(Q)$. Let $u \in H^1_0(Q)$ be the solution of $\Delta u = f$ with $u = 0$ on ∂Q. Prove $u \in C^\infty(\overline{Q})$. Can we say $u \in C^\infty_c(Q)$? do functions f such that $u \in C^\infty_c(Q)$ exist ?

Exercise 34. Let $T \subset \mathbb{R}^2$ be a triangle and $f \in L^p(T)$. Let $u \in H^1_0(T)$ be the solution of $\Delta u = f$ with $u = 0$ on ∂T. Prove $u \in W^{2,p}(T)$.

If we assume $\int_T f = 0$ (why ?) let v be the solution of the Neumann problem $\Delta v = f$ with $\partial v/\partial n = 0$ on ∂T (ain the weak sense : $\int_T \nabla v \cdot \nabla \phi = -\int_T f\phi$ for every $\phi \in H^1(T)$). Prove $v \in W^{2,p}(T)$ and $v \in W^{3,p}(T)$ if we also have $f \in W^{1,p}(T)$.

Exercise 35. Given $u \in W^{1,p}_{loc}(\mathbb{R}^d)$, suppose that we have

$$\int_{B(x_0,r)} |\nabla u|^p dx \le Cr^d f(r),$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is defined as $f(r) = -\log r$ for $r \le e^{-1}$ and $f(r) = 1$ for $r \ge e^{-1}$. Prove that u is continuous and that we have

$$|u(x) - u(y)| \le C|x - y|f(|x - y|)$$

(possibly for a different constant C).

Exercise 36. Let $u \ge 0$ be a continuous function on a bounded domain Ω which is a weak solution of

$$\begin{cases}\Delta u = \sqrt{u} & \text{in } \Omega,\\ u = 1 & \text{on } \partial\Omega.\end{cases}$$

Prove that we have $u \in C^{2,1/2}_{loc}(\Omega)$ and $u \in C^\infty$ if $u > 0$. Prove that we do have $u > 0$ if Ω is contained in a ball of radius R sufficiently small (how much ?). Is it possible, for larger domains Ω, to have $\min u = 0$?

Exercise 37. Given a function $H : \mathbb{R}^d \to \mathbb{R}$ which is both elliptic and $C^{1,1}$ (i.e. threre are two positive constants c_0, c_1 such that $c_0 I \le D^2 H \le c_1 I$, let $u \in H^1_{loc}(\Omega)$ be a weak solution in Ω of $\nabla \cdot (\nabla H(\nabla u)) = 0$.

a) Prove that we have $u \in H^2_{loc}$.

b) Prove the same result for solutions of $\nabla \cdot (\nabla H(\nabla u)) = f$, $f \in L^2_{loc}$.

c) In case Ω is the torus, prove that the same result is global and only uses the lower bound on $D^2 H$.

Exercise 38. Let $u \in H^1_{loc}$ be a weak solution of $\nabla \cdot ((3 + \sin(|\nabla u|^2))\nabla u) = 0$. Prove that we have $u \in C^{0,\alpha}_{loc}$. Also prove $u \in C^\infty$ on any open set where $\sup |\nabla u| < 1$.

Exercise 39. Let $u \in H^1_{loc} \cap L^\infty_{loc}$ be a weak solution of $\nabla \cdot (A(x, u)\nabla u) = 0$, where

$$A_{ij}(x, u) = (1 + u^2|x|^2)\delta_{ij} - u^2 x_i x_j.$$

Prove that we have $u \in C^\infty$.

Exercise 40. Let p_n be a sequence of C^1 functions on a same bounded and smoothh domain $\Omega \subset \mathbb{R}^d$ which is bounded in $L^\infty \cap H^1$ and g a given C^2 function on Ω. Let $u_n \in H^1(\Omega)$ be the unique weak solution of

$$\begin{cases} \Delta u_n = \nabla u \cdot \nabla p_n & \text{in } \Omega, \\ u_n = g & \text{on } \partial\Omega. \end{cases}$$

a) Find a minimization problem solved by u_n.

b) Prove that we have $||u_n||_{L^\infty} \le ||g||_{L^\infty}$.

c) Prove that u_n is bounded in $H^1(\Omega)$.

d) Prove that u_n is locally Hölder continuous, with a modulus of continuity on each subdomain Ω' compactly contained in Ω which independent of n (but can depend on Ω')

e) If p_n weakly converges in H^1 to a function p prove that we have $u_n \to u$ (the convergence being strong in H^1 and locally uniform), where u is the unique solution of the same problem where p_n is replaced by p.

Exercise 41. Let $(F_n)_n$ be a sequence of functionals defined on a common metric space X, such that $F_n \le F_{n+1}$. Suppose that each F_n is l.s.c. and set $F := \sup_n F_n$. Prove $F_n \xrightarrow{\Gamma} F$.

Exercise 42. Let us define the following functionals on $X = L^2([-1, 1])$

$$F_n(u) := \begin{cases} \frac{1}{2n} \int_{-1}^1 |u'(t)|^2 dt + \frac{1}{2} \int_{-1}^1 |u(t) - t|^2 dt & \text{if } u \in H^1_0([-1, 1]), \\ +\infty & \text{otherwise}; \end{cases}$$

together with

$$H(u) = \begin{cases} \frac{1}{2} \int_{-1}^1 |u(t) - t|^2 dt & \text{if } u \in H^1_0([-1, 1]), \\ +\infty & \text{otherwise}; \end{cases}$$

and $F(u) := \frac{1}{2} \int_{-1}^1 |u(t) - t|^2 dt$ for every $u \in X$.

a) Prove that, for each n, the functional F_n is l.s.c. for the L^2 (strong) convergence ;

b) Prove that also F is l.s.c. for the same convergence, but not H ;

c) Find the minimizer u_n of F_n over X ;

d) Find the limit as $n \to \infty$ of u_n. Is it a strong L^2 limit ? is it a uniform limit ? a pointwise a.e. limit ?

e) Find the Γ-limit of F_n (which, without surprise, is one of the functionals F or H), proving Γ-convergence ;

f) Does the functional H admit a minimizer in X ?

Exercise 43. Given a function $a : [0,1] \to \mathbb{R}_+$ consider the functional $F[a]$ defined on $L^2([0,1])$ via

$$F[a](u) := \begin{cases} \frac{1}{2}\int_0^1 a|u'|^2 & \text{if } u \in H^1,\ u(1) = 0 \\ +\infty & \text{if not.} \end{cases}$$

Suppose that $a_n, a : [0,1] \to \mathbb{R}$ is a sequence such that $a_- \le a_n, a \le a_+$ ($a_\pm$ being two strictly positive constants) and $a_n^{-1} \rightharpoonup a^{-1}$, the convergence being a weak$-*$ convergence in L^∞. Prove that $F[a_n]$ Γ-converge in L^2 to $F[a]$.

Exercise 44. Consider the functions $a_n : [0,1] \to \mathbb{R}$ given by $a_n(x) = a(nx)$ where $a = 2\sum_{k\in\mathbb{Z}} \mathbb{1}_{[2k,2k+1]} + \sum_{k\in\mathbb{Z}} \mathbb{1}_{[2k-1,2k]}$. Given $f \in L^1([0,1])$ with $\int_0^1 f(t)dt = 0$ and $p \in]1,+\infty[$, compute

$$\lim_{n\to\infty} \min\left\{ \int_0^1 \left(\frac{1}{p}a_n|u'(t)|^p dt + f(t)u(t)\right) dt \ :\ u \in W^{1,p}([0,1]) \right\}.$$

Exercise 45. Prove the convergence in e) of Exercise 40 using Γ-convergence.

Exercise 46. Given a sequence of functions $a_n : \Omega \to \mathbb{R}$ which are bounded from below and above by two same strictly positive constants a_-, a_+ and a function a bounded by the same constants, prove that we have

$$\begin{cases} a_n \rightharpoonup a \\ a_n^{-1} \rightharpoonup a^{-1} \end{cases}$$

(the weak convergence being weak$-*$ in L^∞) if and only if we have $a_n \to a$ in L^2 (and hence in all L^p spaces for $p < \infty$).

Exercise 47. Let $A \subset \mathbb{R}^2$ be a bounded measurable set. Let $\pi(A) \subset \mathbb{R}$ be defined via $\pi(A) = \{x \in \mathbb{R} : \mathcal{L}^1(\{y \in \mathbb{R} : (x,y) \in A\} > 0\}$. Prove that we have $Per(A) \ge 2\mathcal{L}^1(\pi(A))$. Appy this result to the isoperimetric problem in $\mathbb{R}^2$, proving the existence of a solution to

$$\min\left\{ Per(A) \ :\ A \subset \mathbb{R}^2,\ A \text{ bounded},\ |A| = 1 \right\}.$$

Exercise 48. Let $\Omega \subset \mathbb{R}^d$ be the unit square $\Omega = (0,1)^2$ and $S = \{0\} \times (0,1) \subset \partial\Omega$. Consider the functions $a_n : \Omega \to \mathbb{R}$ defined via

$$a_n(x,y) = \begin{cases} A_0 & \text{if } x \in [\frac{2k}{2n}, \frac{2k+1}{2n}) \text{ for } k \in \mathbb{Z} \\ A_1 & \text{if } x \in [\frac{2k+1}{2n}, \frac{2k+2}{2n}) \text{ for } k \in \mathbb{Z}, \end{cases}$$

where $0 < A_0 < A_1$ are two given values. On the space $L^2(\Omega)$ (endowed with the strong L^2 convergence ; every time that we write $\to$ here below we mean this kind of convergence), consider the sequence of functionals

$$F_n(u) = \begin{cases} \int_\Omega a_n|\nabla u|^2 & \text{if } u \in X, \\ +\infty & \text{if not,} \end{cases}$$

where $X \subset L^2(\Omega)$ is the space of functions $u \in H^1(\Omega)$ satisfying $u = 0$ on S (which can be defined via the condition $u\eta \in H_0^1(\Omega)$ for every cut-off function $\eta \in \mathbb{C}^\infty(\mathbb{R}^2)$ with $\text{spt}(\eta) \cap (\partial\Omega \setminus S) = \emptyset$). The goal is to find the Γ-limit of the sequence $(F_n)_n$. Set

$$A_* := \left(\frac{\frac{1}{A_0} + \frac{1}{A_1}}{2}\right)^{-1} \quad \text{and} \quad A_\diamond := \frac{A_0 + A_1}{2}.$$

a) Given a sequence $(u_n)_n$ with $u_n \to u$ and $F_n(u_n) \le C$, prove $u \in X$ and $\liminf_n \int_\Omega a_n|\partial_x u_n|^2 \ge A_* \int_\Omega |\partial_x u|^2$.

b) For the same sequence, also prove $\liminf_n \int_\Omega a_n|\partial_y u_n|^2 \geq A_\diamond \int_\Omega |\partial_y u|^2$.

c) For any $u \in X$, find a sequence u_n such that $u_n \to u$, $a_n \partial_x u_n \to A_* \partial_x u$ and $\partial_y u_n \to \partial_y u$.

d) Conclude by finding the Γ-limit of F_n. Is it of the form $F(u) = \int a|\nabla u|^2$ for $u \in X$?

Exercise 49. Consider the following sequence of minimization problems, for $n \geq 0$,

$$\min \left\{ \int_0^1 \left(\frac{|u'(t)|^2}{\frac{3}{2} + \sin^2(nt)} + \left(\frac{3}{2} + \sin^2(nt) \right) u(t) \right) dt \ : \ u \in H^1([0,1]), u(0) = 0 \right\},$$

calling u_n their unique minimizers and m_n their minimal values.

Prove that we have $u_n(t) \to t^2 - 2t$ uniformly, and $m_n \to -2/3$.

Exercise 50. Let u_n, v_n be two sequences of functions belonging to $H^1([0,1])$. Suppose

$$u_n \rightharpoonup u_0, \quad v_n \rightharpoonup v_0, \quad E_n(u_n, v_n, [0,1]) \leq C$$

where E_n is the energy defined for every interval $J \subset [0,1]$ through

$$E_n(u, v, J) := \int_J \left(v(t)|u'(t)|^2 + \frac{1}{2n}|v'(t)|^2 + \frac{n}{2}|1 - v(t)|^2 \right) dt,$$

$C \in \mathbb{R}$ is a given constant, the weak convergence of u_n and v_n occurr in $L^2([0,1])$, and u_0 is a function which is piecewise C^1 on $[0,1]$ (i.e. there exists a partition $0 = t_0 < t_1 < \cdots < t_N = 1$ such that $u_0 \in C^1(]t_i, t_{i+1}[)$, and the limits of u_0 exist finite at $t = t_i^\pm$ but $u_0(t_i^-) \neq u_0(t_i^+)$ for $i = 1, \ldots, N-1$).

Denote by $\mathcal{J}$ the family of all the intervals J compactly contained in one of the open intervals $]t_i, t_{i+1}[$. Also suppose, for simplicity, that $u_n \rightharpoonup u_0$ in $H^1(J)$ for every interval $J \in \mathcal{J}$.

Prove that we necessarily have

a) $v = 1$ a.e. and $v_n \to v$ strongly in L^2.

b) $\liminf_{n\to\infty} E_n(u_n, v_n, J) \geq \int_J |u_0'(t)|^2 dt$ for every interval $J \in \mathcal{J}$.

c) $\liminf_{n\to\infty} E_n(u_n, v_n, J) \geq 1$ for every interval J containing one of the points t_i.

d) $C \geq \liminf_{n\to\infty} E_n(u_n, v_n, [0,1]) \geq \int_0^1 |u_0'(t)|^2 dt + (N-1)$.

Exercise 51. Let u_ε be solutions of the minimization problems P_ε given by

$$P_\varepsilon \ := \ \min \left\{ \int_0^\pi \left(\frac{\varepsilon}{2}|u'(t)|^2 + \frac{1}{2\varepsilon}\sin^2(u(t)) + 10^3|u(t) - t| \right) dt \ : \ u \in H^1([0,\pi]) \right\}.$$

Prove that u_ε converges strongly (the whole sequence, not up to subsequences!!) in L^1 to a function u_0 as $\varepsilon \to 0$, find this function, and prove that the convergence is actually strong in all the L^p spaces with $p < \infty$. Is it a strong L^∞ convergence ?