Memo – *Lusin's theorem.*

A well known theorem in measure theory states that every measurable function f on a reasonable measure space X, is actually continuous on a set K which fills almost all the measure of X. This set K can be taken compact. Actually, there can be at least two statements: either we want f to be merely continuous on K (which is easier), or we want f to coincide on K with a continuous function defined on the whole X. This theorem is usually stated for real-valued functions, but we happen to need it for functions valued in more general spaces. Let us give some precise statements and proofs. Take a topological space X endowed with a regular measure μ (i.e. any Borel set $A \subset X$ satisfies $\mu(A) = \sup\{\mu(K) : K \subset A, K \text{ compact}\}$. The arrival space Y will be supposed to be second-countable (i.e. to admit a countable family $(B_i)_i$ of open sets such that any other open set $B \subset Y$ may be expressed as a union of B_i).

Theorem (weak Lusin): If X is a topological measure space endowed with a regular measure μ, if Y is second-countable and $f : X \to Y$ is measurable, then for every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $\mu(X \setminus K) < \varepsilon$ and the restriction of f to K is continuous.

Proof: For every $i \in \mathbb{N}$, set $A_i^+ = f^{-1}(B_i)$ and $A_i^- = f^{-1}(B_i^c)$. Consider compact sets $K_i^+ \subset A_i^+$ such that $\mu(A_i^+ \setminus K_i^+) < \varepsilon 2^{-i}$ and $K_i^- \subset A_i^-$ with $\mu(A_i^- \setminus K_i^-) < \varepsilon 2^{-i}$. Set $K_i = K_i^+ \cup K_i^-$ and $K = \bigcap_i K_i$. For each i we have $\mu(X \setminus K_i) < 2\varepsilon 2^{-i}$. By construction, K is compact and $\mu(X \setminus K) < 4\varepsilon$. To prove that f is continuous on K it is sufficient to check that $f^{-1}(B) \cap K$ is relatively open in K for each open set B, and it is enough to check this for $B = B_i$. Equivalently, it is enough to prove that $f^{-1}(B_i^c) \cap K$ is closed, and this is true since it coincides with $K_i^+ \cap K$.

Theorem (strong Lusin): If X is a metric space endowed with a regular measure μ and $f : X \to \mathbb{R}$ is measurable, then for every $\varepsilon > 0$ there exists a compact set $K \subset X$ and a continuous function $g : X \to \mathbb{R}$ such that $\mu(X \setminus K) < \varepsilon$ and $f = g$ on K.

Proof: First apply weak Lusin's theorem, since $\mathbb{R}$ is second countable. Then we just need to extend $f_{|K}$ to a continuous function g on the whole X. This is possible since $f_{|K}$ is uniformly continuous (as a continuous function on a compact set) and hence has a modulus of continuity ω, such that $|f(x) - f(x')| \le \omega(d(x, x'))$. Then define $g(x) = \inf\{f(x') + \omega(d(x, x')) : x' \in K\}$. It can be easily checked that g is continuous and coincides with f on K.

Notice that this last proof strongly uses the fact that the arrival space is $\mathbb{R}$. It could be adapted to the case of $\mathbb{R}^d$ just by extending componentwise. It is also interesting to remark that if Y is a separable metric space, than it can be embedded into ℓ^∞, and extend componentwise as well (but the extension will not be valued in Y, but in ℓ^∞). On the other hand, it is clear that the strong version of Lusin's Theorem cannot hold (without extension to a bigger space) for any space Y, and the counter-example is easy to build : just take X connected and Y disconnected. A measurable function $f : X \to Y$ taking two different values in two different connected components on two sets of positive measure cannot be approximated by continuous functions in the sense of the strong Lusin's Theorem.

The consequence of all these continuity, semi-continuity, and compactness results is the existence, under very mild assumptions on the cost and the space, of an optimal transport plan γ. Then, if one is interested in the problem of Monge, the question may become "does this minimal γ come from a transport map T?". Actually, if the answer to this question is yes, then it is evident that the problem of Monge has a solution, which also solves a wider problem, that of minimizing among transport plans. This is the object of the next two sections 1.2 and 1.3. On the other hand, in some cases proving that the optimal transport plan comes from a transport map (or proving that there exists at least one optimal plan coming from a map) is equivalent to proving that the problem of Monge has a solution, since very often the infimum among transport plans and among transport maps is the same. This depends on the presence of atoms (see Sections 1.4 and 1.5).

1.2 Duality

Since the problem (PK) is a linear optimization under linear constraints, an important tool will be duality theory, which is typically used for convex problems. We will find a dual problem (PD) for (PK) and exploit the relations between dual and primal.

The first thing we will do is finding a formal dual problem, by means of an inf-sup exchange.

First express the constraint $\gamma \in \Pi(\mu, \nu)$ in the following way : notice that, if γ is a non-negative measure on $X \times Y$, then we have

$$\sup_{\phi,\psi} \int \phi \, d\mu + \int \psi \, d\nu - \int (\phi(x) + \psi(y)) \, d\gamma = \begin{cases} 0 & \text{if } \gamma \in \Pi(\mu,\nu) \\ +\infty & \text{otherwise} \end{cases}.$$

Hence, one can remove the constraints on γ if he adds the previous sup, since if they are satisfied nothing has been added and if they are not one gets $+\infty$ and this will be avoided by the minimization. Hence we may look