VERY SHORT LECTURE NOTES ON CONVEX DUALITY AND REGULARITY IN CALCULUS OF VARIATIONS

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1. Convex duality

Given $H : X \to \mathbb{R} \cup \{+\infty\}$, let us recall the definition of H^*, Legendre transform of H, defined on the dual space X'

$$H^*(w) = \sup_{v \in X} \langle w, v\rangle - H(v).$$

We will use reflexive spaces for simplicitiy. In this case it is clear that H^{**} is also defined on X. We recall the important result stating that, if $H : X \to \mathbb{R}$ is convex and l.s.c.,then $H^{**} = H$. For the main facts about conjugate convex functions, see for instance [9].

Finally, we also recall the formula for the Legendre transform of $H(v) = \frac{1}{q}|v|^q$, defined on $\mathbb{R}^d$, where we get $H^*(w) = \frac{1}{p}|w|^q$, where q is the conjugate exponent of p, i.e. $q = p' = p/(p-1)$, wharacterized by $\frac{1}{p} + \frac{1}{q} = 1$.

Now, consider a function $H : \Omega \times \mathbb{R}^d \to \mathbb{R}$ which is convex in the second variable

$$(Hyp1) \quad \text{for every } x \quad v \mapsto H(x, v) \text{ is convex}$$

and satisfying the following uniform bounds:

$$(Hyp2) \quad \frac{c_0}{q}|v|^q - h_0(x) \leq H(x, v) \leq \frac{c_1}{q}|v|^q + h_1(x),$$

where h_0, h_1 are L^1 functions on Ω and $c_0, c_1 > 0$ are given finite constants, and $p \in]1, +\infty[$ is a given exponent. For functions of this form, when we write $H^*(x, w)$ we mean the Legendre transform in the second variable, i.e. $H^*(x, w) = \sup_v w \cdot v - H(x, v)$.

These notes will be devoted to the study of duality in calculus of variations problems, and to its applications to the regularity of the minimizers. The starting problem that we will consider will be of the form

$$\min \left\{ \int_\Omega H(x, v(x))dx \; : \; \nabla \cdot v = f \right\}.$$

Before giving rigorous results, we want to show how to build its dual problem, with an informal derivation. This can be done in the following way: the constraint $\nabla \cdot v = f$ can be written, in weak form, as $-\int v \cdot \nabla u = \int fu$ for every u (let us be sloppy about the regularity of the test functions now). This means that we can rewrite the above problem in the min-max form

$$\min \left\{ \int H(x, v) + \sup_u - \int fu - \int v \cdot \nabla u \right\},$$

since the last sup is 0 is the constraint is satisfied and $+\infty$ if not. Now, we have a min-max problem and the dual problem can be obtained (without claiming any relation with the original problem, or equality of the two values) just by inverting inf and sup. In this case we get

$$\sup\left\{-\int fu+\inf_v\int H(x,v)-\int v\cdot\nabla u\right\}.$$

Since $\inf_v\int H(x,v)-\int v\cdot\nabla u=-\sup_v\int\nabla u\cdot v-\int H(x,v)=\int H^*(x,\nabla u)$, the problem becomes

$$\sup\left\{-\int fu-\int H^*(x,\nabla u)\right\}.$$

In the following, we will see precise statements about the duality between the two problems, and also provide a variant for the case of Dirichlet conditions. The duality proof, based on the above conve analysis tools, is essentially inspired to the method used in [6] . Other proofs are obviously possible.

1.1. **Neumann boundary conditions.** We define the space $W^{1,p}_\diamond(\Omega)$ as the vector subspace of $W^{1,p}(\Omega)$ composed by functions with zero mean and the space $(W^{1,p})'_\diamond(\Omega)$ as the subspace of the dual of $W^{1,p}$ composed by those f such that $\langle f,1\rangle=0$ (i.e. those f with zero mean as well).

Note that for every $v\in L^q(\Omega;\mathbb{R}^d)$, the distribution $\nabla\cdot v$, defined through

$$\langle\nabla\cdot v,\phi\rangle:=-\int_\Omega v\cdot\nabla\phi$$

belongs naturally to $(W^{1,p})'_\diamond(\Omega)$. This will be by the way the definition we will use of the divergence operator (in weak form), and it includes a natural Neumann boundary condition on $\partial\Omega$. However, consider that we will often use Ω to be the torus, which gets rid of many boundary issues.

We will prove the following duality result.

Theorem 1.1. *Suppose that Ω is smooth enough and that H satisfies Hyp1 and Hyp2. Then, for any $f\in(W^{1,p})'_\diamond(\Omega)$, we have*

$$\min\left\{\int_\Omega H(x,v(x))dx\;:\;v\in L^q(\Omega;\mathbb{R}^d),\nabla\cdot v=f\right\}$$
$$=\max\left\{-\int_\Omega H^*(x,\nabla u(x))dx-\langle f,u\rangle\;:\;u\in W^{1,p}(\Omega)\right\}$$

Proof. We will define a function $\mathcal{F}:(W^{1,p})'\to\mathbb{R}$ in the following way

$$\mathcal{F}(p):=\min\left\{\int_\Omega H(x,v(x))dx\;:\;v\in L^q(\Omega;\mathbb{R}^d),\nabla\cdot v=f+p\right\}.$$

Note that if $p\notin(W^{1,p})'_\diamond\subset(W^{1,p})'$, then $\mathcal{F}(p)=+\infty$, as there is no $v\in L^q$ with $\nabla\cdot v=f+p$. On the other hand, if $p\in(W^{1,p})'_\diamond$, then $\mathcal{F}(p)$ is well-defined and real-valued since $\int_\Omega H(x,v(x))dx$ is comparable to the L^q norm, and we use the following fact: for every $f\in(W^{1,p})'_\diamond$ there exists $v\in L^q$ such that $f=\nabla\cdot v$ and $||v||_{L^q}\le||f||_{(W^{1,p})'_\diamond}$ (see next lemma).

We now compute $\mathcal{F}^*: W^{1,p} \to \mathbb{R}$:

$$\begin{aligned}
\mathcal{F}^*(u) &= \sup_p \langle p, u\rangle - \mathcal{F}(p) \\
&= \sup_{p,v\,:\,\nabla\cdot v = f+p} \langle p, u\rangle - \int_\Omega H(x, v(x))dx \\
&= \sup_{p,v\,:\,\nabla\cdot v = f+p} \langle p+f, u\rangle - \langle f, u\rangle - \int_\Omega H(x, v(x))dx \\
&= \sup_v -\langle f, u\rangle - \int_\Omega H(x, v(x))dx - \int (v\cdot\nabla u)dx \\
&= -\langle f, u\rangle + \int_\Omega H^*(x, -\nabla u(x))dx.
\end{aligned}$$

Now we want to use the fact that $\mathcal{F}^{**}(0) = \sup -\mathcal{F}^*$. Note that $\sup -\mathcal{F}^* = +\infty$ if $f \notin (W^{1,p})'_\diamond$, as it is possible to add an arbitrary constant to u, without changing the gradient term, and letting the term $-\langle f, u\rangle$ tend to $-\infty$. On the other hand, if $f \in (W^{1,p})'_\diamond$, then in the above optimization u can be taken in $W^{1,p}$ or in $W^{1,p}_\diamond$ and the result does not change, as adding a constant does not change neither the integral term (which only depends on ∇u) nor the duality term (as $\langle f, 1\rangle = 0$).

By taking the sup on $-u$ instead of u we also have

$$\mathcal{F}^{**}(0) = \sup_u \ -\langle f, u\rangle - \int_\Omega H^*(x, \nabla u(x))dx = -\inf_u \langle f, u\rangle + \int_\Omega H^*(x, \nabla u(x))dx.$$

Finally, if we prove that $\mathcal{F}$ is convex and l.s.c., then we also have $\mathcal{F}^{**}(0) = \mathcal{F}(0)$, which gives the thesis.

Convexity of $\mathcal{F}$ is easy. We just need to take $p_0, p_1 \in (W^{1,p})'_\diamond(\Omega)$ and define $p_t := (1-t)p_0 + tp_1$. Let v_0, v_1 be optimal in the definition of $\mathcal{F}(p_0)$ and $\mathcal{F}(p_1)$, i.e. $\int H(x, v_i(x))dx = \mathcal{F}(p_i)$ and $\nabla\cdot v_i = f + p_i$. Let $v_t := (1-t)v_0 + tv_1$. Of course we have $\nabla\cdot v_t = f + p_t$ and, by convexity of $H(x,\cdot)$ we have

$$\mathcal{F}(p_t) \le \int H(x, v_t(x))dx \le (1-t)\int H(x, v_0(x))dx + t\int H(x, v_1(x))dx \le (1-t)\mathcal{F}(p_0) + t\mathcal{F}(p_1),$$

and the convexity is proven.

For the semicontinuity, we take a sequence $p_n \to p$ in $(W^{1,p})'$. We can suppose that $\mathcal{F}(p_n) < +\infty$ otherwise there is nothing to prove, hence $p_n \in (W^{1,p})'_\diamond(\Omega)$. Take the corresponding optimal vector fields $v_n \in L^q$, i.e. $\int H(x, v_n(x))dx = \mathcal{F}(p_n)$. We can extract a subsequence such that $\lim_k \mathcal{F}(p_{n_k}) = \liminf_n \mathcal{F}(p_n)$. Moreover, from the bound on H we can see that the L^q norm of v_n is bounded in terms of the values of $\mathcal{F}(p_n)$, which are (use Lemma 1.2) bounded by the $(W^{1,p})'_\diamond$ norms of p_n. Since p_n converges, then we get a bound on $||v_n||_{L^q}$. Hence, up to an extra subsequence extraction, we can assume $v_{n_k} \rightharpoonup v$. Obviously we have $\nabla\cdot v = f + p$ and, by semicontinuity of the integral functional $v \mapsto \int H(x, v)dx$, we get

$$\mathcal{F}(p) \le \int H(x, v(x))dx \le \liminf_k \int H(x, v_{n_k}(x))dx = \lim_k \mathcal{F}(p_{n_k}) = \liminf_n \mathcal{F}(p_n),$$

which gives the desired result. □

The duality result that we proved will be used in the rest of these notes written in the following form

$$\min\{A(v)\} + \min\{B(u)\} = 0, \tag{1.1}$$

where A is defined on $L^q(\Omega;\mathbb{R}^d)$ and B on $W^{1,p}(\Omega)$ via

$$A(v) := \begin{cases} \int_\Omega H(x, v(x))dx & \text{if } \nabla\cdot v = f, \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$B(u) = \int_\Omega H^*(x, \nabla u(x))dx + \langle f, u\rangle.$$

Lemma 1.2. *Given $f \in (W^{1,p})'_\diamond(\Omega)$ there exists $v \in L^q(\Omega;\mathbb{R}^d)$ such that $f = \nabla\cdot v$ and $||v||_{L^q} \leq C||f||_{(W^{1,p})'}$.*

Proof. Consider the minimization problem

$$\min\left\{\frac{1}{p}\int_\Omega |\nabla\phi|^p dx + \langle f, \phi\rangle \ : \ \phi \in W^{1,p}(\Omega)\right\}.$$

It is easy to prove that this problem admits a solution, as the minimization can be restricted to the set $W^{1,p}_\diamond$. This solution ϕ satisfies

$$-\int_\Omega (\nabla\phi)^{p-1}\cdot\nabla\psi = \langle f, \psi\rangle$$

for all $\psi \in W^{1,p}(\Omega)$ (pay attention to the notation: for every vector v we denote by w^α the vector with modulus equal to $|w|^\alpha$, and same direction as w, i.e. $w^\alpha := |w|^{\alpha-1}w$). This exactly means $\nabla\cdot v = f$ for $v = (\nabla\phi)^{p-1}$. Moreover, testing against ϕ, we get

$$\begin{aligned} ||v||^q_{L^q} = \int_\Omega |v|^q = \int_\Omega |\nabla\phi|^p = \langle f, \phi\rangle \leq ||f||_{(W^{1,p})'}||\phi||_{W^{1,p}} \\ \leq C||f||_{(W^{1,p})'}||\nabla\phi||_{L^p} = C||f||_{(W^{1,p})'}||v||^{q-1}_{L^q}, \end{aligned}$$

which gives the desired bound on $||v||_{L^q}$. □

1.2. **Dirichlet boundary conditions.** We also want to provide a variant of Theorem 1.1 in the case where the values of u are prescribed on $\partial\Omega$. In this case we besides the space $W^{1,p}$ and its dual $(W^{1,p})'$, we also need to consider the space $X(\partial\Omega)$ defined as those elements π of $(W^{1,p})'$ such that $\langle\rho, u\rangle = 0$ for all $u \in W^{1,p}_0(\Omega)$ (in practice, these are the elements of $(W^{1,p})'$ which are concentrated on the boundary $\partial\Omega$).

We first note the following fact: for every $f \in (W^{1,p})'$ there exists $\pi \in X(\partial\Omega)$ such that

$$f + \pi \in (W^{1,p})'_\diamond, \ ||\pi||_{(W^{1,p})'} \leq C|f||_{(W^{1,p})'}. \tag{1.2}$$

This can be done either explicitly, by taking

$$\langle\pi, \phi\rangle := -\frac{\int_{\partial\Omega}\phi d\mathcal{H}^{d-1}}{\mathcal{H}^{d-1}(\partial\Omega)}\langle f, 1\rangle,$$

or by using the Hahn-Banach Theorem (see for instance the first chapter in [9]) in the following way: there exists an element $\pi \in (W^{1,p})'$ with the following properties $\langle\pi, \phi\rangle = 0$ for every

$\phi \in W_0^{1,p}$, $\langle \pi, 1\rangle = -\langle f, 1\rangle$ and $||\pi||_{(W^{1,p})'} \leq |\langle f, 1\rangle|/||1||_{W^{1,p}}$ (the only condition which is necessary to do this is $1 \notin W_0^{1,p}$, which is a very mild assumption on Ω).

Theorem 1.3. *Suppose that Ω is smooth enough and that H satisfies Hyp1 and Hyp2. Then, for any $f \in (W^{1,p})'(\Omega)$ and $\bar{u} \in W^{1,p}(\Omega)$, we have*

$$\min\left\{\int_\Omega H(x, v(x))dx + \langle \pi, \bar{u}\rangle \ : \ v \in L^q(\Omega; \mathbb{R}^d), \pi \in X(\partial\Omega), \nabla\cdot v = f + \pi\right\}$$
$$= \max\left\{-\int_\Omega H^*(x, \nabla u(x))dx - \langle f, u\rangle \ : \ u \in W^{1,p}(\Omega), u - \bar{u} \in W_0^{1,p}(\Omega)\right\}$$

Proof. The proof will be very similar to that of Theorem 1.1. We define

$$\mathcal{F}(p) := \min\left\{\int_\Omega H(x, v(x))dx + \langle \pi, \bar{u}\rangle \ : \ v \in L^q(\Omega; \mathbb{R}^d), \pi \in X(\partial\Omega), \nabla\cdot v = f + p + \pi\right\}.$$

We now compute $\mathcal{F}^* : W^{1,p} \to \mathbb{R}$:

$$\begin{aligned}
\mathcal{F}^*(u) &= \sup_p \langle p, u\rangle - \mathcal{F}(p)\\
&= \sup_{p,v,\pi\,:\,\nabla\cdot v = f+p+\pi} \langle p, u\rangle - \int_\Omega H(x, v(x))dx - \langle \pi, \bar{u}\rangle\\
&= \sup_{p,v,\pi\,:\,\nabla\cdot v = f+p} \langle p + f + \pi, u\rangle - \langle f, u\rangle - \int_\Omega H(x, v(x))dx - \langle \pi, u + \bar{u}\rangle\\
&= \sup_{v,\pi} -\langle f, u\rangle - \int_\Omega H(x, v(x))dx - \int (v\cdot\nabla u)dx - \langle \pi, u + \bar{u}\rangle\\
&= \sup_{\pi} -\langle f, u\rangle + \int_\Omega H^*(x, -\nabla u(x))dx - \langle \pi, u + \bar{u}\rangle\\
&= \begin{cases} -\langle f, u\rangle + \int_\Omega H^*(x, -\nabla u(x))dx & \text{if } u + \bar{u} \in W_0^{1,p}(\Omega),\\ +\infty & \text{if not.}\end{cases}
\end{aligned}$$

Agains, we will conclude by using $\mathcal{F}^{**}(0) = \sup -\mathcal{F}^*$ and taking the sup on $-u$ instead of u.

We need to prove that $\mathcal{F}$ is convex and l.s.c.

Convexity of $\mathcal{F}$ follows the same scheme as in Theorem 1.1. Take $p_0, p_1 \in (W^{1,p})'(\Omega)$ and define $p_t := (1-t)p_0 + tp_1$. Let (v_0, π_0) and (v_1, π_1) be optimal in the definition of $\mathcal{F}(p_0)$ and $\mathcal{F}(p_1)$ and use $v_t := (1-t)v_0 + tv_1, \pi_t := (1-t)\pi_0 + t\pi_1$.

For the semicontinuity, we take a sequence $p_n \to p$ in $(W^{1,p})'$, with the corresponding optimal (v_n, π_n). We also define $\tilde{\pi}_n$ as the element of $X(\partial\Omega)$ defined by (1.2) and corresponding to $f + p_n$. Then we can use the vector viend $\tilde{v}_n$ provided by Lemma 1.2 corresponding to $f + p_n + \tilde{\pi}_n$ and obtain a bound on $\mathcal{F}(p_n) \leq C||\tilde{v}_n||^p_{L^q} + C + C||\tilde{\pi}_n|| \leq C$.

From this bound we want to deduce bound on v_n and π_n, which would allow to extract converging subsequences and conclude as in Theorem 1.1.

For these bounds, it is enough to observe that we have

$$\langle \pi_n, \bar{u}\rangle = -\langle f + p_n, \bar{u}\rangle - \int v_n\cdot\nabla\bar{u}\,dx \geq -C - C||v_n||_{L^q}$$

and

$$\frac{c_0}{p}||v_n||^q_{L^q} - C \leq \int H(x, v_n(x))dx,$$

which allows to give a bound on $||v_n||_{L^q}$ in terms of $\mathcal{F}(p_n)$. Once we have a bound on v_n, the bound on π_n comes from the constraint $\nabla \cdot v_n = f + p_n + \pi_n$.

The proof can be completed as in Theorem 1.1. □

2. Regularity via duality

In this section we will use the relation (1.1) to produce Sobolev regularity results for solutions of the minimization problems $\min A$ or $\min B$.

We will start by describing the general strategy. We consider a function H not explicitly depending on x, and we suppose that an inequality of the following form is true

$$(Hyp3) \quad H(v) + H^*(w) \geq v \cdot w + c|F(v) - G(w)|^2$$

for some given functions $F, G : \mathbb{R}^d \to \mathbb{R}^d$. This is an improvement of the Young inequality $H(v) + H^*(w) \geq v \cdot w$(which is just a consequence of the definition of H^*). Of course this is always true taking $F = G = 0$, but the interesting cases are the ones where F and G are non-trivial.

To simplify the computations, we will suppose that Ω is the flat d-dimensional torus $\mathbb{T}^d$ (and we will omit the indication of the domain). We start from the following observations, tha we collect in a lemma. For the sake of the notations, we call v_* and u_* the minimizers (or some minimizers, in case there is no uniqueness) of A and B, respectively, and we denote by u_h the function $u_h(x) : u(x + h)$. We define a function $g : \mathbb{R}^d \to \mathbb{R}$ given by

$$g(h) := \int f(x)u_*(x + h)dx - \int f(x)u_*(x)dx.$$

Lemma 2.1. *Suppose H satisfies $Hyp1, 2, 3$ and let v_* and u_* be optimal. Then*

(1) *$F(v_*) = G(\nabla u_*)$.*
(2) *$c \int |G(\nabla u_h) - G(\nabla u_*)|^2 dx \leq g(h)$.*
(3) *If $g(h) = O(|h|^2)$, then $G(\nabla u_*) \in H^1$.*
(4) *If g is $C^{1,1}$, then $g(h) = O(|h|^2)$ and $G(\nabla u_*) \in H^1$.*
(5) *If $f \in W^{1,p}_\diamond(\Omega)$, then $g \in C^{1,1}$ and hence $G(\nabla u_*) \in H^1$*

Proof. First, we compute for arbitrary v and u admissible in the primal and dual problems (i.e. we need $\nabla \cdot v = f$), the sum $A(v) + B(u)$:

$$A(v)+B(u) = \int (H(v)+H^*(\nabla u)+fu)dx = \int (H(v)+H^*(\nabla u)-v\cdot\nabla u)dx \geq c \int |F(v)-G(\nabla u)|^2 dx.$$

If we take $v : v_*$ and $u = u_*$, then $A(v) = \min A$, $B(u) = \min B$ and $A(v) + B(u) = 0$. Hence, we deduce $F(v_*) = G(\nabla u_*)$, i.e. the Part (1) in the statement.

Now, let us fix $v = v_*$ but $u = u_h$. We obtain

$$c \int |G(\nabla u_*) - G(\nabla u_h)|^2 dx = c \int |F(v_*) - G(\nabla u_h)|^2 dx \leq A(v_*) + B(u_h) = B(u_h) - B(u_*).$$

In computing $B(u_h) - B(u_*)$, we see that the terms $\int H^*(\nabla u_h)$ and $\int H^*(\nabla u_*)$ are equal, as one can see from an easy change-of-variable $x \mapsto x + h$. Hence,

$$B(u_h) - B(u_*) = \int f u_h - \int f u_* = g(h),$$

which gives part (2).

Part (3) of the statement is an easy consequence of classical characterization of Sobolev spaces. Part (4) comes from the optimality of u_*, which means that $g(0) = 0$ and $g(h) \geq 0$ for all h. This implies, as soon as $g \in C^{1,1}$, $\nabla g(0) = 0$ and $g(h) = O(|h|^2)$.

For Part (5), we first differentiate $g(h)$, thus getting

$$\nabla g(h) = \int f(x)\nabla u_*(x+h)dx.$$

If we want to differentiate once more, we use the regularity assumption on f: we write

$$\int f(x)\nabla u_*(x+h)dx = \int f(x-h)\nabla u_*(x)dx$$

and then

$$D^2 g(h) = \int \nabla f(x-h) \otimes \nabla u_*(x)dx,$$

which also gives $|D^2 g| \leq ||\nabla f||_{L^q}||\nabla u_*||_{L^p}$. Note that u_* naturally belongs to $W^{1,p}$, hence the integral is finite and bounded, and $g \in C^{1,1}$. □

Unfortunately, the last assumption ($f \in W^{1,q}$) is quite restrictive, but we want to provide a case where it is reasonable to use it. Before, we find interesting cases of functions H and H^* for which we can provide non-trivial functions F and G.

2.1. **Pointwise vector inequalities.** The first interesting case is the quadratic case. Take $H(v) = \frac{1}{2}|v|^2$ with $H^*(w) = \frac{1}{2}|w|^2$. In this case we have easily

$$H(v) + H^*(w) = \frac{1}{2}|v|^2 + \frac{1}{2}|w|^2 = v \cdot w + \frac{1}{2}|v - w|^2,$$

hnece one can take $F(v) = v$ and $G(w) = w$.

Then, we pass to another interesting case, the case of the powers. Take $H(v) = \frac{1}{q}|v|^q$ with $H^*(w) = \frac{1}{p}|w|^p$. We claim that in this case we can take $F(v) = v^{q/2}$ and $G(w) = w^{p/2}$ (remember the notation for powers of vectors).

Lemma 2.2. *For any $v, w \in \mathbb{R}^d$ we have*

$$\frac{1}{p}|v|^p + \frac{1}{q}|w|^q \geq v \cdot w + \frac{1}{2\max\{p,q\}}|v^{p/2} - w^{q/2}|^2.$$

Proof. First we write $a = v^{p/2}$ and $b = w^{q/2}$ and we express the inequality in terms of a, b. Hence we try to prove $\frac{1}{p}|a|^2 + \frac{1}{q}|b|^2 \geq a^{2/p} \cdot b^{2/q} + \frac{1}{2\max\{p,q\}}|a - b|^2$. In this way the inequality is more homogeneous, as it is of order 2 in all its terms (remember $1/p + 1/q = 1$). Then we notice that we can also write the expression in terms of $|a|, |b|$ and $\cos\theta$, where θ is the angle between a and b (which is the same as the one between $v = a^{2/p}$ and $w = b^{2/q}$. Hence, we want to prove

$$\frac{1}{p}|a|^2 + \frac{1}{q}|b|^2 \geq \cos\theta\left(|a|^{2/p}|b|^{2/q} - \frac{1}{\max\{p,q\}}|a||b|\right) + \frac{1}{2\max\{p,q\}}(|a|^2 + |b|^2).$$

since this depends linearly in $\cos\theta$, it is enough to prove the inequality in the two limit cases $\cos\theta = \pm 1$.

For simplicity, ue to the simmetry in p and q of the claim, we suppose $p \leq 2 \leq q$. We start from the case $\cos\theta = 1$, i.e. $b = ta$, with $t \geq 0$ (the case $a = 0$ is trivial). In this case the l.h.s. of the inequality becomes

$$|a|^2(\frac{1}{p} + \frac{1}{q}t^2) = |a|^2(\frac{1}{p} + \frac{1}{q}(1+(t-1))^2) = |a|^2(1 + \frac{2}{q}(t-1) + \frac{1}{q}(t-1)^2) \geq |a|^2(t^{2/q} + \frac{1}{q}(t-1)^2),$$

where we used the concavity of $t \mapsto t^{2/q}$, which provides $1 + \frac{2}{q}(t-1) \geq t^{2/q}$. This inequality is even stronger than the one we wanted to prove, as we get a factor $1/q$ instead of $1/(2q)$ in the r.h.s..

The factor $1/(2q)$ appears in the case $\cos\theta = -1$, i.e. $b = -ta$, $t \geq 0$ (we do not claim that this coefficient is optimal, anyway). In this case we start from the r.h.s.

$$|a|^2(\frac{1}{2q}(1+t)^2 - t^{2/q}) \leq |a|^2\frac{1}{2q}(1+t)^2 \leq |a|^2\frac{2}{2q}(1+t^2) \leq |a|^2(\frac{1}{p} + \frac{1}{q}t^2),$$

which gives the claim. □

Remark 2.1. *The above inequality replaces, in this duality-based approach, the usual vector inequality that PDE methods require to handle equations involving Δ_p, i.e.*

$$(w_0^{p-1} - w_1^{p-1}) \cdot (w_0 - w_1) \geq c|w_0^{p/2} - w_1^{p/2}|^2,$$

which is an improved version of the monotonicity of the gradient of $w \mapsto \frac{1}{p}|w|^p$. Note on the other hand that the proofs of this other inequality are quite logn, usually dealing with integration over suitable segments and refined change-of-variables. See for instance [15]. Here the inequality we need is proven in half a page.

2.2. **Very degenerate PDEs.** Consider for instance the case $H(v) = |v| + \frac{1}{q}|v|^q$. In this case, we can use $F(v) = v^{q/2}$ and $G(w) = (w-1)_+^{p/2}$ (again, we use this weird notation: the vector $(w-1)_+^{p/2}$ is the vectorn with norm equal to $(|w|-1)_+^{p/2}$ and same direction as w, i.e. $G(w) = (|w|-1)_+^{p/2} w/|w|)$.

Indeed, we have

$$H^*(w) = \sup_v \; v \cdot w - |v| - \frac{1}{q}|v|^q = \frac{1}{p}(|w|-1)_+^p$$

and

$$H(v) + H^*(w) = |v| + \frac{1}{q}|v|^q + \frac{1}{p}(|w|-1)_+^p \geq |v| + v \cdot (w-1)_+ + c|v^{q/2} - (w-1)_+^{p/2}|^2.$$

We only need to prove $|v| + v \cdot (w-1)_+ \geq v \cdot w$. This can be done by writing

$$|v| + v \cdot (w-1)_+ = |v|(1 + (|w|-1)_+ \cos\theta).$$

If $|w| \geq 1$ then we go on with

$$|v|(1 + (|w|-1)_+ \cos\theta) \geq |v|\cos\theta(1 + (|w|-1)_+) = |v|\cos\theta|w| = v \cdot w.$$

If $|w| \leq 1$ then we simply use

$$|v|(1 + (|w|-1)_+ \cos\theta) = |v| \geq v \cdot w.$$

Has a consequence we get the following result.

Proposition 2.3. *Let H be given by $H(v) = |v| + \frac{1}{q}|v|^q$ and $H^*(w) = \frac{1}{p}(|w|-1)_+^p$. Suppose that Ω is the flat torus and $f \in W^{1,q}(\Omega)$. Let v_* is a solution of $\min A$ and u_* a solution of $\min B$ (equivalently, suppose that u_* solves $\nabla\cdot((\nabla u_*-1)_+^{p-1}) = f$). Then $v_*^{q/2} = (\nabla u_* - 1)_+)^{p/2} \in H^1$.*

This result is the same proven in [7], where it was proven with PDE methods, and does not seem easy to improve. The equation $\nabla\cdot((\nabla u - 1)_+^{p-1}) = f$, which can be written,

$$\nabla\cdot(((|\nabla u|-1)_+^{p-1}\frac{\nabla u}{|\nabla u|}) = f,$$

is very degenerate in the sense that the coefficient $(|\nabla u|-1)_+^{p-1}/|\nabla u|$ vanishes on the whole set where $|\nabla u| \leq 1$.

This equation and these minimization problems arise in traffic congestion (see [3, 11, 7]) and the choice of the function H is very natural: we need a superlinear function of the form $H(v) = |v|h(|v|)$, with $h \geq 1$). This automatically implies the degeneracy.

2.3. **The Laplacian case: $\Delta u = f$.** The case of the Poisson equation $\Delta u = f$, corresponding to the minimization of $\int \frac{1}{2}|\nabla u|^2 + fu$, and hence to $H(v) = \frac{1}{2}|v|^2$ and $H^*(w) = \frac{1}{2}|w|^2$, deserves special attention. It is possible to treat this case by the same techniques as in the degenerate case above, but the result is disappointing. Indeed, from these techniques we just obtain $f \in H^1 \Rightarrow \nabla u \in H^1$, while it is well-known that $f \in L^2$ should be enough for the same result. Yet, with some more attention it is also possible to treat the L^2 case.

Proposition 2.4. *Suppose that Ω is the flat torus and $\Delta u_* = f \in L^2(\Omega)$. Then $\nabla u_* \in H^1$.*

Proof. We use the variational framework we presented before, with $H(v) = \frac{1}{2}|v|^2$. We have

$$\frac{1}{2}||\nabla u_h - \nabla u_*||_{L^2}^2 \leq g(h). \tag{2.1}$$

Now, set $\omega_t := \sup\{||\nabla u_h - \nabla u||_{L^2} \ : \ |h| \leq t\}$. From (2.1) we have

$$\omega_t^2 \leq \sup_{h:|h|\leq t} 2g(h) \leq 2t \sup_{h:|h|\leq t} |\nabla g(h)|.$$

From $\nabla g(h)\nabla g(h) - \nabla g(0) = \int f(\nabla u_h - \nabla u_*)$ we deduce $|\nabla g(h)| \leq ||f||_{L^2}||\nabla u_h - \nabla u_*||_{L^2} \leq ||f||_{L^2}\omega_t$, hence $\omega_t^2 \leq 2t||f||_{L^2}\omega_t$, which implies $\omega_t \leq 2t||f||_{L^2}$ and hence $\nabla u_* \in H^1$. □

2.4. **The $p-$Laplacian case: $\Delta_p u = f$.** If we look at the case $H(v) = \frac{1}{q}|v|^q$, we have $H^*(w) = \frac{1}{p}|w|^p$ and the solutions of $\Delta_p u = f$ (where $\Delta_p u := \nabla\cdot(\nabla u^{p-1})$) are the minimizers of $\int \frac{1}{p}|\nabla u|^p + fu$. Classical references on the p-Laplacian regularity question are, for instance, [5, 12].

From the consideration of the previous sections we easily obtain the following.

Proposition 2.5. *Suppose that Ω is the flat torus and $\Delta_p u_* = f \in W^{1,q}(\Omega)$. Then $(\nabla u_*)^{p/2} \in H^1$.*

This result is quite classical (see for instance [15]). Yet, it is not very satisfactory, since if we set $p = q = 2$ we get the result $\Delta u \in H^1 \Rightarrow \nabla u \in H^1$ which, as we said, is very disappointing.

This is why we also look at the following other classical result. We recall before stating it some useful definitions of fractional Sobolev spaces (see, for instance, [1]).

Definition 2.1. *When Ω is bounded and its diameter is R, if $1 < p < +\infty$ and $0 < s < 1$, the space $W^{s,p}(\Omega)$ is defined as*

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) \,:\, [u]^p_{s,p} := \int_{B(0,R)} \frac{||u_\delta - u||^p_{L^p}}{\delta^{d+sp}} d\delta < +\infty \right\}$$

and its norm is given by $||u||_{L^p} + [u]_{s,p}$. The space H^s is defined as $W^{s,2}$.

Note that an inequality of the form $||u_\delta - u||_{L^p} \le C|\delta|^s$ implies $u \in W^{s',p}$ for every $s' < s$. Also note that the Hilbert case $p = 2$ also enjoys an alternative definition in terms of the Fourier transform. indeed, we have $u \in H^s$ if and only if $\xi \mapsto |\xi|^s \hat{u}(\xi) \in L^2$ and $[u]_{s,2}$ is equivalent to the L^2 norm of $\xi \mapsto |\xi|^s \hat{u}(\xi)$.

Proposition 2.6. *Suppose that Ω is the flat torus and $\Delta_p u_* = f \in L^q(\Omega)$, with $p > 2$.Then $||(\nabla u_h)^{p/2} - (\nabla u_*)^{p/2}||_{L^2} \le C|h|^{q/2}$, which implies in particular $(\nabla u_*)^{p/2} \in H^s$ for $s < q/2 < 1$.*

Proof. We use the same strategy as in Proposition 2.4. For simplicity, we set $G := (\nabla u)^{p/2}$. As in Proposition 2.4, we set $\omega_t := \sup_{h:|h|\le t} ||G_h - G||_{L^2}$. We have $||G_h - G||^2_{L^2} \le Cg(h)$, which implies

$$\omega_t^2 \le Ct \sup_{h:|h|\le t} |\nabla g(h) - \nabla g(0)| \le Ct||f||_{L^q} \sup_{h:|h|\le t} ||\nabla u_h - \nabla u_*||_{L^p}.$$

From the α-Hölder behaviour of the vector map $w \mapsto w^\alpha$ in $\mathbb{R}^d$ (see Lemma 2.7 below), with $\alpha = 2/p < 1$, we deduce, using $\nabla u = G^\alpha$,

$$||\nabla u_h - \nabla u_*||^p_{L^p} = \int |\nabla u_h - \nabla u_*|^p dx \le \int |G_h - G|^2 dx = ||G_h - G||^2_{L^2}.$$

Hence, we have

$$\omega_t^2 \le Ct||f||_{L^q} \omega_t^{2/p},$$

which implies

$$\omega_t^{2/q} \le Ct||f||_{L^q},$$

i.e. the claim □

Lemma 2.7. *For $0 < \alpha < 1$, the map $w \mapsto w^\alpha$ is α-Hölder continuous in $\mathbb{R}^d$.*

Proof. Let $a, b \in \mathbb{R}^d$. We write

$$|a^\alpha - b^\alpha| = \left| |a|^\alpha \frac{a}{|a|} - |a|^\alpha \frac{b}{|b|} + |a|^\alpha \frac{b}{|b|} - |b|^\alpha \frac{b}{|b|} \right| \le |a|^\alpha \left| \frac{a}{|a|} - \frac{b}{|b|} \right| + \left| |a|^\alpha - |b|^\alpha \right|.$$

For the second term in the r.h.s., we use the α-Hölder behaviour of $t \mapsto t^\alpha$ in $\mathbb{R}_+$ and get

$$\left| |a|^\alpha - |b|^\alpha \right| \le \left| |a| - |b| \right|^\alpha \le |a - b|^\alpha.$$

For the first term in the r.h.s., we use the inequality

$$\left| \frac{a}{|a|} - \frac{b}{|b|} \right| = \left| \frac{a}{|a|} - \frac{b}{|a|} + \frac{b}{|a|} - \frac{b}{|b|} \right| \le \frac{|a-b|}{|a|} + |b| \frac{||b| - |a||}{|a||b|} \le 2\frac{|a-b|}{|a|}$$

and get

$$|a|^\alpha \left| \frac{a}{|a|} - \frac{b}{|b|} \right| \le 2|a|^{\alpha-1}|a-b|.$$

If we choose a to be such $|a| \ge |b|$ (which is possible w.l.o.g.), we have $2|a| \ge |a-b|$ and hence $2^{\alpha-1}|a|^{\alpha-1} \le |a-b|^{\alpha-1}$, i.e. $2|a|^{\alpha-1}|a-b| \le 2^{2-\alpha}|a-b|^\alpha$.

Summing up, we have (without pretending that this constant is optimal)

$$|a^\alpha - b^\alpha| \le (2^{2-\alpha}+1)|a-b|^\alpha. \quad \square$$

Remark 2.2. *Note that the result of Proposition 2.6 is also classical, and quite sharp. Indeed, one can informally consider the following example. Take $u(x) \approx |x|^r$ as $x \approx 0$ (and then multiply times a cut-off function out of 0). In this case we have*

$$\nabla u(x) \approx |x|^{r-1}, \quad (\nabla u(x))^{p-1} \approx |x|^{(r-1)(p-1)}, \quad f(x) := \Delta_p u(x) \approx |x|^{(r-1)(p-1)-1}.$$

Hence, $f \in L^q$ if and only if $((r-1)(p-1)-1)q > -d$, i.e. $(r-1)p - q > -d$. On the other hand, the fractional Sobolev regularity can be observed by considering that "differentiating s times" means subtracting s from the exponent, hence

$$(\nabla u(x))^{p/2} \approx |x|^{p(r-1)/2} \Rightarrow (\nabla u)^{p/2} \in H^s \Leftrightarrow |x|^{p(r-1)/2-s} \in L^2 \Leftrightarrow p(r-1) - 2s > -d.$$

If we want this last condition to be true for arbitrary $s < q/2$, then it amounts to $p(r-1)-q > -d$, which is the same condition as above.

2.5. **Variant – Local regularity.** In the previous sections, we only provided global Sobolev regularity results on the torus. This guaranteed that we could do translations without boundary problems, and that by change-of-variable, the term $\int H(\nabla u_h)dx$ did not actually depend on h. We now provide a result concerning local regularity. As the result is local, boundary conditions should not be very important. Yet, as the method stays anyway global, we need to fix them and be precise on the variational problems that we use. We will use Dirichlet boundary conditions.

We will only provide the following result, in the easiest case $p = 2$.

Theorem 2.8. *Let H, H^*, F and G satisfy Hyp1, 2, 3 with $p = 2$. Suppose $f \in H^1$. Suppose also $H^* \in C^{1,1}$ and $G \in C^{0,1}$. Suppose $\nabla \cdot (\nabla H^*(\nabla u_*)) = f$ in Ω. Then, $G(\nabla u_*) \in H^1_{loc}$.*

Proof. The condition $\nabla \cdot \nabla H^*(\nabla u_*) = f$ is equivalent to the fact that u_* is solution of

$$\min \left\{ \int_\Omega H^*(\nabla u)dx + \int fu\, dx \;:\; u \in W^{1,p}(\Omega), u - \bar{u} \in W^{1,p}_0(\Omega) \right\},$$

for $\bar{u} = u_*$ (i.e. u_* minimizes under its own optimality conditions. We will also use the dual problem presented in Theorem 1.3. We set $A(v,\pi) := \int H(v) + \langle \pi, \bar{u} \rangle$ with the constraint $\nabla \cdot v = f + \pi$. As usual, we sum $A(v,\pi) + B(u)$ and we get $A(v,\pi) + B(u) = \int (H(v) + H^*(\nabla u) - v \cdot \nabla u)dx \ge c \int |F(v) - G(\nabla u)|^2 dx$.

The strategy is the same: use the optimal v and π together with a translation of u. Yet, in order not to have boundary problems, we need to use a cut-off function $\eta \in C^\infty_c(\Omega)$ and define

$$u_h(x) = u_*(x + h\eta(x))$$

(note that for small h this does not change the boundary value $\bar{u}$). In this case it is no longer true that $\tilde{g}(h) := \int H^*(\nabla u_h)dx = \int H^*(\nabla u_*)dx$. If this term is not constant in h, then we need to prove that it is a $C^{1,1}$ function of h. to do this, and to avoid differentiating ∇u_*, we use a change-of-variable. Set $y = x + h\eta(x)$. We have $\nabla(u_h)(x) = (\nabla u_*)(y)(I + h \otimes \nabla\eta(x))$, hence

$$\tilde{g}(h) = \int H^*(\nabla u_h)dx = \int H^*(\nabla u_*(y) + (\nabla u_*(y) \cdot h)\nabla\eta(x)) \frac{1}{1 + h \cdot \nabla\eta(x)} dy,$$

where $x = X(h,y)$ is a function of h and y obtained by inverting $x \mapsto x + h\eta(x)$ and we used $\det(I + h \otimes \nabla\eta(x)) = 1 + h \cdot \nabla\eta(x)$. The function X is C^∞ by the implicit function theorem, and all the other ingredient of the above integral are at least $C^{1,1}$ in h. This proves that $\tilde{g}$ is $C^{1,1}$. The regularity of the term $g(h) = \int f u_h$ should also be considered. Differentiating once we get $\nabla g(h) = \int f(x)\nabla u_*(x + h\eta(x))\eta(x)dx$. To differentiate once more, we use the same change-of-variable, thus getting

$$\nabla g(h) = \int f(X(h,y))\nabla u_*(y)\eta(X(h,y))\frac{1}{1 + h \cdot \nabla \eta(x)}dy.$$

From $y = X(h,y) + h\eta(X(h,y))$ we get a formula for $D_h X(h,y)$, i.e. $0 = D_h X(h,y) + \eta(X(h,y))I + h \otimes \nabla\eta(\eta(X(h,y))D_h X(h,y)$. This allows to differentiate once more the function g and proves $g \in C^{1,1}$.

Finally, we come back to the duality estimate. What we can easily get is

$$c||G(\nabla(u_h)) - G(\nabla u_*)||^2_{L^2} \le g(h) + \tilde{g}(h) = O(|h|^2).$$

The problem is that $G(\nabla(u_h))$ is not the translation of $G(\nabla u_*)$! Yet, it is almost true. Indeed, if we put the subscript h every time that we compose with $x + h\eta(x)$, we have

$$\nabla(u_h) = (\nabla u_*)_h + h \cdot (\nabla u_*)_h \eta.$$

Since G is supposed to be Lipschitz continuous, then

$$|G(\nabla(u_h)) - G((\nabla u_*)_h)| \le C|h||\nabla u_*|_h \eta.$$

Hence, we have

$$||G((\nabla u_*)_h) - G(\nabla u_*)||_{L^2} \le ||G(\nabla(u_h)) - G(\nabla u_*)||_{L^2} + C|h|||\nabla u||_{L^2},$$

which is enough to show that this increment si of order $|h|$, since $u_* \in H^1$ (this depends on the fact that H^* is quadratic). Hence, as in Lemma 2.1 (4), we get $G(\nabla u_*) \in H^1$. □

2.6. **Variant – Dependence on x.** The duality theory has been presented in the case where H and H^* could also depend on x, while for the moment regularity results have only be prensented under the assumption that they not. In this section, we will see how to handle the following particular case, corresponding to the minimization problem

$$\min\left\{\frac{1}{p}\int_\Omega a(x)|\nabla u(x)|^p dx + \int f(x)u(x)dx \ : \ u \in W^{1,p}(\Omega)\right\}. \tag{2.2}$$

We will use $\Omega = \mathbb{T}^d$ to avoid cumulating difficulites (boundary issues and dependence on x). Note that the PDE corresponding to the above minimization problem is $\nabla \cdot (a(\nabla u)^{p-1}) = f$.

First, we need to compute the transform of $w \mapsto H^*(w) := \frac{a}{p}|w|^p$. Set $b = a^{1/(p-1)}$. It is easy to obtain $H(v) = \frac{1}{bq}|v|^q$. Also, we can check (just by scaling the inequality of Lemma 2.2, that we have

$$\frac{1}{bq}|v|^q + \frac{b^{p-1}}{p}|w|^p \ge v \cdot w + b^{p-1}\left|w^{p/2} - \frac{v^{q/2}}{b^{p/2}}\right|^2.$$

In particular, if we suppose that $a(x)$ is bounded from below by a positive constant, and we set $H^*(x,w) = \frac{a(x)}{p}|w|^p$ then we get

$$H(x,v) + H^*(x,w) \ge v \cdot w + c|F(x,v) - G(w)|^2$$

where $G(w) = w^{p/2}$.

We can now prove the following theorem.

Theorem 2.9. *Suppose* $f \in W^{1,q}$ *and* $a \in \text{Lip}, a \geq a_0$*, and let* u_* *be the minimizer of* (2.2)*. Then* $G = \nabla u_*^{p/2} \in H^1$*.*

Proof. Our usual computations show that

$$c||G_h - G||_{L^2}^2 \leq g(h) + \tilde{g}(h),$$

where $g(h) = \int f u_h - \int f u_*$ and $\tilde{g}(h) = \int \frac{a(x)}{p}|\nabla u_h|^p - \int \frac{a(x)}{p}|\nabla u_*|^p$. With our assumptions, $g \in C^{1,1}$. As for $\tilde{g}(h)$, we write

$$\int \frac{a(x)}{p}|\nabla u_h|^p = \int \frac{a(x-h)}{p}|\nabla u_*|^p$$

and hence

$$\nabla \tilde{g}(h) = \int \frac{\nabla a(x-h)}{p}|\nabla u_*|^p = \int \frac{\nabla a(x)}{p}|\nabla u_h|^p.$$

Hence,

$$|\nabla \tilde{g}(h) - \nabla \tilde{g}(0)| \leq \int \frac{|\nabla a(x)|}{p}\,||\nabla u_h|^p - |\nabla u_*|^p| \leq C \int ||G_h|^2 - |G|^2| \leq C||G_h - G||_{L^2}||G_h + G||_{L^2}.$$

Here we used the L^∞ bound on $|\nabla a|$. Then, from the lower bound on a, we also know $G \in L^2$, hence we get $|\nabla \tilde{g}(h) - \nabla \tilde{g}(0)| \leq C||G_h - G||_{L^2}$.

Now, we define as usual $\omega_t := \sup_{h=|h|\leq t} ||G_h - G||_{L^2}$ and we get

$$\begin{aligned}\omega_t^2 \leq C \sup_{h=|h|\leq t} g(h) + \tilde{g}(h) \leq Ct \sup_{h=|h|\leq t} |\nabla g(h) + \nabla \tilde{g}(h)| \\ = Ct \sup_{h=|h|\leq t} |\nabla g(h) - \nabla g(0) + \nabla \tilde{g}(h) - \nabla \tilde{g}(0)| \leq Ct^2 + Ct\omega_t,\end{aligned}$$

which allows to deduce $\omega_t \leq Ct$ and hence $G \in H^1$. □

We also provide the following theorem, which is also interesting for $p = 2$.

Theorem 2.10. *Suppose* $p \geq 2$*,* $f \in L^q$ *and* $a \in \text{Lip}, a \geq a_0$*, and let* u_* *be the minimizer of* (2.2)*. Then* $G = \nabla u_*^{p/2}$ *satisfies* $||G_h - G||_{L^2} \leq C|h|^{q/2}$*. In particular,* $G \in H^1$ *for* $p = 2$ *and* $G \in H^s$ *for all* $s < q/2$ *for* $p > 2$*.*

Proof. The only difference with the previous case is that we cannot say that g is $C^{1,1}$ but we should stick to the computation of ∇g. We use as usual

$$|\nabla g(h) - \nabla g(0)| \leq ||f||_{L^q}||\nabla u_h - \nabla u_*||_{L^p}.$$

As we are forced to let the norm $||\nabla u_h - \nabla u_*||_{L^p}$ appear, we will use it also in $\tilde{g}$. Indeed, we can observe that we can estimate

$$\begin{aligned}|\nabla \tilde{g}(h) - \nabla \tilde{g}(0)| \leq \int \frac{|\nabla a(x)|}{p}\,||\nabla u_h|^p - |\nabla u_*|^p| \leq C \int (|\nabla u_h|^{p-1} + |\nabla u_*|^{p-1})|\nabla u_h - \nabla u_*| \\ \leq C||\nabla u_*^{p-1}||_{L^q}||\nabla u_h - \nabla u_*||_{L^p}.\end{aligned}$$

We then use $||\nabla u_*^{p-1}||_{L^q} = ||\nabla u_*||_{L^p}^{p-1}$ and conclude

$$|\nabla \tilde{g}(h) - \nabla \tilde{g}(0)| \leq C||\nabla u_h - \nabla u_*||_{L^p}.$$

This gives, defining ω_t as usual,

$$\omega_t \leq Ct \sup_{h=|h|\leq t} |\nabla g(h) - \nabla g(0) + \nabla \tilde{g}(h) - \nabla \tilde{g}(0)| \leq Ct \sup_{h=|h|\leq t} ||\nabla u_h - \nabla u_*||_{L^p}$$

and hence

$$\omega_t^2 \leq Ct\omega_t^{2/p}$$

as in Proposition 2.6. □

2.7. **Time-dependent problems.** The technique that we saw in this section to prove Sobolev regularity provides in general classical results, through a slighty different point of view than the usual PDE-based tools. Yet, it has the advantage that it requires only the optimality, with no need to write a PDE, and could be useful in some very degenerate cases. The first use (to the best of my knowledge) of duality-based methods to prove regularity was in [8] (later improved by [2]), in the study of variational models for the incompressible Euler Equation. This has been later adapted in [10] to density-constrained mean-field games. In this last section we only want to give an idea of where these estimates could be really useful. Without entering into details, we will see what happens in the case of an easier mean-field game. This takes the following form

Consider the following minimization problem

$$\min \left\{ \mathcal{A}(\rho, v) := \int_0^T\!\!\int_\Omega \left(\frac{1}{2}\rho_t |v_t|^2 + H(\rho_t) \right) + \int_\Omega \Psi \rho_T \right\}$$

among pairs (ρ, v) such that $\partial_t \rho + \nabla \cdot (\rho v) = 0$, with given ρ_0, where G is a given convex function.

Note that this problem is convex in the variables $(\rho, E := \rho v)$ (while it is not convex in (ρ, v)) and it recalls the Benamou-Brenier formulation for optimal transport ([4]). And, in these variables, it exactly corresponds to a probem with constraints on the divergence (indeed, $\partial_t \rho + \nabla \cdot E$ is the space-time divergence of (ρ, E)).

As all convex minimization problem, $\min \mathcal{A}$ admits a dual problem, formally obtained by interchanging inf and sup in

$$\min_{\rho, v} \left\{ \mathcal{A}(\rho, v) + \sup_\phi \int_0^T\!\!\int_\Omega (\rho \partial_t \phi + \nabla \phi \cdot \rho v) + \int_\Omega \phi_0 \rho_0 - \int_\Omega \phi_T \rho_T \right\}.$$

We get

$$\sup \left\{ -\mathcal{B}(\phi, p) := \int_\Omega \phi_0 \rho_0 - \int_0^T\!\!\int_\Omega H^*(p_+) \ : \ \phi_T \leq \Psi, \ -\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 = p \right\},$$

where H^* is the Legendre transform of G (the positive part is due to the constraint $\rho \geq 0$). Note that the problem could be written in terms of ϕ only (as p depends on ϕ), but in this way there is more simmetry with the primal problem, as in both case we have two variables ((ρ, v) or (ϕ, p)), linked by a PDE.

Now, we can do our usual computation taking arbitrary (ρ, v) and (ϕ, p) admissible in the primal and dual problem. Compute

$$\mathcal{A}(\rho, v) + \mathcal{B}(\phi, p) = \int_\Omega (\Psi - \phi_T)\rho_T + \int_0^T\!\!\int_\Omega (H(\rho) + H^*(p_+) - p\rho) + \frac{1}{2}\int_0^T\!\!\int_\Omega \rho |v + \nabla \phi|^2. \tag{2.3}$$

Notice $(H(\rho) + H^*(p_+) - p\rho) \geq \frac{\lambda}{2}|\rho - (H')^{-1}(p_+)|^2$ where $\lambda = \inf H''$. Suppose $\lambda > 0$.

Supposing for simplicity $\Omega = \mathbb{T}^d$ to be the flat torus, using

$$\mathcal{A}(\rho, v) + \mathcal{B}(\phi, p) \geq c \int_0^T \int_\Omega |\rho - (H')^{-1}(p_+)|^2$$

we can deduce, with the same technique as in the rest of the section, $\rho \in H^1$ (we can get both regularity in space and local in time). By the way, using the last term in (2.3), we can also get $\iint \rho |D^2 \phi|^2 < \infty$.

The above computation is important as it gives regularity for ρ, and hence for p, and p appears in the Hamilton-Jacobi equation $-\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 = p$. Indeed, the solution (ρ, v) represents the motion of a population ρ, where each individual follows the velocity $v = -\nabla \phi$. But ϕ is the value function of the control problem

$$\min \left\{ \int_0^T \left(\frac{|x'(t)|^2}{2} + p(t, x(t)) \right) dt + \Psi(x(T)) \right\}.$$

This explains the name mean-field games: we look for a global configuration of motion, where each individual chooses his trajectory by optimizing a criterion where p (and hence ρ) appears, i.e. where the criterion depends, through a sort of mean-field effect, on the choice of the others. The mathematical difficulty is that we need to integrate p over the different trajectories, which requires a little bit of regularity.

The situation is even more complicated when we try to study the case where the density penalization $H(\rho)$ is replaced by the constraint $\rho \leq 1$?

If we look at the variational problem

$$\min \left\{ \int_0^T \int_\Omega \frac{1}{2} \rho_t |v_t|^2 + \int_\Omega \Psi \rho_T \ : \ \rho \leq 1 \right\}$$

we can compute the dual

$$\sup \left\{ \int_\Omega \phi_0 \rho_0 - \int_0^T \int_\Omega p_+ \ : \ \phi_T \leq \Psi, \ -\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = p \right\}.$$

Here p is a pressure arising from the incompressibility constraint $\rho \leq 1$ but finally acts as a price. In order to give a meaning to the above problem we need a bit of regularity. The situation is much trickier, but the same kind of duality arguments, as in [2, 8], allow to get

$$p \in L^2_{loc}((0, T); BV(\Omega)).$$

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