Some notes for the course

Convex Duality and Applications in PDEs and Game Theory

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Contents

Chapter 1

Some elements of convex analysis

1.1 Fenchel-Legendre Transform

Let us fix a Banach space X together with its dual X', and denote by $\langle \xi, x\rangle$ the duality between an element $\xi \in X'$ and $x \in X$. More generally, we could fix a pair of normed vector spaces on which we fix a bilinear form which plays the role of the duality between them.

Definition 1.1. We say that a function valued in $\mathbb{R} \cup \{+\infty\}$ is *proper* if it is not identically equal to $+\infty$. The set $\{f < +\infty\}$ is called the *domain* of f.

Definition 1.2. Given a vector space X and its dual X', and a proper function $f : X \to \mathbb{R} \cup \{+\infty\}$ we define its Fenchel-Legendre transform $f^* : X' \to \mathbb{R} \cup \{+\infty\}$ via

$$f^*(\xi) := \sup_x \langle \xi, x\rangle - f(x).$$

Remark 1.3. We observe that we trivially have $f^*(0) = -\inf_X f$.

We note that f^*, as a sup of affine continuous (in the sequel we will just say affine and mean affine and continuous, i.e. of the form $\ell(x) = \langle \xi, x\rangle + c$ for $\xi \in X'$ and $c \in \mathbb{R}$) functions, is both convex and l.s.c., as these two notions are stable by sup.

By abuse of notations, when considering functions defined on X' we will see their Fenchel-Legendre transform as a function defined on X (and not on X'': this is possible since $X \subset X''$ and we can restrict it to X, and by the way in most cases we will use only reflexive spaces).

We prove the following results.

Proposition 1.4.

1. *If $f : X \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and l.s.c. then there exists a continuous affine function ℓ such that $f \geq \ell$.*
2. *If $f : X \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and l.s.c. then it is a sup of continuous affine functions.*

3. *If* $f : X \to \mathbb{R} \cup \{+\infty\}$ *is proper, convex and l.s.c. then there exists* $g : X' \to \mathbb{R} \cup \{+\infty\}$ *such that* $f = g^*$.

4. *If* $f : X \to \mathbb{R} \cup \{+\infty\}$ *is proper, convex and l.s.c. then* $f^{**} = f$.

Proof. We consider the epigraph $\mathrm{Epi}(f) := \{(x,t) \in X \times \mathbb{R} : t \geq f(x)\}$ of f which is a convex and closed set in $X \times \mathbb{R}$. We take a point x_0 such that $f(x_0) < +\infty$ and consider the singleton $\{(x_0, f(x_0) - 1)\}$ which is a convex and compact set in $X \times \mathbb{R}$. The Hahn-Banach separation theorem provides the existence of a pair $(\xi, a) \in X' \times \mathbb{R}$ and a cosntant c such that $\langle \xi, x_0 \rangle + a(f(x_0) - 1) < c$ and $\langle \xi, x \rangle + at > c$ for every $(x,t) \in \mathrm{Epi}(f)$. Note that this last condition implies $a \geq 0$ since we can take $t \to \infty$. Moreover, we should also have $a > 0$ otherwise taking any point $(x,t) \in \mathrm{Epi}(f)$ with $x = x_0$ we have a contradiction. If we then take $t = f(x)$ for all x such that $f(x) < +\infty$ we obtain $af(x) \geq -\langle \xi, x \rangle + \langle \xi, x_0 \rangle + a(f(x_0) - 1)$ and, dividing by $a > 0$, we obtain the first claim.

We now take an arbitrary $x_0 \in X$ and $t_0 < f(x_0)$ and separate again the singleton $\{(x_0, t_0)\}$ from $\mathrm{Epi}(f)$, thus getting a pair $(\xi, a) \in X' \times \mathbb{R}$ and a constant c such that $\langle \xi, x_0 \rangle + at_0 < c$ and $\langle \xi, x \rangle + at > c$ for every $(x,t) \in \mathrm{Epi}(f)$. Again, we have $a \geq 0$. If $f(x_0) < +\infty$ we obtain as before $a > 0$ and the inequality $f(x) > -\frac{\xi}{a} \cdot (x - x_0) + t_0$. We then have an affine function ℓ with $f \geq \ell$ and $\ell(x_0) = x_0$. This shows that the sup of all affine functions smaller than f is, at the point x_0, at least t_0. Hence this sup equals f on $\{f < +\infty\}$. The same argument works for $f(x_0) = +\infty$ if for t_0 arbitrary large the corresponding coefficient a is strictly positive. If not, we have $\langle \xi, x_0 \rangle < 0$ and $\langle \xi, x \rangle \geq 0$ for every x such that $(x,t) \in \mathrm{Epi}(f)$ for at least one $t \in \mathbb{R}$, i.e. for $x \in \{f < +\infty\}$. Consider now $\ell_n = \ell - n\xi$ where ℓ is the affine function smaller than f previously found. We have $f \geq \ell \geq \ell - n\xi$ since ξ is non-negative on $\{f < +\infty\}$ and moreover $\lim_n \ell_n(x_0) = +\infty$. This shows that in such a point x_0 the sup of the affine functions smaller than f equals $+\infty$, and hence $f(x_0)$.

Once that we know that f is a sup of affine functions we can write

$$f(x) = \sup_\alpha \langle \xi_\alpha, x \rangle + c_\alpha$$

for a family of indexes α. We then set $c(\xi) := \sup\{c_\alpha : \xi_\alpha = \xi\}$. The set in the sup can be empty, which means $c(\xi) = -\infty$. Anyway, the sup is always finite: fix a point x_0 with $f(x_0) < +\infty$ and use since $c_\alpha \leq f(x_0) - \langle \xi, x_0 \rangle$. We then define $g = -c$ and we see $f = g^*$.

finally, before proving $f = f^{**}$ we prove that for any function f we have $f \geq f^{**}$ even if f is not convex or lsc. Indeed, we have $f^*(\xi) + f(x) \geq \langle \xi, x \rangle$ which allows to write $f(x) \geq \langle \xi, x \rangle - f^*(\xi)$, an inequality true for every ξ. Taking the sup over ξ we obtain $f \geq f^{**}$. We want now to prove that this inequality is an equality if f is convex and lsc. We write $f = g^*$ and transform this into $f^* = g^{**}$. We then have $f^* \leq g$ and, transforming this inequality (which changes its sign), $f^{**} \geq g^* = f$, which proves $f^{**} = f$. □

Corollary 1.5. *Given an arbitrary proper function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *we have* $f^{**} = \sup\{g : g \leq f,\ g \text{ is convex and lsc}\}$.

Proof. Let us call h the function obtained as a sup on the right hand side. Since f^{**} is convex and lsc and smaller than f, we have $f^{**} \leq h$. Note that h, as a sup of convex and lsc functions, is also convex and lsc, and it is of course smaller than f. We write $f \geq h$ and double transform this inequality, which preserves the sign. We then have $f^{**} \geq h^{**} = h$, and the claim is proven. □

We finally discuss the relations between the behavior at ∞ of a fonction f and its Legendre transform. We give two definitions.

Definition 1.6. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ defined on a normed vector space X is said to be *coercive* if $\lim_{||x|| \to \infty} f(x) = +\infty$; it is said to be *superlinear* if $\lim_{||x|| \to \infty} \frac{f(x)}{||x||} = +\infty$.

We note that the definition of coercive does not include any speed of convergence to ∞, but that for onvex functions this should be at least linear:

Proposition 1.7. *A proper, convex, and l.s.c. function $f : X \to \mathbb{R} \cup \{+\infty\}$ is coercive if and only there exist two constants c_0, c_1 such that $f(x) \geq c_0||x|| - c_1$.*

Proof. We just need to prove that c_0, c_1 exist if f is coercive, the converse being trivial. Take a point x_0 such that $f(x_0) < +\infty$. Using $\lim_{||x|| \to \infty} f(x) = +\infty$ we know that there exists a radius R such that $f(x) \geq f(x_0) + 1$ as soon as $||x - x_0|| \geq R$. By convexity, we have, for each x with $||x - x_0|| > R$, the inequality $f(x) \geq f(x_0) + ||x - x_0||/R$ (it is enough to use the definition of convexity on the three points x_0, x and $x_t = (1-t)x_0 + tx \in \partial B(x_0, R)$). Since f is bounded from below by an affine function, it is bounded from below by a constant on $B(x_0, R)$, so that we can write $f(x) \geq c_2 + ||x - x_0||/R$ for some $c_2 \in \mathbb{R}$ and all $x \in X$. We then use the triangle inequality and obtain the claim with $c_0 = 1/R$ and $c_1 = c_2 - ||x_0||/R$. □

Proposition 1.8. *A proper and convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ is coercive if and only if f^* is bounded in a neighboorhood of* 0*; it is superlinear if and only if f^* is bounded on each bounded ball of X'.*

Proof. We know that f is coercive if and only if there exist two constants c_0, c_1 such that $f \geq g_{c_0,c_1}$ where $g_{c_0,c_1}(x) := c_0||x|| - c_1$. Since both f and g_{c_0,c_1} are convex l.s.c., this inequality is equivalent to the opposite inequality for their transforms, i.e. $f^* \leq g^*_{c_0,c_1}$. We can compute the transform and obtain

$$g^*_{c_0,c_1}(\xi) = \begin{cases} c_1 & \text{if } ||\xi|| \leq c_0^{-1}, \\ +\infty & \text{if not.} \end{cases}$$

This shows that f is coercive if and only if there exist two constants R, C (with $R = c_0^{-1}$, $C = c_1$) such that $f^* \leq C$ on the ball of radius R of X', which is the claim.

We follow a similar procedure for the case of superlinear functions. We first note that a convex l.s.c. function f is superlinear if and only if for every c_0 there exists c_1 such that $f \geq g_{c_0,c_1}$. Indeed, it is clear that, should this condition be satisfied, we would have $\liminf_{||x|| \to \infty} f(x)/||x|| \geq c_0$, and hence $\liminf_{||x|| \to \infty} f(x)/||x|| = +\infty$ because c_0 is arbitrary, so that f would be superlinear. On the other, if f is superlinear, for every

c_0 we have $f(x) \geq c_0||x||$ for large $||x||$, say outside of $B(0,R)$. If we then choose $-c_1 := \min\{\inf_{B(0,R)} f - c_0 R, 0\}$ (a value which is finite since f is bounded from below by an affine function), the inequality $f(x) \geq c_0||x|| - c_1$ is true everywhere.

We then deduce that f is superlinear if and only if for every $R = c_0^{-1}$ there is a constant c_1 such that $f^* \leq c_1$ on the ball of radius R of X', which is, again, the claim. □

1.2 Subdifferentials

The above-the-tangent property of convex functions ispired the definition of an extension of the notion of differential, called sub-differential, as a set-valued map:

Definition 1.9. Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$ we define its subdifferential at x as the set

$$\partial f(x) = \{\xi \in X' : f(y) \geq f(x) + \langle \xi, y - x \rangle \ \forall y \in X\}.$$

We observe that $\partial f(x)$ is always a closed and convex set, whatever is f. Moreover, if f is l.s.c. we easily see that the graph of the subdifferential multi-valued map is closed:

Proposition 1.10. *Suppose that f is l.s.c. and take a sequence $x_n \to x$. Suppose $\xi_n \rightharpoonup \xi$ and $\xi_n \in \partial f(x_n)$. Then $\xi \in \partial f(x)$.*

Proof. for every y we have $f(y) \geq f(x_n) + \langle \xi, y - x_n \rangle$. We can then use the strong convergence of x_n and the weak convergence of ξ_n, together with the lower semicontinuity of f, to pass to the limit and deduce $f(y) \geq f(x) + \langle \xi, y - x \rangle$, i.e. $\xi \in \partial f(x)$. □

Note that in the above proposition we could have exchanged strong convergence for x_n and weak for ξ_n for weak convergence for x_n (but f needed in this case to be weakly l.s.c.) and strong for ξ_n.

When dealing with arbitrary functions f, the subdifferential is in most cases empty, as there is no reason that the inequality defining $\xi \in \partial f(x)$ is satisfied for y very far from x. The situation is completely different when dealing with convex functions, which is the standard case here subdifferentials are defined. In this case we can prove that $\partial f(x)$ is never empty if x lies in the interior of the set $\{f < +\infty\}$ (note that outside $\{f < +\infty\}$ the subdifferential of a proper function is clearly empty).

We first provide an insight about the finite-dimensional case. In this case we simply write $\xi \cdot x$ for the duality product, which coincides with the Euclidean scalar product on $\mathbb{R}^N$.

We start from the following property.

Proposition 1.11. *Given $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ suppose that f is differentiable at a point x_0. Then $\partial f(x_0) \subset \{\nabla f(x_0)\}$. If moreover f is convex, then $\partial f(x_0) = \{\nabla f(x_0)\}$.*

Proof. from the definition of sub-differential we see that $\xi \in \partial f(x_0)$ means that $y \mapsto f(y) - \xi \cdot y$ is minimal at $y = x_0$. Since we are supposing that f is differentiable at such a point, we obtain $\nabla f(x_0) - \xi = 0$, i.e. $\xi = \{\nabla f(x_0)$. This shows $\partial f(x_0) \subset \{\nabla f(x_0)\}$. The inclusion becomes an equality if f is convex since the above-the-tanent property of convex functions exactly provides $f(y) \geq f(x_0) + \nabla f(x_0) \cdot (y - x_0)$ for every y. □

Proposition 1.12. *Suppose that $f:\mathbb{R}^N\to\mathbb{R}\cup\{+\infty\}$ is convex and take a point x_0 in the interior of $\{f<+\infty\}$. Then $\partial f(x_0)\neq\emptyset$.*

Proof. It is well-known that convex functions in finite dimension are locally Lipschitz in the interior of their domain, and Lipschitz functions are differentiable Lebesgue-a.e. because of the Rademacher's theorem. We can then take a sequence of points $x_n\to x_0$ such that f is differentiable at x_n. We then have $\nabla f(x_n)\in\partial f(x_n)$ and the Lipschitz behavior of f around x_0 implies $|\nabla f(x_n)|\leq C$. It is then possible to extract a subsequence such that $\nabla f(x_n)\to v$. Proposition 1.10 implies $v\in\partial f(x_0)$. □

We can easily see, even from the 1D case, that different situations can occurr at the boundary of $\{f<+\infty\}$. If we take for instance the proper function f defined via

$$f(x)=\begin{cases}x^2 & \text{if } x\geq 0,\\ +\infty & \text{if } x<0,\end{cases}$$

we see that we have $\partial f(0)=[-\infty,0]$ so that the subdifferential can be "fat" on these boundary points. If we take, instead, the proper function f defined via

$$f(x)=\begin{cases}-\sqrt{x} & \text{if } x\geq 0,\\ +\infty & \text{if } x<0,\end{cases}$$

we see that we have $\partial f(0)=\emptyset$, a fact related to the infinite slope of f at 0, an inifnite slope that can of course only appaer on boundary points.

The proof of the fact that the sub-differential is non-empty[1] in the interior of the domain is more involved in the general (infinite-dimensional) case, and also based on the use of the Hahn-Banach theorem. It will require the function to be convex and l.s.c., this second assumption being useless in the finitdimensional case, since convex functions are locally Lipschitz in the interior of their domain. It also requires to discuss whether the function is indeed locally bounded around some points. We state the following clarifying proposition.

Proposition 1.13. *If $f:X\to\mathbb{R}\cup\{+\infty\}$ is a proper convex and l.s.c. function, the following facts are equivalent*

1. *f is locally Lipschitz continuous on the interior of its domain (i.e. for every point in the interior of the domain there exists a ball centered at such a point where f is Lipschitz continuous);*
2. *there exists a non-empty ball where f is finite-valued and Lipschitz continuous;*
3. *there exists a point where f is continuous and finite-valued;*
4. *there exists a point which has a neighborhood where f is bounded from above by a finite constant.*

[1] Note that, instead, we do not discuss the relations between subdifferential and gradients as we decided not to discuss the differentiability in the inifnite-dimensional case.

Moreover, all the above facts hold if X is a Banach space.

Proof. It is clear that 1. implies 2., which implies 3., which implies 4. Let us note that, if f is bounded from above on a ball $B(x_0,R)$, then necessarily f is Lipschitz continuous on $B(x_0,R/2)$. Indeed, we know (point 1 in Proposition 1.4) that f is also bounded from beow by an affine function, and hence by a constant on $B(0,R)$. Then, if there are two points $x_1,x_2 \in B(x_0,R/2)$ with an incremental ratio equal to L and $f(x_1) > f(x_2)$, then, following the half-line going from x_2 to x_1 we find two points $x_3 \in \partial B(x_0,R/2)$ and $x_4 \in \partial B(x_0,3R/4)$ with $|x_3 - x_4| \geq R/4$ and an incremental ration which is also at least L. This implies $f(x_4) > f(x_3) + LR/4$ but the upper and lower bounds on f on the ball $B(0,R)$ imply that L cannot be too large, so f is Lipschitz continuous on $B(x_0,R/2)$.

Hence, in order to prove that 4. implies 1. we just need to prove that every point in the interior of the domain admits a ball centered at such a point where f is bounded from above. We start from the existence of a ball $B(x_0,R)$ where f is bounded from above and we take another point x_1 in the interior of the domain of f. for small $\varepsilon > 0$, the point $x_2 := x_1 - \varepsilon(x_0 - x_1)$ also belongs to the domain of f and every point of the ball $B(x_1,r)$ with $r = \frac{\varepsilon R}{1+\varepsilon}$ can be written as a convex combination of x_2 and a point in $B(x_0,R)$: indeed we have

$$x_1 + v = \frac{1}{1+\varepsilon}x_2 + \frac{\varepsilon}{1+\varepsilon}(x_0 + \frac{1+\varepsilon}{\varepsilon}v)$$

so that $|v| < r$ implies $x_0 + \frac{1+\varepsilon}{\varepsilon}v \in B(x_0,R)$. Then, f is bounded from above on $B(x_1,r)$ by $\max\{f(x_2), \sup_{B(x_0,R)} f\}$ which shows the local bound around x_1.

finally, we want to prove that f is necessarily locally bounded around every point of the interior of its domain if X is complete. Consdier a closed ball $\overline{B}$ contained in $\{f < +\infty\}$ and write $\overline{B} = \bigcup_n \{f \leq n\} \cap \overline{B}$. Since f is l.s.c., each set $\{f \leq n\} \cap \overline{B}$ is closed. Since their countable union has non-empty interior Baire's theorem (which is valid in complete metric spaces) implies that at least one of these sets also has non-empty interior, so there exists a ball contained in a set $\{f \leq n\}$, hence a point where f is locally bounded from above. Then we satisfy condition 4., and consequently also condition 1., 2. and 3. □

We can now prove the following theorem.

Theorem 1.14. *Suppose that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and l.s.c. function and take a point x_0 in the interior of $\{f < +\infty\}$. Also suppose that there exists at least a point x_1 (possibly different from x_0) where f is continuous. Then $\partial f(x_0) \neq \emptyset$.*

Proof. Let us consider the set A given by the interior of $\mathrm{Epi}(f)$ in $X \times \mathbb{R}$ and $B = \{(x_0, f(x_0))\}$. They are two disjoint convex sets, and a is open. Hence, there exists a pair $(\xi, a) \in X' \times \mathbb{R}$ and a constant such that $\langle \xi, x_0 \rangle + af(x_0) \leq c$ and $\langle \xi, x \rangle + at > c$ for every $(x,t) \in A$.

The assumption on the exisntence of a point x_1 where f is continuous implies that f is bounded (say by a constant M) on a ball around x_1 (say $B(x_1,R)$ and hence the set A is non-empty since it includes $B(x_1,R) \times (M, \infty)$. Then, the convex set $\mathrm{Epi}(f)$ has non-empty interior and it is thus the closure of its interior (to see this it is enough to connect every point of a closed convex set to the center of a ball contained in the set, and see that there is a cone composed of small balls contained in the convex set approximating such a point). In particular, since $(x_0, f(x_0))$ belongs to $\mathrm{Epi}(f)$, it is in the closure of A, so necessarily we have $\langle \xi, x_0\rangle + af(x_0) \geq c$ and hence $\langle \xi, x_0\rangle + af(x_0) = c$.

As we did iin Proposition 1.4, we must have $a > 0$. Indeed, we use Proposition 1.13 to see that f is also locally bounded around x_0 so that points of the form $(x,t) = (x_0,t)$ for t large enough should belong to A and should satisfy $a(t - f(x_0)) > 0$, which implies $a > 0$.

Then, we can write $\langle \xi, x\rangle + at > c = \langle \xi, x_0\rangle + af(x_0)$, dividing by a and using $\tilde{\xi} := \xi/a$ as

$$\langle \tilde{\xi}, x\rangle + t > \langle \tilde{\xi}, x_0\rangle + f(x_0) \text{ for every } (x,t) \in A.$$

The inequality becomes large on the clouse of A but implies, when applied to $(x,t) = (x, f(x)) \in \mathrm{Epi}(f) = \overline{A}$,

$$\langle \tilde{\xi}, x\rangle + f(x) \geq \langle \tilde{\xi}, x_0\rangle + f(x_0),$$

which exactly means $-\tilde{\xi} \in \partial f(x_0)$. □

Of course, thanks to the last claim in Proposition 1.13, when x is a Banach space we obtain $\partial f(x_0) \neq \emptyset$ for every x_0 in the interior of $\{f < +\infty\}$.

We list some other properties of subdifferentials.

Proposition 1.15. *1. A point x_0 solves $\min\{f(x) : x \in X\}$ if and only if $0 \in \partial f(x_0)$.*

2. The subdifferential satisfies the monotonicity property

$$\xi_i \in \partial f(x_i) \text{ for } i = 1,2 \Rightarrow \langle \xi_1 - \xi_2, x_1 - x_2\rangle \geq 0.$$

3. If f is convex and l.s.c., the subdifferentials of f and f^* are related through

$$\xi \in \partial f(x) \Leftrightarrow x \in \partial f^*(\xi) \Leftrightarrow f(x) + f^*(\xi) = \langle \xi, x\rangle.$$

Proof. 1. is a straightforward consequence of the definition of subdifferential, since $0 \in \partial f(x_0)$ means that for every y we have $f(y) \geq f(x_0)$.

2. is also straightforward if we sum up the inequalities

$$f(x_2) \geq f(x_1) + \langle \xi, x_2 - x_1\rangle; \quad f(x_1) \geq f(x_2) + \langle \xi, x_1 - x_2\rangle.$$

for part 3., once we know that for convex and l.s.c. functions we have $f^{**} = f$, it is enough to prove $\xi \in \partial f(x) \Leftrightarrow f(x) + f^*(\xi) = \langle \xi, x\rangle$ since then, by symmetry, we can also obtain $x \in \partial f^*(\xi) \Leftrightarrow f(x) + f^*(\xi) = \langle \xi, x\rangle$. We now look at the definition of subdifferential, and we have

$$\begin{aligned}
\xi \in \partial f(x) \quad &\Leftrightarrow \quad \text{for every } y \in X \text{ we have } f(y) \geq f(x) + \langle \xi, y - x\rangle \\
&\Leftrightarrow \quad \text{for every } y \in X \text{ we have } \langle \xi, x\rangle - f(x) \geq \langle \xi, y\rangle - f(y) \\
&\Leftrightarrow \quad \langle \xi, x\rangle - f(x) \geq \sup_y \langle \xi, y\rangle - f(y) \\
&\Leftrightarrow \quad \langle \xi, x\rangle - f(x) \geq f^*(\xi).
\end{aligned}$$

This shows that $\xi \in \partial f(x)$ is equivalent to $\langle \xi, x\rangle \geq f(x) + f^*(\xi)$, which is in turn equivalent to to $\langle \xi, x\rangle = f(x) + f^*(\xi)$, since the opposite inequality is always true by definition of f^*. □

We also state another property, but we prefer to stick to the finite-dimensional case for simplicity.

Proposition 1.16. *A function* $f:\mathbb{R}^N\to\mathbb{R}\cup\{+\infty\}$ *is strictly convex if and only if* $\partial f(x_0)\cap\partial f(x_1)=\emptyset$ *for all* $x_0\neq x_1$.

Proof. If we have $\partial f(x_0)\cap\partial f(x_1)\neq\emptyset$ for two points $x_0\neq x_1$, take $\xi\in\partial f(x_0)\cap\partial f(x_1)$ and define $\tilde f(x):=f(x)-\langle\xi,x\rangle$. Then both x_0 and x_1 minimize $\tilde f$ and this implies that f is not strictly convex. If, instead, we suppose that f is not strictly convex, we need to find two points with a same vector in the subdifferential. Consider two points x_0 and x_1 on which the strict convexity fails, i.e. f is affine on $[x_0,x_1]$. We then have a segment $S=\{(x,f(x)) : x\in[x_0,x_1]\}$ in the graph of f and we can separate it from the interior A of $\text{Epi}(f)$ exactly as we did in Theorem 1.14. Up to restricting the space where f is defined to the minimal affine space containing its domain, we can always suppose that the domain has non-empty interior and hence the epigraph as well. This is not restrictive, since if we want then to define subdifferentials in the original space it is enough to add arbitrary components orthogonal to the affine space containing the domain.

Following the same arguments as in Theorem 1.14, we obtain the existence of a pair $(\xi,a)i\in X'\times\mathbb{R}$ such that

$$\langle\xi,x\rangle+af(x)<\langle\xi,x'\rangle+at \text{ for every } (x',t)\in A \text{ and every } x\in[x_0,x_1].$$

We then prove $a>0$, divide by a, pass to a large inequality on the closure of A, and deduce $-\xi/a\in\partial f(x)$ for every $x\in[x_0,x_1]$. □

Limiting once more to the finite-dimensional case (also because we do not want to discuss differentiability in other settings) we can deduce from the previous proposition and from part 3. of Proposition 1.15 the following fact.

Proposition 1.17. *Take two proper, convex and l.s.c. conjugate functions* f *and* f^* *(with* $f=f^{**}$*); then* f *is a real-valued* C^1 *function on* $\mathbb{R}^N$ *if and only if* f^* *is strictly convex and superlinear.*

Proof. We first prove that a convex function is C^1 if and only if $\partial f(x)$ is a singleton for every x. If it is C^1, and then differentiable, we already saw that this implies $\partial f(x)=\{\nabla f(x)\}$. The converse implication can be proven as follows: first we observe that, if we have a map $v:\mathbb{R}^N\to\mathbb{R}^N$ with $\partial f(x)=\{v(x)\}$, then v is necessarily locally bounded and continuous. Indeed, the function f is necessaril finite everywhere (since the subdifferential is non-empty at every point) and hence locally Lipschitz; local boundedess of v comes from $v(x)\cdot e\leq f(x+e)-f(x)\leq C|e|$ (where we use the Local Lipschitz behavior of f): using e oriented as $v(x)$ we obtain a bound on $|v(x)|$. Continuity comes from local boundedness and from Proposition 1.10: when $x_n\to x$ then $v(x_n)$ admits a converging subsequence, but the limit can only belong to $\partial f(x)$, i.e. it must equal $v(x)$. Then we prove $v(x)=\nabla f(x)$, and for this we use the definition of $\partial f(x)$ and $\partial f(y)$ so as to get

$$v(y)\cdot(y-x)\geq f(y)-f(x)\geq v(x)\cdot(y-x).$$

We then write $v(y)\cdot(y-x) = v(x)\cdot(y-x) + o(|y-x|)$ and we obtain the first-order development $f(y)-f(x) = v(x)\cdot(y-x)+o(|y-x|)$ which characterizes $v(x) = \nabla f(x)$.

This shows that f is C^1 as soon as subdifferentials are singletons. On the other hand, point 3. in 1.15 shows that this is equivalent to having, for each $x \in \mathbb{R}^N$, exacty one point ξ with $x \in \partial f^*(\xi)$. The fact that no more than one point ξ has the same vector in the subdifferential is equivalent (Proposition 1.16) to being strictly convex. The fact that each point is taken at least once as a subdifferential is, instead, equivalent to being superlinear (see Lemma 1.18 below) □

Lemma 1.18. *A convex and l.s.c. function $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is superlinear if and only if ∂f is surjective.*

Proof. Let us suppose that f is convex, l.s.c, and superlinear. Then ofr every ξ the function $\tilde{f}$ given by $\tilde{f}(x) := f(x) - \xi \cdot x$ is also l.s.c, and superlinear, and it admitds a minizer x_0. Such a point satisfies $\xi \in \partial f(x_0)$, which proves that ξ is in the image of ∂f, which is then surjective.

Let us suppose now that ∂f is surjective. Let us fix a number $L > 0$ and take the $2N$ vectors $\xi_i^\pm := \pm L e_i$, where the vectors e_i are the canonical basis of $\mathbb{R}^N$. Since each of these vectors belong to the image of the subdifferential, there exist points $x_i^\pm$ such that $\xi_i^\pm \in \partial f(x_i^\pm)$. This implies that f satisfies $2N$ inequalities of the form $f(x) \geq \xi_i^\pm \cdot x - C_i^\pm$ for some constants $C_i^\pm$. We then obtain $f(x) \geq L||x||_\infty - C$, where $||x||_\infty = \max\{|x_i|\} = \max_i x\cdot(\pm e_i)$ is the norm on $\mathbb{R}^N$ given by the maximal modulus of the components, and $C = \max C_i^\pm$. from the equivalence of the norms in finite dimension we get $f(x) \geq C(N)L||x|| - C$, which shows $\liminf_{||x||\to\infty} \frac{f(x)}{||x||} \geq C(N)L$. The arbitrariness of L concludes the proof. □

Remark 1.19. Note that this very last lemma would be more delicate in infinite dimension, even if it still holds in Banach spaces (the proof is proposed as an Exercise (see Exercise 6.7) and combines several ingredients in this Section. Exercise 6.8 also discusses some subtle points about why some counter-examples fail.

1.3 Formal duality for constrained and penalized optimization problems

In this section we introduce the notion of dual problem of a convex optimization problem through an inf-sup exchange procedure. This often requires to write possible constraints as a sup penalization, and we will then see how to adapt to more general problems. No proof of duality results will be given, and we will analyze in details in the next section the most relevant examples in the calculus of variations. The proofs presented in Section 4.3 will then inspire Section 4.5 for a more general theory.

We start from the following problem:

$$\min\{f(x) \,:\, x \in X, Ax = b\},$$

where $A : X \to Y$ is a linear map between two normed vector spaces, $b \in Y$ is a fixed vector and $f : X \to \mathbb{R} \cup \{+\infty\}$ is a given convex and l.s.c. function. We will denote by A^t the transpose operator of A, a linear mapping defined on Y', the dual of Y, and valued into X', dual of X, and characterized by

$$\langle A^t\xi, x\rangle := \langle \xi, Ax\rangle \ \text{ for all } \xi \in Y' \text{ and } x \in X.$$

We can see that the above problem is equivalent to

$$\min\left\{ f(x) + \sup_{\xi\in Y'}\langle \xi, Ax-b\rangle \ : \ x \in X \right\},$$

since we can compute the value of the expression $\sup_{\xi\in Y'}\langle \xi, Ax-b\rangle$ by distinguishing two cases: either $Ax = b$, in which case $\langle \xi, Ax-b\rangle = 0$ for every ξ and the sup equals 0, or $Ax \neq b$, in which case there exists an element $\xi \in Y'$ such that $\langle \xi, Ax-b\rangle \neq 0$ and, by multiplying ξ times arbitrarily large constants, positive or negative depending on the sign of $\langle \xi, Ax-b\rangle$, we can see that the sup is $+\infty$. Hence, adding this sup means adding 0 if the constraint is satisfied or adding $+\infty$ if not; since in a minimization problem the value $+\infty$ is the same as a constraint, we can see the equivalence between the problem with the constraint $Ax = b$ and the problem with the sup over ξ.

We get now to a problem of the form

$$\inf_x \sup_\xi L(x,\xi), \qquad \text{where } L(x,\xi) = f(x) + \langle \xi, Ax-b\rangle.$$

This is an inf-sup problem, and we can associate with it a second optimization problem, obtained by switching the order of the inf and the sup. We can consider

$$\sup_\xi \inf_x L(x,\xi),$$

which means maximizing over ξ the function obtained as the value of the inf over x. Remember that we have forgotten the constraint $Ax = b$, since it took part in the definition of L, and we now minimize over all x. We can then give a better expression to this new problem, that we will call dual problem. Indeed we have

$$\sup_\xi \inf_x L(x,\xi) = \sup_\xi -\langle \xi, b\rangle + \inf_x f(x) + \langle \xi, Ax\rangle.$$

We then rewrite $\langle \xi, Ax\rangle$ as $\langle A^t\xi, x\rangle$ and change the ign in the inf so as to write it as a sup. We do obtain

$$\sup_\xi \inf_x L(x,\xi) = \sup_\xi -\langle \xi, b\rangle - \sup_x -f(x) + \langle -A^t\xi, x\rangle.$$

We now recognize in the sup over x the form of a Legendre transform and we finally obtain

$$\sup_\xi \inf_x L(x,\xi) = \sup_\xi -\langle \xi, b\rangle - f^*(-A^t\xi).$$

This is a convex optimization problem in the variable ξ (the maximization of the sum of a linear functional and the opposite of a convex function, f^*, applied to a linear function of ξ, involving the Legendre transform of the original objective function f.

We would like the two above optimization problems ("inf sup" and "sup inf") to be related to each other, and for instance their values to be the same. Given an arbitrary function L the values of inf sup and of sup inf are in genrel different, as w can see from this very simple example: take $L : A\times B \to \mathbb{R}$ with $A = B = \{\pm 1\}$ and $L(a,b) = \text{sign}(ab)$. In this case we have $\inf\sup = 1 > \sup\inf = -1$. Indeed, we always have an inequality, that we prove here.

Proposition 1.20. *Given an arbitrary function $L = A\times B \to \mathbb{R}$ we have*

$$\inf_a \sup_b L(a,b) \geq \sup_b \inf_a L(a,b).$$

Proof. Take $(a_0,b_0) \in A\times B$ and write $L(a_0,b_0) \geq \inf_a L(a,b_0)$. We then take the sup over b_0 on both sides, thus obtaining $\sup_{b_0} L(a_0,b_0) \geq \sup_{b_0}\inf_a L(a,b_0)$. We have now a number on the right-hand side, and a function of a_0 on the left-hand side. We then take the inf over a_0 and get

$$\inf_{a_0}\sup_{b_0} L(a_0,b_0) \geq \sup_{b_0}\inf_a L(a,b_0),$$

which is exactly the same as the claim up to renaming the variables. □

If in general it is not possible to connect the two problems obtained as inf-sup and sup-inf of a same function, it can be the case when some conditions are met. The main tool to do it is a theorem by Rockafellar (see [29], Section 37) requiring concavity in the variable on which we maximize, convexity in the other one, and some compactness assumption. In our precise case concavity and convexity are met, since L is convex in x and linear in ξ, and hence concave. Yet, Rockafellar's statement concerns finite-dimensional spaces, and moreover we should still deal with the compactness properties we would need. Hence, we will not provide any proof here that $\min\{f(x) : Ax = b\}$ and $\max\{-\langle \xi, b\rangle - f^*(-A^t\xi) : \xi \in Y'\}$ are equal, and we will wait till the next section for a proof in a very particular case.

We only discuss here some consequences and some variants of this duality approach.

A first consequence concerns sufficient optimality conditions. Suppose to consider two optimization problems issued by an inf-sup/sup-inf procedure, i.e. $\min\{f(a) : a \in A\}$ and $\max\{g(b) : b \in B\}$ with $f(a) := \sup_b L(a,b)$ and $g(b) := \inf_a L(a,b)$. Then, if $a_0 \in A$ and $b_0 \in B$ are such that $f(a_0) = g(b_0)$, automatically a_0 minimizes f and b_0 maximizes g, just because Proposition 1.20 guarantees $f(a_0) \geq \inf f \geq \sup g \geq g(b_0)$ and all inequalities must be equalities here. In our precise case the functional f should be replaced with f plus the constraint $Ax = b$, and g is given by $g(\xi) := -\langle \xi, b\rangle - f^*(-A^t\xi)$

A second consequence concers instead necessary optimality conditions. We need now to believe in $\inf f = \sup g$, and we suppose that we have a pair (a_0,b_0) where a_0 minimizes f and b_0 maximizes g. Then we deduce $f(a_0) = g(b_0)$, but this equality is a very strong piece of information in many cases. for instance, in our case this means that, if x_0 and ξ_0 are optimal, then we have

$$f(x_0) = -\langle \xi_0, b\rangle - f^*(-A^t\xi_0) \quad \text{and} \quad Ax_0 = b.$$

This can be re-written as

$$f(x_0) + f^*(-A^t \xi_0) = -\langle \xi_0, b\rangle = -\langle \xi_0, Ax_0\rangle = -\langle A^t \xi_0, x_0\rangle,$$

i.e. we have equality in the inequality $f(x) + f^*(y) \geq \langle x, y\rangle$. This is equivalent to

$$x_0 \in \partial f^*(-A^t \xi_0) \quad \text{and} \quad -A^t \xi_0 \in \partial f(x_0).$$

We can note the similarity with Lagrange multipliers, where optimizing a function f under a linear constraint of the form $Ax = b$ can be translated into the fact that ∇f should belong to a subspace, orthogonal to the affine space of the constraints, which is indeed the image of A^t.

Before moving on to variants of the previous pair of dual problems we want to insist that writing an equality constraint as a sup over test elements of a dual space is exactly what is always done in the weak formulation of PDEs. In the next section we will see as an example what happens when the constraint is of the form $\nabla \cdot \mathbf{v} = f$, which can be written as $\int \nabla\phi \cdot \mathbf{v} + \phi f = 0$ for every test function ϕ, and it is very natural to replace the constraint with a sup over ϕ. In this case, the dual problem turns out to be a maximization over scalar functions ϕ. Moreover, the transpose of the divergence operator $\nabla\cdot$ is the opposite of the gradient, since $\int \phi \nabla \cdot \mathbf{v} = -\int \nabla\phi \cdot \mathbf{v}$ by integration by parts, as soon as boundary conditions are taken care of. In this way, the functional $f^*(-A^t \phi)$ will be a very classical functional in calculus of variations.

Among variants, we first want to discuss the case of inequality constraints instead of equalities. A constraint of the form $Ax \leq b$ only has a meaning if we give a notion of inequality among vectors, which is general is not canonically defined. In finite dimension a general convention, mainly used by computer scientists in optimization problem, is that we can consider the inequality component-wise. In calculus of variations, we can expect both Ax and b to be functions in a certain functional space, and we can require the inequality to be satisfied pointwise (or a.e.). What is important is that we should characterize the inequality in terms of test functions. for instance, the inequality $f \leq g$ a.e. is equivalent to $\int \phi(f - g) \leq 0$ for every $\phi \geq 0$ and a similar equivalence can be stated in the finite-dimensional componentwise case. We then write

$$\min\{f(x) \,:\, x \in X, Ax \leq b\} = \min\left\{ f(x) + \sup_{\xi \in Y', \xi \geq 0} \langle \xi, Ax - b\rangle \,:\, x \in X \right\},$$

since

$$\sup_{\xi \in Y', \xi \geq 0} \langle \xi, y\rangle = \begin{cases} 0 & \text{if } y \leq 0, \\ +\infty & \text{if not.} \end{cases}$$

We can then go on with the very same procedures and obtain the dual problem

$$\max\{-\langle \xi, b\rangle - f^*(-A^t \xi) \,:\, \xi \in Y, \xi \geq 0\}.$$

Finally, once that we know how to build dual problems out of constrained optimization problems, we could consider a more general case, such as

$$\min\{f(x) + g(Ax)\},$$

where $g = \mathbb{1}_{\{b\}}$ corresponds to the previous example. In this case we do not have constraints to write as a sup, but we can decide to write one of the two functions f or g as a sup thanks to the double Legendre transform. We then set

$$L(x,\xi) := f(x) + \langle \xi, Ax \rangle - g^*(\xi)$$

and we easily see that we have

$$\min \{f(x) + g(Ax)\} = \inf_x \sup_\xi L(x,\xi).$$

We then interchange inf and sup thus obtaining the dual problem

$$\begin{aligned}\sup_\xi \inf_x L(x,\xi) &= \sup_\xi -g^*(\xi) + \inf_x f(x) + \langle \xi, Ax \rangle \\ &= \sup_\xi -g^*(\xi) - \sup_x -f(x) + \langle -A^t\xi, x \rangle = \sup_\xi -g^*(\xi) - f^*(-A^t\xi).\end{aligned}$$

As we said, the equality constraint $Ax = b$ corresponds to $g = \mathbb{1}_{\{b\}}$, so that we have $g^*(\xi) = \langle \xi, b \rangle$.

The duality between

$$\min \{f(x) + g(Ax)\} \quad \text{and} \quad \sup_\xi -g^*(\xi) - f^*(-A^t\xi),$$

is a classical object in convex analysis and a theorem guaranteeing, under some conditions, that the the values are actually equal is known as Fenchel-Rockafellar's theorem. We will see in Section 4.5 a proof, in a simplified setting, of this theorem, inspired by the precise proof of a concrete duality result presented in Section 4.3.

Chapter 2

An example of convex duality: minimal flows and optimal compliance

We start this section from two classical examples of convex optimization probems, in strong connection with PDEs, one from traffic congesion modelling and one from the mechanics of deformation.

2.1 Formal duality

The first problem consists in finding a flow transporting some mass from an original configuration f^+ (to be intended as a distribution of mass on given domain Ω, so, for instance, one could take $f^+ \in \mathscr{P}(\Omega)$, the set of probability measures on Ω) to a target configuration f^-. The flow will be a vector field $\mathbf{v}$ which described, in Eulerian variables, the motion: $\mathbf{v}(x)$ stands for the intensity and the direction of the movement of the mass at the point x; the motion is supposed to be stationary (i.e. the mass follows a permanent movement, and new mass is always injected in the points belonging to the support of f^+, and distributed according to f^-, and withdrawn according to f^-) so that there is no explicit time dependence. The condition to move mass from f^+ to f^- can be written in terms od $\nabla \cdot \mathbf{v}$, thanks to the following heuristic consideration: for ecery small subdomain $A \subset \Omega$, the conservation of the mass imposes that the mass exiting A, which equals $\int_{\partial A} \mathbf{v} \cdot \mathbf{n}$ since mass can only leave A through the boundary, and tangential movement on the boundary does not contribute to this, should be equal to the mass which has been inserted minus that which has been removed, i.e. to $\int_A f^+ - f^-$; writing $\int_{\partial A} \mathbf{v} \cdot \mathbf{n} = \int_A \nabla \cdot \mathbf{v}$ thanks to the divergence theorem and assuming that this equality holds for any A means $\nabla \cdot \mathbf{v} = f^+ - f^-$. Note that, if Ω itself has a boundary and no mass is injected/removed at points of the boundary, one shuld also impose $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$ so that no mass leaves Ω. A convenient way to express all this set of constraints is the weak formulation

$$\int \nabla\phi \cdot \mathbf{v} + \phi f = 0 \text{ for all } \phi \in C^\infty(\overline{\Omega}),$$

where $f = f^+ - f^-$. This includes the possibility that $f^\pm$ are singular measures with a part on the boundary, which should compensate $\mathbf{v}\cdot\mathbf{n}$.

Among all the possibile flows $\mathbf{v}$ satisfying these constraints, we minimize a cost which penalizes the total movement which is realized, i.e. an integral cost which is increasing in $|\mathbf{v}|$. We will discuss in Section 4.6 the connection of this problem with optimal transport; here we just say that a natural choice would be to minimize $\int |\mathbf{v}|\,\mathrm{d}x$ or $\int k(x)|\mathbf{v}|\,\mathrm{d}x$, where $k > 0$ is a weight (exactly as we did in Section 1.4.3 for weighted geodesics); on the other hand, when we want to model traffic congestion, we can suppose that the weight k is not given a priori but depends on the traffic itself and, in a very simplified setting, on $|\mathbf{v}|$. The simplest choice is to choose $k = |\mathbf{v}|$ and hence solve (putting for simplicity of computations a factor $\frac{1}{2}$ in front of the integral)

$$\min\left\{\int_\Omega \frac{1}{2}|\mathbf{v}|^2\,\mathrm{d}x \;:\; \nabla\cdot\mathbf{v} = f\right\}. \tag{2.1}$$

As we said, the second problem comes from mechanics, and is much more standard in its modelling. suppose that a membrane is originally at rest in horizontal position, that we model through the constant function $u = 0$ on Ω: the domain Ω stands for the projection on the horizontal plan of the membrane, and the value of $u(x) \in \mathbb{R}$ stands for its vertical displacement. If no force acts on the membrane, the configuration $u = 0$ is stable and is what can be observed in reality. Suppose now that in some points the membrane is pushed down, in others it is pushed up, thus applying a force $f = f^+ - f^-$ with positive and negative parts. If the two parts equilibrate (i.e. $\int f^+ = \int f^-$) we can imagine that the barycenter of the membrane does not move, but the shape of the membrane deformates. The configuration which is realized by the membrane is the one which minimizes a sum of a potential energy given by the work of the force f (i.e. $\int uf$) and an elastic energy related to the deformation. The simplest choice is to choose the Dirichlet energy $\int \frac{1}{2}|\nabla u|^2$ as an elastic energy. One of the reason for this choice is that it is equivalent to the Taylor expansion (up to the first non-zero term) of the energy $\int \sqrt{1+|\nabla u|^2}$, which represents the surface area of the deformed membrane (and hence approximates well such an area if u is small in C^1 norm). Alternatively, one can also describe the equiliobrium shape of the membrane in PDE terms through $\Delta u = f$. We will discuss later how to insert Dirichlet data on u (i.e. regions where the displacement u is prescribed, for instance $u = 0$), and in particular we will see in Chapter 6 that an interesting question is how to optimize the Dirichlet regions, which can be considered as reinforcements for the membrane, under possible constraints or penalizations on their size. So far, we stick to the case where no value is prescribed for u, which corresponds to Neumann boundary conditions for the PDE $\Delta u = f$ if Ω has a boundary. The variational problem reads

$$\min\left\{\int_\Omega \frac{1}{2}|\nabla u|^2\,\mathrm{d}x + \int uf \;:\; u \in H^1(\Omega)\right\}. \tag{2.2}$$

In the case where f is a function instead of a more general distribution in the dual of H^1 (in which case one should replace $\int uf$ with $\langle f,u\rangle$), this is a particular case of the problems presented in Section 2.1. The minimal value is necessarily negative (since $u=0$ is a competitor) and one can see that, using the optimality condition given by the Euler-Lagrange equation $\Delta u = f$, it equals $-\int \frac{1}{2}|\nabla u|^2$. Hence, if we consider that this minimal values depends on the domain Ω and on the force f we see that how much it differs from 0 is a measure of how stiff is the configuration, since it measures the effective total deformation (in terms of elastic energy). The opposition of this minimal value is called *compliance* (and, as we said, it can be optimized in order to find the stiffest configuration, usually among domains Ω for given f, and usually adding Dirichlet boundary data).

The main goal of this section is to show that Problems of the form (2.1) and (2.2) are actually dual to each other, and that the same holds for a large class of variantss of the same problems, replacing the quadratic costs with other convex functions.

To be mo precise we will consider a function $H:\Omega\times\mathbb{R}^d\to\mathbb{R}$ which is convex in the second variable

$$(Hyp1) \quad \text{for every } x \quad v\mapsto H(x,v) \text{ is convex}$$

and satisfying the following uniform bounds:

$$(Hyp2) \quad \frac{c_0}{q}|v|^q - h_0(x) \le H(x,v) \le \frac{c_1}{q}|v|^q + h_1(x),$$

where h_0,h_1 are L^1 functions on Ω, $c_0,c_1>0$ are given finite constants, and $p\in(1,+\infty)$ is a given exponent. For functions of this form, when we write $H^*(x,w)$ we mean the Legendre transform in the second variable, i.e. $H^*(x,w)=\sup_v w\cdot v - H(x,v)$.

We consider the problem

$$\min\left\{\int_\Omega H(x,\mathbf{v}(x))\,\mathrm{d}x \;:\; \nabla\cdot\mathbf{v}=f\right\}.$$

Before giving rigorous results, we will formally build its dual problem, with the same informal derivation as we didi in Section 4.2.3. This can be done in the following way: the constraint $\nabla\cdot\mathbf{v}=f$ can be written, in weak form, as $-\int\mathbf{v}\cdot\nabla u=\int fu$ for every u in a suitable space of test functions This means that we can rewrite the above problem in the min-max form

$$\min\left\{\int H(x,\mathbf{v})+\sup_u -\int fu-\int\mathbf{v}\cdot\nabla u\right\},$$

since the last sup is 0 is the constraint is satisfied and $+\infty$ if not. Now, we have a min-max problem and the dual problem can be obtained just by inverting inf and sup. In this case we get

$$\sup\left\{-\int fu+\inf_{\mathbf{v}}\int H(x,\mathbf{v})-\int\mathbf{v}\cdot\nabla u\right\}.$$

Since $\inf_{\mathbf{v}}\int H(x,\mathbf{v})-\int\mathbf{v}\cdot\nabla u=-\sup_{\mathbf{v}}\int\nabla u\cdot\mathbf{v}-\int H(x,v)=\int H^*(x,\nabla u)$, the problem becomes

$$\sup\left\{-\int fu-\int H^*(x,\nabla u)\right\}.$$

In the following, we will see precise statements about the duality between the two problems. The duality proof, based on the above conve analysis tools, is essentially inspired by the method used in [7] . Other proofs are obviously possible.

2.2 Rigourous duality with no-flux boundary conditions

We define the space $W^{1,p}_\diamond(\Omega)$ as the vector subspace of $W^{1,p}(\Omega)$ composed by functions with zero mean and the space $(W^{1,p})'_\diamond(\Omega)$ as the subspace of the dual of $W^{1,p}$ composed by those f such that $\langle f,1\rangle = 0$ (i.e. those f with zero mean as well).

Note that for every $\mathbf{v} \in L^{p'}(\Omega;\mathbb{R}^d)$, the distribution $\nabla\cdot\mathbf{v}$, defined through

$$\langle \nabla\cdot\mathbf{v},\phi\rangle := -\int_\Omega v\cdot\nabla\phi$$

belongs naturally to $(W^{1,p})'_\diamond(\Omega)$. This will be by the way the definition we will use of the divergence operator (in weak form), and it includes a natural Neumann (no-flux) boundary condition on $\partial\Omega$. However, consider that we will often use Ω to be the torus, which gets rid of many boundary issues.

We will prove the following duality result.

Theorem 2.1. *Suppose that Ω is smooth enough and that H satisfies Hyp1 and Hyp2. Then, for any $f \in (W^{1,p})'_\diamond(\Omega)$, we have*

$$\min\left\{\int_\Omega H(x,\mathbf{v}(x))\,\mathrm{d}x \;:\; \mathbf{v}\in L^{p'}(\Omega;\mathbb{R}^d), \nabla\cdot\mathbf{v} = f\right\}$$
$$= \max\left\{-\int_\Omega H^*(x,\nabla u(x))\,\mathrm{d}x - \langle f,u\rangle \;:\; u\in W^{1,p}(\Omega)\right\}$$

Proof. We will define a function $\mathscr{F} : (W^{1,p})' \to \mathbb{R}$ in the following way

$$\mathscr{F}(p) := \min\left\{\int_\Omega H(x,\mathbf{v}(x))\,\mathrm{d}x \;:\; \mathbf{v}\in L^{p'}(\Omega;\mathbb{R}^d), \nabla\cdot\mathbf{v} = f+p\right\}.$$

Note that if $p \notin (W^{1,p})'_\diamond \subset (W^{1,p})'$, then $\mathscr{F}(p) = +\infty$, as there is no $\mathbf{v}\in L^{p'}$ with $\nabla\cdot\mathbf{v} = f+p$. On the other hand, if $p \in (W^{1,p})'_\diamond$, then $\mathscr{F}(p)$ is well-defined and real-valued since $\int_\Omega H(x,\mathbf{v}(x))\,\mathrm{d}x$ is comparable to the $L^{p'}$ norm, and we use the following fact: for every $f \in (W^{1,p})'_\diamond$ there exists $\mathbf{v}\in L^{p'}$ such that $f = \nabla\cdot\mathbf{v}$ and $||\mathbf{v}||_{L^{p'}} \le ||f||_{(W^{1,p})'_\diamond}$ (see next lemma).

We now compute $\mathscr{F}^*: W^{1,p} \to \mathbb{R}$:

$$\begin{aligned}
\mathscr{F}^*(u) &= \sup_p \langle p,u\rangle - \mathscr{F}(p) \\
&= \sup_{p,\mathbf{v}:\nabla\cdot\mathbf{v}=f+p} \langle p,u\rangle - \int_\Omega H(x,\mathbf{v}(x))\,\mathrm{d}x \\
&= \sup_{p,\mathbf{v}:\nabla\cdot\mathbf{v}=f+p} \langle p+f,u\rangle - \langle f,u\rangle - \int_\Omega H(x,\mathbf{v}(x))\,\mathrm{d}x \\
&= \sup_{\mathbf{v}} -\langle f,u\rangle - \int_\Omega H(x,\mathbf{v}(x))\,\mathrm{d}x - \int (\mathbf{v}\cdot\nabla u)\,\mathrm{d}x \\
&= -\langle f,u\rangle + \int_\Omega H^*(x,-\nabla u(x))\,\mathrm{d}x.
\end{aligned}$$

Now we want to use the fact that $\mathscr{F}^{**}(0) = \sup -\mathscr{F}^*$. Note that $\sup -\mathscr{F}^* = +\infty$ if $f \notin (W^{1,p})'_\diamond$, as it is possible to add an arbitrary constant to u, without changing the gradient term, and letting the term $-\langle f,u\rangle$ tend to $-\infty$. On the other hand, if $f \in (W^{1,p})'_\diamond$, then in the above optimization u can be taken in $W^{1,p}$ or in $W^{1,p}_\diamond$ and the result does not change, as adding a constant does not change neither the integral term (which only depends on ∇u) nor the duality term (as $\langle f,1\rangle = 0$).

By taking the sup on $-u$ instead of u we also have

$$\mathscr{F}^{**}(0) = \sup_u -\langle f,u\rangle - \int_\Omega H^*(x,\nabla u(x))\,\mathrm{d}x = -\inf_u \langle f,u\rangle + \int_\Omega H^*(x,\nabla u(x))\,\mathrm{d}x.$$

Finally, if we prove that $\mathscr{F}$ is convex and l.s.c., then we also have $\mathscr{F}^{**}(0) = \mathscr{F}(0)$, which proves the claim.

The convexity of $\mathscr{F}$ is easy. We just need to take $p_0, p_1 \in (W^{1,p})'_\diamond(\Omega)$ and define $p_t := (1-t)p_0 + tp_1$. Let $\mathbf{v}_0, \mathbf{v}_1$ be optimal in the definition of $\mathscr{F}(p_0)$ and $\mathscr{F}(p_1)$, i.e. $\int H(x,\mathbf{v}_i(x))\,\mathrm{d}x = \mathscr{F}(p_i)$ and $\nabla\cdot\mathbf{v}_i = f + p_i$. Let $\mathbf{v}_t := (1-t)\mathbf{v}_0 + t\mathbf{v}_1$. Of course we have $\nabla\cdot\mathbf{v}_t = f + p_t$ and, by convexity of $H(x,\cdot)$ we have

$$\begin{aligned}
\mathscr{F}(p_t) \le \int H(x,\mathbf{v}_t(x))\,\mathrm{d}x &\le (1-t)\int H(x,\mathbf{v}_0(x))\,\mathrm{d}x + t\int H(x,\mathbf{v}_1(x))\,\mathrm{d}x \\
&\le (1-t)\mathscr{F}(p_0) + t\mathscr{F}(p_1),
\end{aligned}$$

and the convexity is proven.

For the semicontinuity, we take a sequence $p_n \to p$ in $(W^{1,p})'$. We can suppose that $\mathscr{F}(p_n) < +\infty$ otherwise there is nothing to prove, hence $p_n \in (W^{1,p})'_\diamond(\Omega)$. Take the corresponding optimal vector fields $\mathbf{v}_n \in L^{p'}$, i.e. $\int H(x,\mathbf{v}_n(x))\,\mathrm{d}x = \mathscr{F}(p_n)$. We can extract a subsequence such that $\lim_k \mathscr{F}(p_{n_k}) = \liminf_n \mathscr{F}(p_n)$. Moreover, from the bound on H we can see that the $L^{p'}$ norm of v_n is bounded in terms of the values of $\mathscr{F}(p_n)$, which are (use Lemma 2.2) bounded by the $(W^{1,p})'_\diamond$ norms of p_n. Since p_n converges, then we get a bound on $||\mathbf{v}_n||_{L^{p'}}$. Hence, up to an extra subsequence extraction, we can assume $v_{n_k} \rightharpoonup v$. Obviously we have $\nabla\cdot\mathbf{v} = f + p$ and, by semicontinuity of the integral functional $\mathbf{v} \mapsto \int H(x,\mathbf{v})\,\mathrm{d}x$, we get

$$\mathscr{F}(p) \le \int H(x,\mathbf{v}(x))\,\mathrm{d}x \le \liminf_k \int H(x,\mathbf{v}_{n_k}(x))\,\mathrm{d}x = \lim_k \mathscr{F}(p_{n_k}) = \liminf_n \mathscr{F}(p_n),$$

which gives the desired result. □

The duality result that we proved can be written in the following form

$$\min\{A(\mathbf{v})\}+\min\{B(u)\}=0, \tag{2.3}$$

where A is defined on $L^{p'}(\Omega;\mathbb{R}^d)$ and B on $W^{1,p}(\Omega)$ via

$$A(\mathbf{v}):=\begin{cases}\int_\Omega H(x,\mathbf{v}(x))\,\mathrm{d}x & \text{if } \nabla\cdot\mathbf{v}=f,\\ +\infty & \text{otherwise,}\end{cases}$$

and

$$B(u)=\int_\Omega H^*(x,\nabla u(x))\,\mathrm{d}x+\langle f,u\rangle.$$

Lemma 2.2. *Given $f\in(W^{1,p})'_\diamond(\Omega)$ there exists $\mathbf{v}\in L^{p'}(\Omega;\mathbb{R}^d)$ such that $f=\nabla\cdot\mathbf{v}$ and $||\mathbf{v}||_{L^{p'}}\le C||f||_{(W^{1,p})'}$.*

Proof. Consider the minimization problem

$$\min\left\{\frac{1}{p}\int_\Omega|\nabla\phi|^p\,\mathrm{d}x+\langle f,\phi\rangle\ :\ \phi\in W^{1,p}(\Omega)\right\}.$$

It is easy to prove that this problem admits a solution, as the minimization can be restricted to the set $W^{1,p}_\diamond$. This solution ϕ satisfies[1] (see Section 2.4)

$$-\int_\Omega(\nabla\phi)^{p-1}\cdot\nabla\psi=\langle f,\psi\rangle$$

for all $\psi\in W^{1,p}(\Omega)$. This exactly means $\nabla\cdot\mathbf{v}=f$ for $\mathbf{v}=(\nabla\phi)^{p-1}$. Moreover, testing against ϕ, we get

$$\begin{aligned}||\mathbf{v}||^{p'}_{L^{p'}}=\int_\Omega|\mathbf{v}|^{p'}=\int_\Omega|\nabla\phi|^p=\langle f,\phi\rangle\le||f||_{(W^{1,p})'}||\phi||_{W^{1,p}}\\ \le C||f||_{(W^{1,p})'}||\nabla\phi||_{L^p}=C||f||_{(W^{1,p})'}||\mathbf{v}||^{p'-1}_{L^{p'}},\end{aligned}$$

which gives the desired bound on $||\mathbf{v}||_{L^{p'}}$. □

[1] pay attention to the notation: for every vector v and $\alpha>0$ we denote by w^α the vector with modulus equal to $|w|^\alpha$, and same direction as w, i.e. $w^\alpha:=|w|^{\alpha-1}w$.

Chapter 3

Regularity via duality

In this section we will use the relation (2.3) to produce Sobolev regularity results for solutions of the minimization problems $\min A$ or $\min B$.

We will start by describing the general strategy. We consider a function H not explicitly depending on x, and we suppose that an inequality of the following form is true

$$(Hyp3) \quad H(v) + H^*(w) \geq v \cdot w + c|j(v) - j_*(w)|^2$$

for some given functions $j, j_* : \mathbb{R}^d \to \mathbb{R}^d$. This is an improvement of the Young inequality $H(v) + H^*(w) \geq v \cdot w$ (which is just a consequence of the definition of H^*). Of course this is always true taking $j = j_* = 0$, but the interesting cases are the ones where j and j_* are non-trivial.

To simplify the computations, we will suppose that Ω is the flat d-dimensional torus $\mathbb{T}^d$ (and we will omit the indication of the domain). We start from the following observations, tha we collect in a lemma. For the sake of the notations, we call $\bar{\mathbf{v}}$ and $\bar{u}$ the minimizers (or some minimizers, in case there is no uniqueness) of A and B, respectively, and we denote by u_h the function $u_h(x) := \bar{u}(x+h)$. We define a function $g : \mathbb{R}^d \to \mathbb{R}$ given by

$$g(h) := \int f(x)\bar{u}(x+h)\,\mathrm{d}x - \int f(x)\bar{u}(x)\,\mathrm{d}x.$$

Lemma 3.1. *Suppose H satisfies $Hyp1,2,3$ and let $\bar{\mathbf{v}}$ and $\bar{u}$ be optimal. Then*

1. $j(\bar{\mathbf{v}}) = j_*(\nabla \bar{u})$.
2. $c \int |j_*(\nabla u_h) - j_*(\nabla \bar{u})|^2\,\mathrm{d}x \leq g(h)$.
3. *If $g(h) = O(|h|^2)$, then $j_*(\nabla \bar{u}) \in H^1$.*
4. *If g is $C^{1,1}$, then $g(h) = O(|h|^2)$ and $j_*(\nabla \bar{u}) \in H^1$.*
5. *If $f \in W^{1,p}_{\diamond}(\Omega)$, then $g \in C^{1,1}$ and hence $j_*(\nabla \bar{u}) \in H^1$*

Proof. First, we compute for arbitrary $\mathbf{v}$ and u admissible in the primal and dual problems (i.e. we need $\nabla\cdot\mathbf{v}=f$), the sum $A(\mathbf{v})+B(u)$:

$$A(\mathbf{v})+B(u)=\int(H(\mathbf{v})+H^*(\nabla u)+fu)\,\mathrm{d}x=\int(H(\mathbf{v})+H^*(\nabla u)-\mathbf{v}\cdot\nabla u)\,\mathrm{d}x\geq c\int|j(\mathbf{v})-j_*(\nabla u)|^2\,\mathrm{d}x.$$

If we take $\mathbf{v}=\bar{\mathbf{v}}$ and $u=\bar{u}$, then $A(\mathbf{v})=\min A$, $B(u)=\min B$ and $A(\mathbf{v})+B(u)=0$. Hence, we deduce $j(\bar{\mathbf{v}})=j_*(\nabla\bar{u})$, i.e. the Part (1) in the statement.

Now, let us fix $\mathbf{v}=\bar{\mathbf{v}}$ but $u=u_h$. We obtain

$$c\int|j_*(\nabla\bar{u})-j_*(\nabla u_h)|^2\,\mathrm{d}x=c\int|j(\bar{\mathbf{v}})-j_*(\nabla u_h)|^2\,\mathrm{d}x\leq A(\bar{\mathbf{v}})+B(u_h)=B(u_h)-B(\bar{u}).$$

In computing $B(u_h)-B(\bar{u})$, we see that the terms $\int H^*(\nabla u_h)$ and $\int H^*(\nabla\bar{u})$ are equal, as one can see from an easy change-of-variable $x\mapsto x+h$. Hence,

$$B(u_h)-B(\bar{u})=\int fu_h-\int f\bar{u}=g(h),$$

which gives part (2).

Part (3) of the statement is an easy consequence of classical characterization of Sobolev spaces. Part (4) comes from the optimality of $\bar{u}$, which means that $g(0)=0$ and $g(h)\geq 0$ for all h. This implies, as soon as $g\in C^{1,1}$, $\nabla g(0)=0$ and $g(h)=O(|h|^2)$.

For Part (5), we first differentiate $g(h)$, thus getting

$$\nabla g(h)=\int f(x)\nabla\bar{u}(x+h)\,\mathrm{d}x.$$

If we want to differentiate once more, we use the regularity assumption on f: we write

$$\int f(x)\nabla\bar{u}(x+h)\,\mathrm{d}x=\int f(x-h)\nabla\bar{u}(x)\,\mathrm{d}x$$

and then

$$D^2g(h)=\int\nabla f(x-h)\otimes\nabla\bar{u}(x)\,\mathrm{d}x,$$

which also gives $|D^2g|\leq||\nabla f||_{L^q}||\nabla\bar{u}||_{L^p}$. Note that $\bar{u}$ naturally belongs to $W^{1,p}$, hence the integral is finite and bounded, and $g\in C^{1,1}$. □

Unfortunately, the last assumption ($f\in W^{1,p'}$) is quite restrictive, but we want to provide a case where it is reasonable to use it. Before, we find interesting cases of functions H and H^* for which we can provide non-trivial functions j and j_*.

3.1 Pointwise vector inequalities

The first interesting case is the quadratic case. Take $H(v)=\frac{1}{2}|v|^2$ with $H^*(w)=\frac{1}{2}|w|^2$. In this case we have easily

$$H(v)+H^*(w)=\frac{1}{2}|v|^2+\frac{1}{2}|w|^2=v\cdot w+\frac{1}{2}|v-w|^2,$$

hnece one can take $j(v) = v$ and $j_*(w) = w$.

Then, we pass to another interesting case, the case of the powers. Take $H(v) = \frac{1}{q}|v|^q$ with $H^*(w) = \frac{1}{p}|w|^p$. We claim that in this case we can take $j(v) = v^{q/2}$ and $j_*(w) = w^{p/2}$ (remember the notation for powers of vectors).

Lemma 3.2. *For any $v, w \in \mathbb{R}^d$ we have*

$$\frac{1}{p}|v|^p + \frac{1}{q}|w|^q \geq v \cdot w + \frac{1}{2\max\{p,q\}}|v^{p/2} - w^{q/2}|^2.$$

Proof. First we write $a = v^{p/2}$ and $b = w^{q/2}$ and we express the inequality in terms of a, b. Hence we try to prove $\frac{1}{p}|a|^2 + \frac{1}{q}|b|^2 \geq a^{2/p} \cdot b^{2/q} + \frac{1}{2\max\{p,q\}}|a-b|^2$. In this way the inequality is more homogeneous, as it is of order 2 in all its terms (remember $1/p + 1/q = 1$). Then we notice that we can also write the expression in terms of $|a|, |b|$ and $\cos\theta$, where θ is the angle between a and b (which is the same as the one between $v = a^{2/p}$ and $w = b^{2/q}$. Hence, we want to prove

$$\frac{1}{p}|a|^2 + \frac{1}{q}|b|^2 \geq \cos\theta\left(|a|^{2/p}|b|^{2/q} - \frac{1}{\max\{p,q\}}|a||b|\right) + \frac{1}{2\max\{p,q\}}(|a|^2 + |b|^2).$$

since this depends linearly in $\cos\theta$, it is enough to prove the inequality in the two limit cases $\cos\theta = \pm 1$.

For simplicity, ue to the simmetry in p and q of the claim, we suppose $p \leq 2 \leq q$. We start from the case $\cos\theta = 1$, i.e. $b = ta$, with $t \geq 0$ (the case $a = 0$ is trivial). In this case the l.h.s. of the inequality becomes

$$|a|^2(\frac{1}{p} + \frac{1}{q}t^2) = |a|^2(\frac{1}{p} + \frac{1}{q}(1+(t-1))^2) = |a|^2(1 + \frac{2}{q}(t-1) + \frac{1}{q}(t-1)^2) \geq |a|^2(t^{2/q} + \frac{1}{q}(t-1)^2),$$

where we used the concavity of $t \mapsto t^{2/q}$, which provides $1 + \frac{2}{q}(t-1) \geq t^{2/q}$. This inequality is even stronger than the one we wanted to prove, as we get a factor $1/q$ instead of $1/(2q)$ in the r.h.s..

The factor $1/(2q)$ appears in the case $\cos\theta = -1$, i.e. $b = -ta$, $t \geq 0$ (we do not claim that this coefficient is optimal, anyway). In this case we start from the r.h.s.

$$|a|^2(\frac{1}{2q}(1+t)^2 - t^{2/q}) \leq |a|^2\frac{1}{2q}(1+t)^2 \leq |a|^2\frac{2}{2q}(1+t^2) \leq |a|^2(\frac{1}{p} + \frac{1}{q}t^2),$$

which gives the claim. □

As a more involved variant of the power cost functions, we also consider these other convex functions which will be natural in the study of very degenerate elliptic PDEs and in congestion models.

Consider the case $H(v) = |v| + \frac{1}{q}|v|^q$. In this case, we can use $j(v) = v^{q/2}$ and $j_*(w) = (w-1)_+^{p/2}$ (again, we use this weird notation: the vector $(w-1)_+^{p/2}$ is the vectorn with norm equal to $(|w|-1)_+^{p/2}$ and same direction as w, i.e. $j_*(w) = (|w| - 1)_+^{p/2} w/|w|$).

Indeed, we have

$$H^*(w) = \sup_v \; v \cdot w - |v| - \frac{1}{q}|v|^q = \frac{1}{p}(|w|-1)_+^p$$

and

$$H(v) + H^*(w) = |v| + \frac{1}{q}|v|^q + \frac{1}{p}(|w|-1)_+^p \geq |v| + v \cdot (w-1)_+ + c|v^{q/2} - (w-1)_+^{p/2}|^2.$$

We only need to prove $|v| + v \cdot (w-1)_+ \geq v \cdot w$. This can be done by writing

$$|v| + v \cdot (w-1)_+ = |v|(1 + (|w|-1)_+ \cos\theta).$$

If $|w| \geq 1$ then we go on with

$$|v|(1 + (|w|-1)_+ \cos\theta) \geq |v|\cos\theta(1 + (|w|-1)_+) = |v|\cos\theta|w| = v \cdot w.$$

If $|w| \leq 1$ then we simply use

$$|v|(1 + (|w|-1)_+ \cos\theta) = |v| \geq v \cdot w.$$

3.2 Applications to (degenerate) elliptic PDEs

If we look at the case $H(v) = \frac{1}{q}|v|^q$, we have $H^*(w) = \frac{1}{p}|w|^p$ and the solutions of $\Delta_p u = f$ (where $\Delta_p u := \nabla \cdot ((\nabla u)^{p-1})$) are the minimizers of $\int \frac{1}{p}|\nabla u|^p + fu$. We already mentioned in Section 2.4 some regularity issues about p-harmonic functions, and the same classical references [6, 18] also provide many results about solutions of $\Delta_p u = f$. In this section we first underline the simplest result that we can obtain from the consideration of the previous sections and these duality methods. It is indeed easy to obtain the following.

Proposition 3.3. *Suppose that Ω is the flat torus and $\Delta_p \bar{u} = f \in W^{1,p'}(\Omega)$. Then $(\nabla \bar{u})^{p/2} \in H^1$.*

Proof. This statement can be proven by combining Lemma 3.1 and Lemma 3.2. □

Remark 3.4. The above result is very classical (see for instance [22]), even if usually obtained through a slightly different technique. In particular, the pointwise inequality of Lemma 3.2 replaces, in this duality-based approach, the usual vector inequality that PDE methods require to handle equations involving Δ_p, i.e.

$$(w_0^{p-1} - w_1^{p-1}) \cdot (w_0 - w_1) \geq c|w_0^{p/2} - w_1^{p/2}|^2,$$

which is an improved version of the monotonicity of the gradient of $w \mapsto \frac{1}{p}|w|^p$.

Instead, if we consider $H(v) = |v| + \frac{1}{q}|v|^q$, we get the following result.

Proposition 3.5. *Let H be given by $H(v) = |v| + \frac{1}{q}|v|^q$ and $H^*(w) = \frac{1}{p}(|w|-1)_+^p$. Suppose that Ω is the flat torus and $f \in W^{1,p'}(\Omega)$. Let $\bar{\mathbf{v}}$ is a solution of* $\min A$ *and $\bar{u}$ a solution of* $\min B$ *(equivalently, suppose that $\bar{u}$ solves $\nabla \cdot ((\nabla \bar{u} - 1)_+^{p-1}) = f$). Then $\bar{\mathbf{v}}^{q/2} = (\nabla \bar{u} - 1)_+)^{p/2} \in H^1$.*

This result is the same proven in [8], where it was proven with PDE methods, and does not seem easy to improve. The equation $\nabla \cdot ((\nabla u - 1)_+^{p-1}) = f$, which can be written,

$$\nabla \cdot ((|\nabla u| - 1)_+^{p-1} \frac{\nabla u}{|\nabla u|}) = f,$$

is very degenerate in the sense that the coefficient $(|\nabla u| - 1)_+^{p-1}/|\nabla u|$ vanishes on the whole set where $|\nabla u| \leq 1$.

This equation and the corresponding minimization problems arise in traffic congestion (see [3, 13, 8] and Section 4.6) and the choice of the function H is very natural: we need a superlinear function of the form $H(v) = |v|h(|v|)$, with $h \geq 1$). This automatically implies the degeneracy.

We now move to the Poisson equation $\Delta u = f$, corresponding to the minimization of $\int \frac{1}{2}|\nabla u|^2 + fu$, and hence to $H(v) = \frac{1}{2}|v|^2$ and $H^*(w) = \frac{1}{2}|w|^2$. It is possible to treat this case by the same techniques as in the degenerate case above, but the result is disappointing. Indeed, from these techniques we just obtain $f \in H^1 \Rightarrow \nabla u \in H^1$, while it is well-known that $f \in L^2$ should be enough for the same result. Yet, with some more attention it is also possible to treat the L^2 case.

Proposition 3.6. *Suppose that Ω is the flat torus and $\Delta \bar{u} = f \in L^2(\Omega)$. Then $\nabla \bar{u} \in H^1$.*

Proof. We use the variational framework we presented before, with $H(v) = \frac{1}{2}|v|^2$. We have

$$\frac{1}{2}||\nabla u_h - \nabla \bar{u}||_{L^2}^2 \leq g(h). \tag{3.1}$$

Now, set $\omega_t := \sup\{||\nabla u_h - \nabla u||_{L^2} \ : \ |h| \leq t\}$. From (3.1) we have

$$\omega_t^2 \leq \sup_{h:|h|\leq t} 2g(h) \leq 2t \sup_{h:|h|\leq t} |\nabla g(h)|.$$

From $\nabla g(h)\nabla g(h) - \nabla g(0) = \int f(\nabla u_h - \nabla \bar{u})$ we deduce $|\nabla g(h)| \leq ||f||_{L^2}||\nabla u_h - \nabla \bar{u}||_{L^2} \leq ||f||_{L^2}\omega_t$, hence $\omega_t^2 \leq 2t||f||_{L^2}\omega_t$, which implies $\omega_t \leq 2t||f||_{L^2}$ and hence $\nabla \bar{u} \in H^1$. □

As we already pointed out, the result stating that solutions u of $\Delta_p \bar{u} = f \in W^{1,p'}(\Omega)$ satisfy $(\nabla \bar{u})^{p/2} \in H^1$ is very classical but not very satisfactory in the limit case $p = 2$. This is why we also look at the following other classical result. We recall before stating it some useful definitions of fractional Sobolev spaces (see, for instance, [**?**]).

Box 3.1. **– Memo** – Fractional Sobolev spaces

When Ω is bounded and its diameter is R, if $1 < p < +\infty$ and $0 < s < 1$, the space $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) \, : \, [u]^p_{s,p} := \int_{B(0,R)} \frac{||u_\delta - u||^p_{L^p}}{\delta^{d+sp}} d\delta < +\infty \right\}$$

and its norm is given by $||u||_{L^p} + [u]_{s,p}$. The space H^s is defined as $W^{s,2}$.

Note that an inequality of the form $||u_\delta - u||_{L^p} \le C|\delta|^s$ implies $u \in W^{s',p}$ for every $s' < s$. Also note that the Hilbert case $p = 2$ also enjoys an alternative definition in terms of the Fourier transform. indeed, we have $u \in H^s$ if and only if $\xi \mapsto |\xi|^s \hat{u}(\xi) \in L^2$ and $[u]_{s,2}$ is equivalent to the L^2 norm of $\xi \mapsto |\xi|^s \hat{u}(\xi)$.

Proposition 3.7. *Suppose that Ω is the flat torus and $\Delta_p \bar{u} = f \in L^q(\Omega)$, with $p > 2$.Then $||(\nabla u_h)^{p/2} - (\nabla \bar{u})^{p/2}||_{L^2} \le C|h|^{q/2}$, which implies in particular $(\nabla \bar{u})^{p/2} \in H^s$ for $s < q/2 < 1$.*

Proof. We use the same strategy as in Proposition 3.6. For simplicity, we set $G := (\nabla u)^{p/2}$. As in Proposition 3.6, we set $\omega_t := \sup_{h:|h|\le t} ||G_h - G||_{L^2}$. We have $||G_h - G||^2_{L^2} \le Cg(h)$, which implies

$$\omega_t^2 \le Ct \sup_{h:|h|\le t} |\nabla g(h) - \nabla g(0)| \le Ct||f||_{L^q} \sup_{h:|h|\le t} ||\nabla u_h - \nabla \bar{u}||_{L^p}.$$

From the α-Hölder behaviour of the vector map $w \mapsto w^\alpha$ in $\mathbb{R}^d$ (see Lemma 3.8 below), with $\alpha = 2/p < 1$, we deduce, using $\nabla u = G^\alpha$,

$$||\nabla u_h - \nabla \bar{u}||^p_{L^p} = \int |\nabla u_h - \nabla \bar{u}|^p \, dx \le \int |G_h - G|^2 \, dx = ||G_h - G||^2_{L^2}.$$

Hence, we have

$$\omega_t^2 \le Ct||f||_{L^q} \omega_t^{2/p},$$

which implies

$$\omega_t^{2/q} \le Ct||f||_{L^q},$$

i.e. the claim □

Lemma 3.8. *For $0 < \alpha < 1$, the map $w \mapsto w^\alpha$ is α-Hölder continuous in $\mathbb{R}^d$.*

Proof. Let $a, b \in \mathbb{R}^d$. We write

$$|a^\alpha - b^\alpha| = \left| |a|^\alpha \frac{a}{|a|} - |a|^\alpha \frac{b}{|b|} + |a|^\alpha \frac{b}{|b|} - |b|^\alpha \frac{b}{|b|} \right| \le |a|^\alpha \left| \frac{a}{|a|} - \frac{b}{|b|} \right| + ||a|^\alpha - |b|^\alpha| .$$

For the second term in the r.h.s., we use the α-Hölder behaviour of $t \mapsto t^\alpha$ in $\mathbb{R}_+$ and get

$$||a|^\alpha - |b|^\alpha| \le ||a| - |b||^\alpha \le |a - b|^\alpha.$$

For the first term in the r.h.s., we use the inequality

$$\left|\frac{a}{|a|}-\frac{b}{|b|}\right| = \left|\frac{a}{|a|}-\frac{b}{|a|}+\frac{b}{|a|}-\frac{b}{|b|}\right| \le \frac{|a-b|}{|a|}+|b|\frac{||b|-|a||}{|a||b|} \le 2\frac{|a-b|}{|a|}$$

and get

$$|a|^{\alpha}\left|\frac{a}{|a|}-\frac{b}{|b|}\right| \le 2|a|^{\alpha-1}|a-b|.$$

If we choose a to be such $|a|\ge|b|$ (which is possible w.l.o.g.), we have $2|a|\ge|a-b|$ and hence $2^{\alpha-1}|a|^{\alpha-1}\le|a-b|^{\alpha-1}$, i.e. $2|a|^{\alpha-1}|a-b|\le 2^{2-\alpha}|a-b|^{\alpha}$.

Summing up, we have (without pretending that this constant is optimal)

$$|a^{\alpha}-b^{\alpha}| \le (2^{2-\alpha}+1)|a-b|^{\alpha}. \quad \square$$

Remark 3.9. Note that the result of Proposition 3.7 is also classical, and quite sharp. Indeed, one can informally consider the following example. Take $u(x)\approx|x|^r$ as $x\approx 0$ (and then multiply times a cut-off function out of 0). In this case we have

$$\nabla u(x)\approx|x|^{r-1}, \quad (\nabla u(x))^{p-1}\approx|x|^{(r-1)(p-1)}, \quad f(x):=\Delta_p u(x)\approx|x|^{(r-1)(p-1)-1}.$$

Hence, $f\in L^q$ if and only if $((r-1)(p-1)-1)q>-d$, i.e. $(r-1)p-q>-d$. On the other hand, the fractional Sobolev regularity can be observed by considering that "differentiating s times" means subtracting s from the exponent, hence

$$(\nabla u(x))^{p/2}\approx|x|^{p(r-1)/2}\Rightarrow(\nabla u)^{p/2}\in H^s\Leftrightarrow|x|^{p(r-1)/2-s}\in L^2\Leftrightarrow p(r-1)-2s>-d.$$

If we want this last condition to be true for arbitrary $s<q/2$, then it amounts to $p(r-1)-q>-d$, which is the same condition as above.

3.3 Variants – Local regularity and dependence on x

In the previous section we only provided global Sobolev regularity results on the torus. This guaranteed that we could do translations without boundary problems, and that by change-of-variable, the term $\int H(\nabla u_h)\,dx$ did not actually depend on h. We now provide a result concerning local regularity. As the result is local, boundary conditions should not be very important. Yet, as the method stays anyway global, we need to fix them and be precise on the variational problems that we use. We will consider, as usual, variational problems without any boundary constraint, which correspond to PDEs with no-flux boundary conditions. Since, as we said, boundary conditins should play no role, with some work it is also possible to modify them, but we will not develop these considerations here.

We will only provide the following result, in the easiest case $p=2$.

Theorem 3.10. *Let H,H^*,j and j_* satisfy Hyp1,2,3 with $p=2$. Suppose $f\in H^1$. Suppose also $H^*\in C^{1,1}$ and $j_*\in C^{0,1}$. Suppose $\nabla\cdot(\nabla H^*(\nabla\bar u))=f$ in Ω with no-flux boundary conditions on $\partial\Omega$. Then, $j_*(\nabla\bar u)\in H^1_{loc}$.*

Proof. The condition $\nabla\cdot\nabla H^*(\nabla\bar u)=f$ is equivalent to the fact that $\bar u$ is solution of

$$\min\left\{\int_\Omega H^*(\nabla u)\,\mathrm{d}x+\int fu\,\mathrm{d}x\;:\;u\in W^{1,p}(\Omega)\right\}.$$

We will follow the usual duality strategy as in the rest of the section. Yet, in order not to have boundary problems, we need to use a cut-off function $\eta\in C^\infty_c(\Omega)$ and define

$$u_h(x)=\bar u(x+h\eta(x))$$

(note that for small h we have $x+h\eta(x)\in\Omega$ for all $x\in\Omega$). In this case it is no longer true that $\tilde g(h):=\int H^*(\nabla u_h)\,\mathrm{d}x=\int H^*(\nabla\bar u)\,\mathrm{d}x$. If this term is not constant in h, then we need to prove that it is a $C^{1,1}$ function of h. To do this, and to avoid differentiating $\nabla\bar u$, we use a change-of-variable. Set $y=x+h\eta(x)$. We have $\nabla(u_h)(x)=(\nabla\bar u)(y)(I+h\otimes\nabla\eta(x))$, hence

$$\tilde g(h)=\int H^*(\nabla u_h)\,\mathrm{d}x=\int H^*(\nabla\bar u(y)+(\nabla\bar u(y)\cdot h)\nabla\eta(x))\frac{1}{1+h\cdot\nabla\eta(x)}\,\mathrm{d}y,$$

where $x=X(h,y)$ is a function of h and y obtained by inverting $x\mapsto x+h\eta(x)$ and we used $\det(I+h\otimes\nabla\eta(x))=1+h\cdot\nabla\eta(x)$. The function X is C^∞ by the implicit function theorem, and all the other ingredient of the above integral are at least $C^{1,1}$ in h. This proves that $\tilde g$ is $C^{1,1}$. The regularity of the term $g(h)=\int fu_h$ should also be considered. Differentiating once we get $\nabla g(h)=\int f(x)\nabla\bar u(x+h\eta(x))\eta(x)\,\mathrm{d}x$. To differentiate once more, we use the same change-of-variable, thus getting

$$\nabla g(h)=\int f(X(h,y))\nabla\bar u(y)\eta(X(h,y))\frac{1}{1+h\cdot\nabla\eta(x)}\,\mathrm{d}y.$$

From $y=X(h,y)+h\eta(X(h,y))$ we get a formula for $D_hX(h,y)$, i.e. $0=D_hX(h,y)+\eta(X(h,y))I+h\otimes\nabla\eta(\eta(X(h,y))D_hX(h,y)$. This allows to differentiate once more the function g and proves $g\in C^{1,1}$.

Finally, we come back to the duality estimate. What we can easily get is

$$c||j_*(\nabla(u_h))-j_*(\nabla\bar u)||^2_{L^2}\le g(h)+\tilde g(h)=O(|h|^2).$$

The problem is that $j_*(\nabla(u_h))$ is not the translation of $j_*(\nabla\bar u)$! Yet, it is almost as if it was a translation. Indeed, if we put the subscript h every time that we compose with $x+h\eta(x)$, we have

$$\nabla(u_h)=(\nabla\bar u)_h+h\cdot(\nabla\bar u)_h\eta.$$

Since j_* is supposed to be Lipschitz continuous, then

$$|j_*(\nabla(u_h))-j_*((\nabla\bar u)_h)|\le C|h||\nabla\bar u|_h\eta.$$

Hence, we have

$$||j_*((\nabla\bar u)_h)-j_*(\nabla\bar u)||_{L^2}\le||j_*(\nabla(u_h))-j_*(\nabla\bar u)||_{L^2}+C|h|||\nabla u||_{L^2},$$

which is enough to show that this increment si of order $|h|$, since $\bar u\in H^1$ (this depends on the fact that H^* is quadratic). Hence, as in Lemma 3.1 (4), we get $j_*(\nabla\bar u)\in H^1$. □

The duality theory has been presented in the case where H and H^* could also depend on x, while for the moment regularity results have only be prensented under the assumption that they not. We now consider a small variant and look at how to handle the following particular case, corresponding to the minimization problem

$$\min\left\{\frac{1}{p}\int_\Omega a(x)|\nabla u(x)|^p\,\mathrm{d}x+\int f(x)u(x)\,\mathrm{d}x\;:\;u\in W^{1,p}(\Omega)\right\}. \tag{3.2}$$

We will use $\Omega=\mathbb{T}^d$ to avoid cumulating difficulties (boundary issues and dependence on x). Note that the PDE corresponding to the above minimization problem is $\nabla\cdot(a(\nabla u)^{p-1})=f$.

First, we need to compute the transform of $w\mapsto H^*(w):=\frac{a}{p}|w|^p$. Set $b=a^{1/(p-1)}$. It is easy to obtain $H(v)=\frac{1}{bq}|v|^q$. Also, we can check (just by scaling the inequality of Lemma 3.2, that we have

$$\frac{1}{bq}|v|^q+\frac{b^{p-1}}{p}|w|^p\geq v\cdot w+b^{p-1}\left|w^{p/2}-\frac{v^{q/2}}{b^{p/2}}\right|^2.$$

In particular, if we suppose that $a(x)$ is bounded from below by a positive constant, and we set $H^*(x,w)=\frac{a(x)}{p}|w|^p$ then we get

$$H(x,v)+H^*(x,w)\geq v\cdot w+c|j(x,v)-j_*(w)|^2$$

where $j_*(w)=w^{p/2}$.

We can now prove the following theorem.

Theorem 3.11. *Suppose $f\in W^{1,p'}$ and $a\in\mathrm{Lip}, a\geq a_0$, and let $\bar u$ be the minimizer of (3.2). Then $G:=\nabla\bar u^{p/2}\in H^1$.*

Proof. Our usual computations show that

$$c||G_h-G||^2_{L^2}\leq g(h)+\tilde g(h),$$

where $g(h)=\int fu_h-\int f\bar u$ and $\tilde g(h)=\int\frac{a(x)}{p}|\nabla u_h|^p-\int\frac{a(x)}{p}|\nabla\bar u|^p$. With our assumptions, $g\in C^{1,1}$. As for $\tilde g(h)$, we write

$$\int\frac{a(x)}{p}|\nabla u_h|^p=\int\frac{a(x-h)}{p}|\nabla\bar u|^p$$

and hence

$$\nabla\tilde g(h)=\int\frac{\nabla a(x-h)}{p}|\nabla\bar u|^p=\int\frac{\nabla a(x)}{p}|\nabla u_h|^p.$$

Hence,

$$|\nabla\tilde g(h)-\nabla\tilde g(0)|\leq\int\frac{|\nabla a(x)|}{p}\,||\nabla u_h|^p-|\nabla\bar u|^p|\leq C\int||G_h|^2-|j_*|^2|\leq C||G_h-G||_{L^2}||G_h+G||_{L^2}.$$

Here we used the L^∞ bound on $|\nabla a|$. Then, from the lower bound on a, we also know $G \in L^2$, hence we get $|\nabla\tilde{g}(h) - \nabla\tilde{g}(0)| \le C||G_h - G||_{L^2}$.

Now, we define as usual $\omega_t := \sup_{h=|h|\le t} ||G_h - G||_{L^2}$ and we get

$$\omega_t^2 \le C \sup_{h=|h|\le t} g(h) + \tilde{g}(h) \le Ct \sup_{h=|h|\le t} |\nabla g(h) + \nabla\tilde{g}(h)|$$
$$= Ct \sup_{h=|h|\le t} |\nabla g(h) - \nabla g(0) + \nabla\tilde{g}(h) - \nabla\tilde{g}(0)| \le Ct^2 + Ct\omega_t,$$

which allows to deduce $\omega_t \le Ct$ and hence $G \in H^1$. □

We also provide the following theorem, which is also interesting for $p = 2$.

Theorem 3.12. *Suppose $p \ge 2$, $f \in L^q$ and $a \in \mathrm{Lip}, a \ge a_0$, and let $\bar{u}$ be the minimizer of* (3.2)*. Then $jG = \nabla\bar{u}^{p/2}$ satisfies $||G_h - G||_{L^2} \le C|h|^{q/2}$. In particular, $G \in H^1$ for $p = 2$ and $G \in H^s$ for all $s < q/2$ for $p > 2$.*

Proof. The only difference with the previous case is that we cannot say that g is $C^{1,1}$ but we should stick to the computation of ∇g. We use as usual

$$|\nabla g(h) - \nabla g(0)| \le ||f||_{L^q}||\nabla u_h - \nabla\bar{u}||_{L^p}.$$

As we are forced to let the norm $||\nabla u_h - \nabla\bar{u}||_{L^p}$ appear, we will use it also in $\tilde{g}$. Indeed, we can observe that we can estimate

$$|\nabla\tilde{g}(h) - \nabla\tilde{g}(0)| \le \int \frac{|\nabla a(x)|}{p} ||\nabla u_h|^p - |\nabla\bar{u}|^p| \le C\int (|\nabla u_h|^{p-1} + |\nabla\bar{u}|^{p-1})|\nabla u_h - \nabla\bar{u}|$$
$$\le C||\nabla\bar{u}^{p-1}||_{L^q}||\nabla u_h - \nabla\bar{u}||_{L^p}.$$

We then use $||\nabla\bar{u}^{p-1}||_{L^q} = ||\nabla\bar{u}||_{L^p}^{p-1}$ and conclude

$$|\nabla\tilde{g}(h) - \nabla\tilde{g}(0)| \le C||\nabla u_h - \nabla\bar{u}||_{L^p}.$$

This gives, defining ω_t as usual,

$$\omega_t \le Ct \sup_{h=|h|\le t} |\nabla g(h) - \nabla g(0) + \nabla\tilde{g}(h) - \nabla\tilde{g}(0)| \le Ct \sup_{h=|h|\le t} ||\nabla u_h - \nabla\bar{u}||_{L^p}$$

and hence

$$\omega_t^2 \le Ct\,\omega_t^{2/p}$$

as in Proposition 3.7. □

Chapter 4

A proof of Fenchel-Rockafellar's duality

In this section we want to take advantage of the technique developed in Section 4.3 for the precisa case of minimal flow problems in order to prove a general abstract version of the Fenchel-Rockafellar duality theorem. For simplicity, we will stick to the case where all spaces are reflexive, so that the role of the function in the primal and in the dual problems are completely symmetric. We will start from the following statement.

Theorem 4.1. *Suppose that X and Y are separable reflexive normed vector spaes, that $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ are convex and lower-semicontinuous functions, and that $A : X \to Y$ is a continuous linear mapping. Suppose that g is bounded from below and f coercive. Then we have*

$$\min \{f(x) + g(Ax) : x \in X\} = \sup \left\{-g^*(\xi) - f^*(-A^t\xi) : \xi \in Y'\right\},$$

where the existence of the minimum on the right-hand side is part of the claim.

Proof. We will define a function $\mathscr{F} : Y \to \mathbb{R} \cup \{+\infty\}$ via

$$\mathscr{F}(p) := \min \{f(x) + g(Ax + p) : x \in X\}.$$

The existence of the minimum is a consequence of the following fact: for any sequence (x_n, p_n) with $f(x_n) + g(Ax_n + p_n) \leq C$ the sequence x_n is bounded. This boundedness comes from the lower bound on g and from the coercive behavior of f. Once we know this, we can use $p_n = p$, take a minimizing sequence x_n for fixed p, and extract a weakly converging subsequence $x_n \rightharpoonup x$ using the Banach-Alaoglu Theorem. We also have $Ax_n + p \rightharpoonup Ax + p$ and the semicontinuity of f and g provide the minimality of x (since, being convex, f and g are both l.s.c. for the strong and the weak convergence, in X and Y, respectively).

We now compute $\mathscr{F}^*: Y' \to \mathbb{R}\cup\{+\infty\}$:

$$\begin{aligned}\mathscr{F}^*(\xi) &= \sup_p \langle \xi, p\rangle - \mathscr{F}(p)\\ &= \sup_{p,x}\langle \xi, p\rangle - f(x) - g(Ax+p)\\ &= \sup_{y,x}\langle \xi, y-Ax\rangle - f(x) - g(y)\\ &= \sup_y\langle \xi, y\rangle - g(y) + \sup_x\langle -A^t\xi, x\rangle - f(x)\\ &= g^*(\xi) + f^*(-A^t\xi).\end{aligned}$$

Now we use, as we did in Section 4.3, $\mathscr{F}^{**}(0) = \sup -\mathscr{F}^*$, which proves the claim, as soon as we prove that $\mathscr{F}$ is convex and l.s.c.

The convexity of $\mathscr{F}$ is easy. We just need to take $p_0, p_1 \in Y$, and define $p_t := (1-t)p_0 + tp_1$. Let x_0, x_1 be optimal in the definition of $\mathscr{F}(p_0)$ and $\mathscr{F}(p_1)$, i.e. $\int f(x_i) + g(Ax_i + p_i) = \mathscr{F}(p_i)$, and set $x_t := (1-t)x_0 + tx_1$. We have

$$\mathscr{F}(p_t) \leq f(x_t) + g(Ax_t + p_t) \leq (1-t)\mathscr{F}(p_0) + t\mathscr{F}(p_1),$$

and the convexity is proven.

For the semicontinuity, we take a sequence $p_n \to p$ in Y. We can suppose $\mathscr{F}(p_n) \leq C$ otherwise there is nothing to prove. Take the corresponding optimal points x_n and, applying the very first observation of this proof, we obtain $||x_n|| \leq C$. We can extract a subsequence such that $\lim_k \mathscr{F}(p_{n_k}) = \liminf_n \mathscr{F}(p_n)$ and $x_{n_k} \rightharpoonup x$. The semicontinuity of f and g provides

$$\mathscr{F}(p) \leq f(x) + g(Ax+p) \leq \liminf_k f(x_{n_k}) + g(Ax_{n_k} + p_{n_k}) = \lim_k \mathscr{F}(p_{n_k}) = \liminf_n \mathscr{F}(p_n),$$

which gives the desired result. □

We now note that, if g is not bounded from below, it is always possible to remove a suitable linear function from it so as to make it bounded from below, since all convex and l.s.c. functions are bounded from below by an affine function. We can then define $\tilde{g}$ via $\tilde{g}(y) = g - \langle \xi_0, y\rangle$ for a suitable ξ_0, and guarantee $\inf \tilde{g} > -\infty$. In order not to change the value of the primal problem we also need to modify f into $\tilde{f}$ defined via $\tilde{f}(x) := f + \langle A^t\xi_0, x\rangle$, so that

$$\tilde{f}(x) + \tilde{g}(Ax) = f(x) + \langle A^t\xi_0, x\rangle + g(Ax) - \langle \xi_0, Ax\rangle = f(x) + g(Ax).$$

Moreover, we can compute what changes in the dual problem. Is it true that we have $\sup_\xi -g^*(\xi) - f^*(-A^t\xi) = \sup_\xi -\tilde{g}^*(\xi) - \tilde{f}^*(-A^t\xi)$?

In order to do this, we need to compute the Legendre transform of $\tilde{f}$ and $\tilde{g}$. A general, and easy, fact, that is proposed as an exercise (see Exercise 6.5) states that subtracting a linear function translates into a translation on the Legendre transform. We then have

$$\tilde{g}^*(\xi) = g^*(\xi + \xi_0); \quad \tilde{f}^*(\zeta) = f^*(\zeta - A^t\xi_0)$$

and then

$$\tilde{g}^*(\xi) + \tilde{f}^*(-A^t\xi) = g^*(\xi + \xi_0) + f^*(-A^t(\xi + \xi_0))$$

and a simple change of variable $\xi \mapsto \xi + \xi_0$ shows that the sup has not changed. This shows that the duality result is not affected by this reformulation in terms of $\tilde{f}$ and $\tilde{g}$. It is then enough, for the duality to hold, that the assumptions of Theorem 4.1 are satisfied by $(\tilde{f}, \tilde{g})$ instead of (f, g). Since we chose ξ_0 on purpose in order to have $\tilde{g}$ lower bounded, we only need now to require that $\tilde{f}$ is coercive. Not that this would be the case if f was superlinear, as it would stay superlinear after adding any linear function, but it is not automatic when speaking of a generic coercive function.

The condition on ξ_0 such that, at the same time $\tilde{g}$ is bounded from below and $\tilde{f}$ superlinear can be more easily translated in terms of f^* and g^*. We can indeed state the following propositon.

Proposition 4.2. *Suppose that X and Y are separable reflexive normed vector spaes, that $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ are convex and lower-semicontinuous functions, and that $A : X \to Y$ is a continuous linear mapping. Suppose that there exists $\xi_0 \in Y'$ such that $g^*(\xi_0) < +\infty$ and that f^* is continuous and finite at $A^t\xi_0$. Then we have*

$$\min \{f(x) + g(Ax) \,:\, x \in X\} = \sup \left\{-g^*(\xi) - f^*(-A^t\xi) \,:\, \xi \in Y'\right\},$$

where the existence of the minimum on the right-hand side is part of the claim.

Proof. The condition $g^*(\xi_0) < +\infty$ means $\tilde{g}^*(0) < +\infty$, which means that $\tilde{g}$ is bounded from below.

The condition on f^* at $A^t\xi_0$ translates into the same condition for $\tilde{f}^*$ at 0, and we know that a function is coercive if and only if its Legendre transform is bounded on a neighborhood of 0. This means that $\tilde{f}$ is coercive.

We then conclude that we can apply Theorem 4.1 to $(\tilde{f}, \tilde{g})$ instead of (f, g), which provides the desired result. □

We can also deduce the following statement, which is probably the most standard formulation of the Fenchel-Rockafellar duality theorem, even if we only state it for reflexive spaces.

Theorem 4.3. *Suppose that X and Y are separable reflexive normed vector spaes, that $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ are convex and lower-semicontinuous functions, and that $A : X \to Y$ is a continuous linear mapping. Suppose that there exists $x_0 \in X$ such that $f(x_0) < +\infty$ and that g is continuous and finite at Ax_0. Then we have*

$$\inf \{f(x) + g(Ax) \,:\, x \in X\} = \max \left\{-g^*(\xi) - f^*(-A^t\xi) \,:\, \xi \in Y'\right\},$$

where the existence of the minimum on the right-hand side is part of the claim.

Proof. The proof is straightforward once we realize that we can interchange f with g^*, g with f^*, and A with A^t in the statement of Proposition 4.2. □

Chapter 5

Discussion – From Optimal Transport to congested traffic and Mean field Games

In this section we want to underline the connections of some of the problems studied in this chapter with problems form optimal transport theory, and with their variants involving traffic congestions. We will also see that related problems can be dealt with in similar ways.

We start from a very brief presentation of the optimal transport problem, for which we refer for instance to the books [32] and [30].

The starting point for the whole theory is the following problem proposed by the French mathemacian Gaspard Monge in 1781 ([26]), that we present here in modern language. Given two metric spaces X and Y, two probability measures $\mu \in \mathscr{P}(X), \nu \in \mathscr{P}(Y)$, and a cost function $c : X \times Y \to \mathbb{R}$, we look for a map $T : X \to Y$ pushing the first one onto the other, i.e. $T_\#\mu = \nu$, and minimizing, among such maps, the integral

$$M(T) := \int_X c(x, T(x))\, \mathrm{d}\mu(x).$$

Box 5.1. **– Memo** – Image measures.

An important notion in measure theory is that of image measure. Given a measure $\mu \in \mathscr{M}(X)$ on a space X and a measurable function $T : X \to Y$, we define the *image measure* or *push-forward*of μ through T as a measure $T_\#\mu$ on Y characterized by

$$(T_\#\mu)(A) := \mu(T^{-1}(A))$$

for every measurable set $A \subset Y$. Equivalently, the definition can be given in terms of test functions, and in this case we require

$$\int_Y \phi \; \mathrm{d}(T_\#\mu) := \int_X \phi \circ T \, \mathrm{d}\mu$$

for every bounded measurable function $\phi : Y \to \mathbb{R}$ or, equialently (by approximation) for every $\phi \in C_b(Y)$. The case of sets can be recovered using $\phi = \mathbb{1}_A$.

An important observation si that image measures pass to the limit through a.e. convergence (if $T_n \to T$ μ-a.e. then $(T_n)_\# \mu \overset{*}{\rightharpoonup} T_\# \mu$) but not through weak convergence (for instance consider $X = [0, 2\pi]$, $\mu = \mathscr{L}^1$, $T_n(x) = \sin(nx)$ and $T(x) = 0$).

In probabilistic language, the image measure is the law of a random variable (X plays the role of the probability space and Y is the space where the random value takes its values).

In the problem considered by Monge X and Y were subsets of the Euclidean space, μ and ν were absolutely continuous, and $c(x,y) = |x-y|$ was the Euclidean distance. Roughly speaking, this means that we have a collection of particles, we know how they are distributed (this is the measure μ), and they have to be moved so that they arrange according to a new prescribed distribution ν. The movement has to be chosen so as to minimize the average displacement, and the map T describes the movement.

Monge proved many properties of the optimal transport map T in the case he was interested in, but never cared to prove that such a map existed. In this sense, the problem stayed with no solution till the reformulation that Leonid Kantorovich gave in 1942, [17]. His formulation consists in the problem

$$\text{(K)} \qquad \min \left\{ \int_{X \times Y} c \, \mathrm{d}\gamma \,:\, \gamma \in \Pi(\mu, \nu) \right\}, \tag{5.1}$$

where $\Pi(\mu,\nu)$ is the set of the so-called *transport plans*, i.e. $\Pi(\mu,\nu) = \{\gamma \in \mathscr{P}(X \times Y) : (\pi_x)_\# \gamma = \mu, (\pi_y)_\# \gamma = \nu, \}$ where π_x and π_y are the two projections of $X \times Y$ onto the two factors X and Y. These probability measures over $X \times Y$ are an alternative way to describe the displacement of the particles of μ: instead of saying, for each x, which is the destination $T(x)$ of the particle originally located at x, we say for each pair (x,y) how many particles go from x to y. It is clear that this description allows for more general movements, since from a single point x particles can a priori move to different destinations y. If multiple destinations really occur, then this movement cannot be described through a map T.

If we define the map $(id,T) : X \to X \times Y$ via $(id,T)(x) := (x, T(x))$, it can be easily checked that $\gamma_T := (id,T)_\# \mu$ belongs to $\Pi(\mu,\nu)$ if and only if T pushes μ onto ν and the functional $\int c \, \mathrm{d}\gamma_T$ takes the form $\int c(x, T(x)) \, \mathrm{d}\mu(x)$, thus generalizing Monge's problem.

This generalized problem by Kantorovich is much easier to handle than the original one proposed by Monge and the Direct Method of Calculus of Variations proves that a minimum does exist (at least when c is l.s.c. and bounded from below). This is not the case for the original Monge problem, since in general we can only obtain weakly converging minimizing sequences $T_n \rightharpoonup T$ but the limit T could have a different image measure than ν. Hence, the general strategy to prove the existence of an optimizer in the Monge problem consists in first considering the minimizer γ of the Kantorovich problem and then trying to prove that it is actually of the form $(id,T)_\# \mu$. This is possible (see, for instance, Chapter 1 in [30]) under some conditions on μ and on the cost c. Anyway, we will not consider this quesiton here. Another important fact which makes the Katorovich problem easier to consider is that it is convex (it is a linear optimization problem under linear constraints) and hence an important tool will be duality theory.

As we already did many times in this chapter, we first find a formal dual problem, by means of an inf-sup exchange. To this aim, we first express the constraint $\gamma \in \Pi(\mu,\nu)$ in the following way : notice that, if γ is a non-negative measure on $X \times Y$, then we have

$$\sup_{\phi \in C(X),\, \psi \in C(Y)} \int \phi \,\mathrm{d}\mu + \int \psi \,\mathrm{d}\nu - \int (\phi(x) + \psi(y))\,\mathrm{d}\gamma = \begin{cases} 0 & \text{if } \gamma \in \Pi(\mu,\nu) \\ +\infty & \text{otherwise} \end{cases}.$$

Hence, we can remove the constraints on γ if we add the previous sup. Then, we look at the problem we get by interchanging the inf in γ and the sup in ϕ, ψ:

$$\sup_{\phi,\psi} \int \phi \,\mathrm{d}\mu + \int \psi \,\mathrm{d}\nu + \inf_{\gamma} \int (c(x,y) - (\phi(x) + \psi(y)))\,\mathrm{d}\gamma.$$

We can then re-write the inf in γ as a constraint on ϕ and ψ, since we have

$$\inf_{\gamma \geq 0} \int (c(x,y) - (\phi(x) + \psi(y)))\,\mathrm{d}\gamma = \begin{cases} 0 & \text{if } \phi(x) + \psi(y) \leq c(x,y) \text{ for all } (x,y) \\ -\infty & \text{otherwise} \end{cases}.$$

The validity of the exchange of inf and sup can be proven with the techniques that we developed in Sections 4.3 and 4.5. A precise proof can be found, for instance, in Section 1.6 of [30]. The spaces are not reflexive but the proof works since it is (magically) possible to prove a compactness result for the functions ϕ and ψ when x and Y are compact and c is continous (indeed, it is possible to restrict to functions sharing the same modulus of continuity of c).

We then have the following dual optimization problem:

$$(\mathrm{D}) \qquad \max\left\{ \int_X \phi \,\mathrm{d}\mu + \int_Y \psi \,\mathrm{d}\nu \;:\; \phi \in C(X), \psi \in C(Y), \phi \oplus \psi \leq c \right\}, \tag{5.2}$$

where the notation $\phi \oplus \psi$ stands for the two-variable function $(x,y) \mapsto \phi(x) + \psi(y)$.

We now look in particular at the case where c is the Euclidean distance (i.e. $X = Y \subset \mathbb{R}^d$ and $c(x,y) = |x-y|$). A first observation concerning the dual problem is that for any ϕ one can choose the best possible (the largest) function ψ which satisfies, together with ϕ, the constraint $\phi(x)+\psi(y) \leq c(x,y)$. Such a function is given by

$$\psi(y) = \inf_x |x-y| - \phi(x)$$

and it belongs to Lip_1. This means that we can restrict the maximization in the dual problems to $\psi \in \mathrm{Lip}_1$ and, analoguously, to $\phi \in \mathrm{Lip}_1$. Not only, if we now know $\phi \in \mathrm{Lip}_1$ there is an easy epxression for the function $\psi(y) = \inf_x |x-y| - \phi(x)$, which is just $\psi = -\phi$ (see Exercise 6.16). In this case the dual problem hence becomes

$$(\mathrm{D-dist}) \qquad \max\left\{ \int_X \phi \,\mathrm{d}(\mu - \nu) \;:\; \phi \in \mathrm{Lip}_1(X) \right\}.$$

Another important point of the optimal transport problem when the cost is given by the Euclidean distance is its connection with the minimal-flow problem

$$\text{(B)} \qquad \min\left\{||\mathbf{v}|| \;:\; \mathbf{v}\in\mathscr{M}^d(X);\; \nabla\cdot\mathbf{v}=\mu-\nu\right\}, \tag{5.3}$$

where $||\mathbf{v}||$ denotes the mass of the vector measure $\mathbf{v}$ and the divergence condition is to be read in the weak sense, with no-flux boundary conditions, i.e. $-\int\nabla\phi\cdot \mathrm{d}\mathbf{v} = \int\phi\,\mathrm{d}(\mu-\nu)$ for any $\phi\in C^1(X)$. Indeed, if X is convex then the value of this problem is equal to that of the optimal transport problem with $c(x,y)=|x-y|$ (in case X is not convex then the Euclidean distance has to be replaced with the geodesic distance inside X)

Box 5.2. **– Memo** – Vector measures

Definition - A finite vector measure λ on a set Ω is a map associating to every Borel subset $A\subset\Omega$ a value $\lambda(A)\in\mathbb{R}^d$ such that, for every disjoint union $A=\bigcup_i A_i$ (with $A_i\cap A_j=\emptyset$ for $i\neq j$), we have

$$\sum_i|\lambda(A_i)|<+\infty \quad \text{and} \quad \lambda(A)=\sum_i\lambda(A_i).$$

We denote by $\mathscr{M}^d(\Omega)$ the set of finite vector measures on Ω. To such measures we can associate a positive scalar measure $|\lambda|\in\mathscr{M}_+(\Omega)$ through

$$|\lambda|(A):=\sup\left\{\sum_i|\lambda(A_i)| \;:\; A=\bigcup_i A_i \text{ with } A_i\cap A_j=\emptyset \text{ for } i\neq j\right\}.$$

This scalar measure is called *total variation measure* of λ. Note that for simplicity we only consider the Euclidean norm on $\mathbb{R}^d$, and write $|\lambda|$ instead of $||\lambda||$ (a notation that we keep for the total mass of the total variation measure, see below), but the same could be defined for other norms as well.

The integral of a Borel function $\xi:\Omega\to\mathbb{R}^d$ w.r.t. λ is well-defined if $|\xi|\in L^1(\Omega,|\lambda|)$, is denoted $\int\xi\cdot\,\mathrm{d}\lambda$ and can be computed as $\sum_{i=1}^d\int\xi_i\,\mathrm{d}\lambda_i$, thus reducing to integrals of scalar functions according to scalar measures. It could also be defined as a limit of integral of piecewise constant functions.

Functional analysis facts - The quantity $||\lambda||:=|\lambda|(\Omega)$ is a norm on $\mathscr{M}^d(\Omega)$, and this normed space is the dual of $C_0(\Omega;\mathbb{R}^d)$, the space of continuous function on Ω vanishing at infinity, through the duality $(\xi,\lambda)\mapsto\int\xi\cdot\,\mathrm{d}\lambda$. This gives a notion of $\overset{*}{\rightharpoonup}$ convergence for which bounded sets in $\mathscr{M}^d(\Omega)$ are compact. As for scalar measures, we denote by $\rightharpoonup$ the weak convergence in duality with C_b functions.

A clarifying fact is the following.

Proposition - For every $\lambda\in\mathscr{M}^d(\Omega)$ there exists a Borel function $u:\Omega\to\mathbb{R}^d$ such that $\lambda=u\cdot|\lambda|$ and $|u|=1$ a.e. (for the measure $|\lambda|$). In particular, $\int\xi\cdot\,\mathrm{d}\lambda=\int(\xi\cdot u)\,\mathrm{d}|\lambda|$.

Sketch of proof - The existence of a function u is a consequence, via Radon-Nikodym Theorem, of $\lambda\ll|\lambda|$ (every A set such that $|\lambda|(A)=0$ obviously satisfies $\lambda(A)=0$). The condition $|u|=1$ may be proven by considering the sets $\{|u|<1-\varepsilon\}$ and $\{u\cdot e> a+\varepsilon\}$ for all hyperplane such that the unit ball B_1 is contained in $\{x\in\mathbb{R}^d\,:\,x\cdot e\leq a\}$ (and, actually, we have $B_1=\bigcap_{e,a}\{x\in\mathbb{R}^d\,:\,x\cdot e\leq a\}$, the intersection being reduced to a countable intersection). These sets must be negligible otherwise we have a contradiction on the definition of $|\lambda|$.

Last point: note that L^1 vector functions can be idenditifed with vector measures which are absolutely continuous w.r.t. the Lebesgue measure. Their L^1 norm coincides in this case with the norm in $\mathscr{M}^n(X)$

This minimal flow problem has been first proposed by Beckmann in [3], under the name of *continuous transportation model*. Such a problem is actually strongly related to the Kantorovich problem with the distance cost $c(x,y) = |x-y|$, as we will show in a while, even if Beckman was not aware of this, the two theories being developed essentially at the same time.

In order to see the connection between the two problems, we start from a formal computation. We re-write the constraint on $\mathbf{v}$ in (5.3) by means of the equality

$$\sup_\phi \int -\nabla\phi \cdot \,\mathrm{d}\mathbf{v} + \int \phi \,\mathrm{d}(\mu-\nu) = \begin{cases} 0 & \text{if } \nabla\cdot\mathbf{v} = \mu-\nu \\ +\infty & \text{otherwise} \end{cases}.$$

Hence one can write (5.3) as

$$\min_{\mathbf{v}} ||\mathbf{v}|| + \sup_\phi \int -\nabla\phi \cdot \,\mathrm{d}\mathbf{v} + \int \phi \,\mathrm{d}(\mu-\nu) = \sup_\phi \int \phi \,\mathrm{d}(\mu-\nu) + \inf_{\mathbf{v}} M(\mathbf{v}) - \int \nabla\phi \cdot \,\mathrm{d}\mathbf{v},$$

where inf and sup have been exchanged formally as in the previous computations. After that one notices that

$$\inf_{\mathbf{v}} ||\mathbf{v}|| - \int \nabla\phi \cdot \,\mathrm{d}\mathbf{v} = \inf_{\mathbf{v}} \int d|\mathbf{v}| \left(1 - \nabla\phi \cdot \frac{\mathrm{d}\mathbf{v}}{d|\mathbf{v}|}\right) = \begin{cases} 0 & \text{if } |\nabla\phi| \le 1 \\ -\infty & \text{otherwise} \end{cases}$$

and this leads to the dual formulation for (B) which gives

$$\sup_{\phi\,:\,|\nabla\phi|\le 1} \int_\Omega \phi \,\mathrm{d}(\mu-\nu).$$

Since this problem is exactly the same as (D-dist) (a consequence of the fact that Lip_1 functions are exactly those functions whose gradient is smaller than 1, when the domain is convex), this gives the equivalence between (B) and (K) (when we use $c(x,y) = |x-y|$ in (K)).

The proof of the duality for (B) is proposed as an exercise (Exercise 6.17), in the more general case

$$(\mathrm{B}{-}K) \qquad \min\left\{ \int K \,\mathrm{d}|\mathbf{v}| \;:\; \mathbf{v} \in \mathscr{M}^d(X);\, \nabla\cdot\mathbf{v} = \mu-\nu \right\},$$

where a given continuous weight $K : X \to \mathbb{R}_+$ is integrated against the total variation measure $|\mathbf{v}|$. In this case the dual is given by

$$\sup\left\{ \int_\Omega \phi \,\mathrm{d}(\mu-\nu) \;:\; \phi \in \mathrm{Lip}, |\nabla\phi| \le K \right\}$$

and the corresponding Kantorovich problem is the one with cost given by $c = d_K$, the weighted distance with weight K (see Section 1.4.3), since the functions ϕ satisfying $|\nabla\phi| \le K$ are exactly those which are Lip_1 w.r.t. this distance.

As a transition to congestion problems, we will now see how to provide a new equivalent formulation to (K) and (B) with geodesic cost functions.

We will use absolutely continuous curves $\omega : [0,1] \mapsto \Omega$. Given $\omega \in \mathrm{AC}(\Omega)$ and a continuous function ϕ, we write $L_\phi(\omega) := \int_0^1 \phi(\omega(t))|\omega'(t)|\,\mathrm{d}t$, which coincides with the weighted length if $\phi \ge 0$. We will write $\mathscr{C}$ (the space of "curves") for $\mathrm{AC}(\Omega)$, when there is no ambiguity on the domain, and consider probability measures Q on the space $\mathscr{C}$, which is endowed with the uniform convergence. Note that Ascoli-Arzelà's theorem guarantees that the sets $\{\omega \in \mathscr{C} : \mathrm{Lip}(\omega) \le \ell\}$ are compact for every ℓ. We will associate with Q two measures on Ω. The first is a scalar one, called *traffic intensity* and denoted by $i_Q \in \mathscr{M}_+(\Omega)$; it is defined (see [13]) via

$$\int_\Omega \phi \,\mathrm{d}i_Q := \int_{\mathscr{C}} \Big(\int_0^1 \phi(\omega(t))|\omega'(t)|\,\mathrm{d}t\Big)\,\mathrm{d}Q(\omega) = \int_{\mathscr{C}} L_\phi(\omega)\,\mathrm{d}Q(\omega).$$

for all $\phi \in C(\Omega, \mathbb{R}_+)$. The interpretation is the following: for a subregion A, $i_Q(A)$ represents the total cumulated traffic in A induced by Q, i.e. for every path we compute "how long" does it stay in A, and then we average on paths.

We also associate with any traffic plan $Q \in \mathscr{P}(\mathscr{C})$ a vector measure $\mathbf{v}_Q$ via

$$\forall \xi \in C(\Omega;\mathbb{R}^d) \quad \int_\Omega \xi \cdot \mathrm{d}\mathbf{v}_Q := \int_{\mathscr{C}} \left(\int_0^1 \xi(\omega(t)) \cdot \omega'(t)\,\mathrm{d}t\right)\mathrm{d}Q(\omega).$$

We will call $\mathbf{v}_Q$ *traffic flow* induced by Q. Taking a gradient field $\xi = \nabla\phi$ in the previous definition yields

$$\int_\Omega \nabla\phi \cdot \mathrm{d}\mathbf{v}_Q = \int_{\mathscr{C}} [\phi(\omega(1)) - \phi(\omega(0))]\,\mathrm{d}Q(\omega) = \int_\Omega \phi \,\mathrm{d}((e_1)_\# Q - (e_0)_\# Q)$$

(e_t denotes the evaluation map at time t, i.e. $e_t(\omega) := \omega(t)$) so that, if we set $(e_0)_\# Q = \mu$ and $(e_1)_\# Q = \nu$, we have

$$\nabla \cdot \mathbf{v}_Q = \mu - \nu$$

in the usual sense with no-flux boundary conditions.

It is easy to check that we have $|\mathbf{v}_Q| \le i_Q$, where $|\mathbf{v}_Q|$ is the total variation measure of the vector measure $\mathbf{v}_Q$. This last inequality is in general not an equality, since the curves of Q could produce some cancellations.

A re-formulation of the optimal transport problem with cost d_K can then be given in the following way:

$$\min\left\{\int_{\mathscr{C}} L_K(\omega)\,\mathrm{d}Q(\omega) \;:\; (e_0)_\# Q = \mu,\, (e_1)_\# Q = \nu\right\}.$$

It it then possible to see that, given a optimal Q, the measure $\gamma := (e_0, e_1)_\# Q$ is an optimal transport plan in (K) for the cost $c = d_K$, and the vector measure $\mathbf{v}_Q$ is optimal in (B-K).

Yet, this language allows to consider many other interesting situations, and in particular the case of congested traffic, where the weight function K is in general not given a priori but depends on the traffic intensity itself. We present here the extension to the continuous framework of the notion of Wardrop equilibrium (a very common definition of equilibrium in traffic problems, introduced in [33], and generally used on networks) proposed in [13].

Congestion effects are captured by the metric associated with Q via its traffic intensity: suppose $i_Q \ll \mathscr{L}^d$ and set

$$K_Q(x) := g(x, i_Q(x))$$

for a given increasing function $g(x,.) : \mathbb{R}_+ \to \mathbb{R}_+$. We then consider the weighted length L_{K_Q}, as well as the corresponding weighted distance d_{K_Q}, and define a Wardrop equilbrium as a measure $Q \in \mathscr{P}(\mathscr{C})$ such that

$$Q(\{\omega \,:\, L_{K_Q}(\omega) = d_{K_Q}(\omega(0), \omega(1))\}) = 1. \tag{5.4}$$

Of course this requires some technicalities, to take into account the case where i_Q is not absolutely continuous or its density is not smooth enough.

This is a typical situation in game theory, and a particular case of *Nash equilibrium*: a configuration where every agent makes a choice, has a cost depending on his own choice and on the others', and no agent will change his mind after knowing what the others did choose. Here the choice of every agent consists in a trajectory ω, the configuration of choices is described via a measure Q on the set of choices (since all agents are indistinguishable, so we only care at how many of them made each choice), the cost which is paid is $L_{K_Q}(\omega)$, which depends on the global configuration of choices Q and, of course, on ω, and we require that Q is concentrated on optimal trajectories, so that nobody wil change his mind.

Box 5.3. **– Important notion** – Nash equilibria

Definition - Consider a game where several players $i = 1, \dots, n$ must choose a strategy among a set of possibilities S_i and suppose that the pay-off of each player (i.e. how much he gains out of the game) depends on what everybody chooses, i.e. it is given by a function $p_i : S_1 \times \cdots \times S_n \to \mathbb{R}$. We say that a configuration $(s_1, \dots, s_n)$ (where $s_i \in S_i$) is an equilibrium (a *Nash equilibrium*) if, for every i, the choice s_i optimizes $S_i \ni s \mapsto f_i(s_1, \dots, s_{i-1}, s, s_{i+1}, \dots, s_n)$ (i.e. s_i is optimal for player i under the assumption that the other players freeze their choice).

Nash equilibria need not exist in all situations, but Nash proved (via convex analysis and fixed point arguments) that they always exist when we consider the so-called *mixed strategies*. This means that we accept that every player instead of choosing an element $s_i \in S_i$, only chooses a probability on S_i and then randomly picks a strategy according to the law he has chosen.

The notion of Nash equilibrium, first introduced by J. Nash in [27, 28] in the case of a finite number of players, can be easily extended to a continuum of players where each one is negligible compared to the others (*non-atomic games*). Considering for simplicity the case of identical players, we have a common space S of possible strategies and we look for a measure $Q \in \mathscr{P}(S)$. This measure induces a payoff function $f_Q : S \to \mathbb{R}$ and we want the following condition to be satisfied: there exists $C \in \mathbb{R}$ such that $f_Q(x) = C$ for Q-a.e. x, and $f_Q(x) \geq C$ everywhere (if the players want to minmize the payoff f_Q, otherwise, if it has to be minimized, we impose $f_Q(x) \leq C$), i.e. f_Q must be optimal Q-a.e.

A typical problem in road traffic is to find a Wardrop equilibrium Q with prescribed transport plan $\gamma = (e_0, e_1)_\# Q$ (what is usually called *origin-destination* matrix in the engineering community, in the discrete case). Another possibility is to only prescribe its marginals $\mu = (e_0)_\# Q$ and $\nu = (e_1)_\# Q$. More generally, we impose a constraint $(e_0, e_1)_\# Q \in \Gamma \subset \mathscr{P}(\Omega \times \Omega)$. A way to find such equilibria is the following.

Let us consider the (convex) variational problem

$$(\mathrm{W}) \qquad \min\left\{\int_\Omega H(x, i_Q(x))\,\mathrm{d}x : Q \in \mathscr{P}(\mathscr{C}), (e_0, e_1)_\# Q \in \Gamma\right\} \tag{5.5}$$

where $H'(x,.) = g(x,.)$, $H(x,0) = 0$. Under some technical assumptions, the main result of [13] is that (W) admits at least one minimizer, and that such a minimizer is a Wardrop equilibrium. Moreover, $\gamma_Q := (e_0, e_1)_\# Q$ solves the optimization problem

$$\min\left\{\int_{\Omega\times\Omega} d_{K_Q}(x,y)\,\mathrm{d}\gamma(x,y) \;:\; \gamma \in \Gamma\right\}.$$

In particular, if Γ is a singleton, this last condition does not play any role (there is only one competitor) and we have existence of a Wardrop equilibrium corresponding to any given transport plan γ. If, on the contrary, $\Gamma = \Pi(\mu, \nu)$, then the second condition means that γ solves a Monge-Kantorovich problem for a distance cost depending on Q itself, which is a new equilibrium condition.

In the case where $\Gamma = \Pi(\mu, \nu)$ it is possible to prove that (W) is indeed equivalent to a variational divergence constrained problem *à la* Beckmann , i.e.

$$(\mathrm{B-cong}) \qquad \min\left\{\int_\Omega \mathscr{H}(x, \mathbf{v}(x))\,\mathrm{d}x : \nabla\cdot\mathbf{v} = \mu - \nu\right\}, \tag{5.6}$$

where $\mathscr{H}(x,v) = H(x,|v|)$. It is then possible to prove that the optimizers of this problem are the vector fields of the form $\mathbf{v}_Q$ where Q solves (W). And this class of problems is exactly the one we considered in sections 4.3 and 4.4!

We now switch to optimal transport problems with other costs, and in particular $c(x,y) = |x-y|^p$. Since these costs do not satisfy the triangle inequality the dual formulation is more involved than that in (D-dist) and no Beckmann formulation exists[1]. Instead, there exists a dynamic formulation, known under the name of *Benamou-Brenier* problem, [5]. We can indeed consider the optimization problem

$$(\mathrm{BB}_p) \quad \min\left\{\int_0^1\int_\Omega |\mathbf{v}_t|^p\,\mathrm{d}\rho_t\,\mathrm{d}t \;:\; \partial_t\rho_t + \nabla\cdot(\rho_t\mathbf{v}_t) = 0,\, \rho_0 = \mu,\, \rho_1 = \nu\right\}$$

[1] We mention anyway [16] which refomulates optimal transport problems with costs which are convex but not 1-homogeneous in $x-y$ into transport costs with 1-homogeneous costs, for which a Beckmann formulation exists, after adding a time variable.

which consists in minimizing the integral in time and space of a p-variant of the kinetic energy among solutions of the continuity equation with prescribed intial and final data.

Box 5.4. **– Good to know!** – The continuity equation in mathematical physics

Suppose that a family of particles moves according to a velocity field $\mathbf{v}$, depending on time and space: $\mathbf{v}(t,x)$ stands for the velocity at time t of any particle which is located at point x at such an instant of time. The position of the particle originally located at x will be given by the solution of the ODE

$$\begin{cases} y_x'(t) = \mathbf{v}_t(y_x(t)) \\ y_x(0) = x. \end{cases} \tag{5.7}$$

We then define the map Y_t through $Y_t(x) = y_x(t)$, and, if we are given the distribution of particles at $t = 0$, we look for the measure $\rho_t := (Y_t)_\#\rho_0$. We can then prove that ρ_t and $\mathbf{v}_t$ solve together the so-called continuity equation

$$\partial_t\rho_t + \nabla\cdot(\rho_t\mathbf{v}_t) = 0$$

(as usual, with no-flux boundary condition on $\partial\Omega$ if we suppose that we have $y_x(t) \in \Omega$ for all (t,x).

It is possible to prove that the minimal value in (BB$_p$) equals that of (K) with cost $|x-y|^p$ and that the optimal solutions of (BB$_p$) are obtained by taking an optimal transport map T for this same cost in the Monge problem, defining $T_t := (1-t)id + tT$, and setting $\rho_t = (T_t)_\#\mu$ and $\mathbf{v}_t = (T - id)\circ(T_t)^{-1}$. A corresponding formula may be given using the optimizers of (K) if they are not given by transport maps.

Concerning Problem (BB$_p$), we observe that it is not a convex optimization problem in the variables $(\rho,\mathbf{v})$, because of the product term $\rho|\mathbf{v}|^p$ in the functional and of the product $\rho\mathbf{v}$ in the differential constraint. But if one changes variable, defining $\mathbf{w} = \rho\mathbf{v}$ and using the variables $(\rho,\mathbf{w})$, then the constraint becomes linear and the functional convex. The important point for convexity is that the function

$$\mathbb{R}\times\mathbb{R}^d \ni (s,\mathbf{w}) \mapsto \begin{cases} \frac{|\mathbf{w}|^p}{ps^{p-1}} & \text{if } s > 0, \\ 0 & \text{if } (s,\mathbf{w}) = (0,0), \\ +\infty & \text{otherwise} \end{cases} \tag{5.8}$$

is convex (and it is actually obtained as $\sup\{as + b\cdot\mathbf{w} \,:\, a + \frac{1}{p'}|b|^{p'} \le 0\}$).

Some analogies and some differences may be underlined between the optimal flow formulation à la Beckmann of the OT problem with $p = 1$ and the dynamic formulation à la Benamou-Brenier of the case $p > 1$. Both involve differential constraints on the divergence, and actually we can also look at the continuity equation as a time-space constraint on $\nabla_{t,x}\cdot(\rho,\mathbf{w})$ making the analogy even stronger. Yet, there is no time in the Beckmann problem and even when time appears in the Lagrangian formulation with measures Q on curves it is a fictitious time parameter since everything is invariant under reparametrization. On the other hand, time plays an important role in the case $p > 1$.

This is also the reason why the models that we will present in a while, for congestion problems where the congestion effect is evaluated at each time, will be so close to the Benamou-Brenier problem.

The models that we want to introduce are known as Mean Field Games, and this theory was introduced around 2006 at the same time by Lasry and Lions, [19, 20, 21], and by Caines, Huang and Malhamé, [15], in order to describe the evolution of a population in a game with a continuum of players where the effect on each player of the presence of the others recalls what is called in physics *mean field*. This class of games, called Mean Field Games (MFG for short), are very particular differential games: typically, in a differential game the role of the time variable is crucial since if a player decides to deviate from a given strategy (a notion which is at the basis of the Nash equilibrium definition), the other can react to this change, so that the choice of a strategy is usually not defined as the choice of a path, but of a function selecting a path according to the information the player has at each given time. Yet, when each player is considered as negligible, any deviation he/she performs will have no effect on the other players, so that they will not react. In this way we have a static game where the space of strategies is a space of paths.

We can give a Lagrangian description of the equilibria by using again measures on paths, or an Eulerian one through a system of PDEs, where the key ingredients are the density ρ and the value function φ of the control problem solved by each player, the velocity $\mathbf{v}(t,x)$ of the agents at (t,x) being, by optimality, related to the gradient $\nabla\varphi(t,x)$. For a general overview of the MFG theory, it is possible to refer to the lecture notes by P. Cardaliaguet [11], based on the course by Lions at Collège de France [23].

Let us describe in a more precise way the simplest MFG models. We look at a population of agents moving inside a domain Ω and suppose that every agent chooses his own trajectory ω solving a minimization problem

$$\min \int_0^T \left(\frac{|\omega'(t)|^2}{2} + h[\rho_t](\omega(t)) \right) \mathrm{d}t + \Psi(\omega(T)),$$

with given initial point $x(0)$. The mean-field effect will be modeled through the fact that the function h depends on the density ρ_t of the agents at time t. The dependence of the cost on the velocity ω' could of course be more general than a simple quadratic function.

For the moment, we consider the evolution of the density ρ_t as an input, i.e. we suppose that agents know it. Hence, we can suppose the function h to be given, and we want to study the above optimization problem. The main tool to analyze it, coming from optimal control theory, is the value function φ (see Section 1.7): we know that it solves the *Hamilton-Jacobi equation* $-\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 = h$ with final datum $\varphi(T,x) = \Psi(x)$ and that the optimal trajectories $\omega(t)$ can be computed using φ, since they are the solutions of

$$\omega'(t) = -\nabla\varphi(t,\omega(t)).$$

This means that the velocity field which advects the particles when each agent follows the optimal curves will be given by $\mathbf{v} = -\nabla\varphi$. If we want to find an equilibrium, the density ρ_t that we fixed at the beginning should also coincide with that which is obtained by following this optimal velocity field $\mathbf{v}$ and so ρ should evolve according to the continuity equation with such a velocity field. This means solving the following coupled (HJ)+(CE) system:

$$\begin{cases} -\partial_t \varphi + \frac{|\nabla \varphi|^2}{2} = h[\rho] \\ \partial_t \rho - \nabla \cdot (\rho \nabla \varphi) = 0, \\ \varphi(T,x) = \Psi(x), \quad \rho(0,x) = \rho_0(x). \end{cases} \tag{5.9}$$

The above system is a PDE, and Eulerian, description, of the equilibrium we look for. In Lagrangian terms we could express the equilibrium condition in a similar way to what we did for Wardrop equilibria. Consider again the space of curves $\mathscr{C}$ and define the kinetic energy functional $K : \mathscr{C} \to \mathbb{R}$ given by

$$K(\omega) = \frac{1}{2} \int_0^T |\omega'|^2(t)\,\mathrm{d}t.$$

For notational simplicity, we write K_Ψ for the kinetic energy augmented by a final cost: $K_\Psi(\omega) := K(\omega) + \Psi(\omega(T))$; similarly, we denote by $K_{\Psi,h}$ the same quantity when also a running cost h is included: $K_{\Psi,h}(\omega) := K_\Psi(\omega) + \int_0^T h(t,\omega(t))\,\mathrm{d}t$.

We now define a MFG equilibrium as a measure $Q \in \mathscr{P}(\mathscr{C})$ such that Q-a.e. curve ω minimizes $K_{\Psi,h}$ with given initial point when $h(t,\cdot) := h[(e_t)_\# Q]$. Whenever $(x,\rho) \mapsto h[\rho](x)$ has some continuity properties it is possible to prove the existence of an equilibrium by the Kakutani fixed point method (see, for instance, [14] or [24] where such a theorem is used to prove the existence of a MFG equilibrium ina slihtly different seting).

On the other hand, the assumption that h is continuous does not cover a very natural case, which is the local case, where $h[\rho](x)$ directly depends on the density of ρ at the point x. We can focus for instance on the case $h[\rho](x) = V(x) + g(\rho(x))$, where we identify the measure ρ with its density w.r.t. the Lebesgue measure on Ω. The function $V : \Omega \to \mathbb{R}_+$ is a potential taking into account different local costs of different points in Ω.

In this case it is not possible to prove the existence of an equilibrium using a ixed point theorem, but luckily there exists a variational formulation. Indeed, we can consider all the possible population evolutions, i.e. pairs $(\rho,\mathbf{v})$ satisfying $\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$ and we minimize the following energy

$$\mathscr{A}(\rho,\mathbf{v}) := \int_0^T \int_\Omega \left(\frac{1}{2} \rho_t |\mathbf{v}_t|^2 + \rho_t V + G(\rho_t) \right) \mathrm{d}x\,\mathrm{d}t + \int_\Omega \Psi\,\mathrm{d}\rho_T,$$

where G is the anti-derivative of g, i.e. $G'(s) = g(s)$ for $s \in \mathbb{R}^+$ with $G(0) = 0$. We fix by convention $G(s) = +\infty$ for $\rho < 0$. Note in particular that G is convex, as its derivative is the increasing function g.

The above minimization problem recalls, in particular when $V = 0$, the Benamou-Brenier dynamic formulation for optimal transport: tThe main difference with the Benamou-Brenier problem is that here we add to the kinetic energy a congestion cost G; also note that usually in optimal transport the target measure ρ_T is fixed, and here it is part of the optimization (but this is not a crucial difference).

As is often the case in congestion games, the quantity $\mathscr{A}(\rho,\mathbf{v})$ is not the total cost for all the agents. Indeed, the term $\int\int \frac{1}{2}\rho|\mathbf{v}|^2$ is exactly the total kinetic energy, and the last term $\int \Psi \, d\rho_T$ is the total final cost, as well as the cost $\int V \, d\rho_t$ exactly coincides with the total cost enduced by the potential V; yet, the term $\int G(\rho)$ is not the total congestion cost, which should be instead $\int \rho g(\rho)$. This means that the equilibrium minimizes an overall energy (we have what is called a potential game), but not the total cost; this gives rise to the so-called *price of anarchy*.

In order to convince the reader of the connection between the minization of $\mathscr{A}(\rho,\mathbf{v})$ and the equilibrium system (5.9), we will use some formal argument from convex duality. A rigorous proof of duality would, by the way, require to re-wirte the problem as a convex optimization problem, which requires to change variables and use again $\mathbf{w} = \rho\mathbf{v}$.

In order to formally produce a dual problem to $\min \mathscr{A}$, we wil use a min-max exchange procedure. First, we write the constraint $\partial_t \rho + \nabla \cdot (\rho\mathbf{v}) = 0$ in weak form, i.e.

$$\int_0^T \int_\Omega (\rho \partial_t \phi + \nabla\phi \cdot \rho\mathbf{v}) + \int_\Omega \phi_0 \rho_0 - \int_\Omega \phi_T \rho_T = 0 \tag{5.10}$$

for every function $\phi \in C^1([0,T]\times\Omega)$ (note that we do not impose conditions on the values of ϕ on $\partial\Omega$, hence this is equivalent to completing (CE) with a no-flux boundary condition $\rho\mathbf{v}\cdot n = 0$).

Using (5.10) , we can re-write our problem as

$$\min_{\rho,\mathbf{v}} \mathscr{A}(\rho,\mathbf{v}) + \sup_\phi \int_0^T \int_\Omega (\rho \partial_t \phi + \nabla\phi \cdot \rho\mathbf{v}) + \int_\Omega \phi_0 \rho_0 - \int_\Omega \phi_T \rho_T,$$

since the sup in ϕ takes value 0 if the constraint is satisfied and $+\infty$ if not. We now switch the inf and the sup and get

$$\sup_\phi \int_\Omega \phi_0\rho_0 + \inf_{\rho,\mathbf{v}} \int_\Omega (\Psi - \phi_T)\rho_T + \int_0^T \int_\Omega \left(\frac{1}{2}\rho_t|\mathbf{v}_t|^2 + \rho_t V + G(\rho_t) + \rho\partial_t\phi + \nabla\phi\cdot\rho\mathbf{v} \right) dx\, dt.$$

First, we minimize w.r.t. $\mathbf{v}$, thus obtaining $\mathbf{v} = -\nabla\phi$ (on $\{\rho_t > 0\}$) and we replace $\frac{1}{2}\rho|\mathbf{v}|^2 + \nabla\phi\cdot\rho\mathbf{v}$ with $-\frac{1}{2}\rho|\nabla\phi|^2$. Then we get, in the double integral,

$$\inf_\rho \{G(\rho) - \rho(-V - \partial_t\phi + \frac{1}{2}|\nabla\phi|^2)\} = -\sup_\rho \{p\rho - G(\rho)\} = -G^*(p),$$

where we set $p := -V - \partial_t\phi + \frac{1}{2}|\nabla\phi|^2$ and G^* is defined as the Legendre transform of G. Then, we observe that the minimization in the final cost simply gives as a result 0 if $\Psi \geq \phi_T$ (since the minimization is only performed among positive ρ_T) and $-\infty$ otherwise. Hence we obtain a dual problem of the form

$$\sup \left\{ -\mathscr{B}(\phi,p) := \int_\Omega \phi_0\rho_0 - \int_0^T \int_\Omega G^*(p) \; : \; \phi_T \leq \Psi, -\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 = V + p \right\}.$$

Note that the condition $G(s) = +\infty$ for $s < 0$ implies $G^*(p) = 0$ for $p \leq 0$. This in particular means that in the above maximization problem one can suppose $p \geq 0$ (indeed, replacing p with p_+ does not change the G^* part, but improves the value of ϕ_0, considered as a function depending on p). The choice of using two variables (ϕ, p) connected by a PDE constraint instead of only ϕ is purely conventional, and it allows for a dual problem which has a sort of symmetry w.r.t. the primal one.

Now, standard arguments in convex duality would allow to say that optimal pairs $(\rho, \mathbf{v})$ are obtained by looking at saddle points $((\rho, \mathbf{v}), (\phi, p))$, provided that there is no duality gap between the primal and the dual problems, and that both problems admit a solution. This would mean that, whenever $(\rho, \mathbf{v})$ minimizes $\mathscr{A}$, then there exists a pair (ϕ, p), solution of the dual problem, such that

- $\mathbf{v}$ minimizes $\frac{1}{2}\rho|\mathbf{v}|^2 + \nabla\phi \cdot \rho\mathbf{v}$, i.e. $\mathbf{v} = -\nabla\phi$ ρ-a.e. This gives (CE): $\partial_t \rho - \nabla \cdot (\rho \nabla \phi) = 0$.

- ρ minimizes $G(\rho) - p\rho$, i.e. $g(\rho) = p$ if $\rho > 0$ or $g(\rho) \geq p$ if $\rho = 0$ (in particular, when we have $g(0) = 0$, we can write $g(\rho) = p_+$); this gives (HJ): $-\partial_t \phi + \frac{1}{2}|\nabla\phi|^2 = V + g(\rho)$ on $\{\rho > 0\}$ (as the reader can see, there are some subtleties where the mass ρ vanishes;).

- ρ_T minimizes $(\Psi - \phi_T)\rho_T$ among $\rho_T \geq 0$. But this is not a condition on ρ_T, but rather on ϕ_T: we must have $\phi_T = \Psi$ on $\{\rho_T > 0\}$, otherwise there is no minimizer. This gives the final condition in (HJ).

This provides an informal justification for the equivalence between the equilibrium and the global optimization. It is only informal because we have not discussed whether we have or not duality gaps and whether or not the maximization in (ϕ, p) admits a solution. Moreover, even once these issues are clarified, what we will get will only be a very weak solution to the coupled system (CE)+(HJ). Nothing guaranteees that this solution actually encodes the individual minimization problem of each agent.

It is also possible to provide a Lagrangian version of the variational problem, which has then the following form:

$$\min \left\{ J(Q) := \int_{\mathscr{C}} K \, \mathrm{d}Q + \int_0^T \mathscr{G}((e_t)_\# Q) \, \mathrm{d}t + \int_\Omega \Psi \, \mathrm{d}(e_T)_\# Q, \; Q \in \mathscr{P}(\mathscr{C}), (e_0)_\# Q = \rho_0 \right\}, \tag{5.11}$$

where $\mathscr{G} : \mathscr{P}(\Omega) \to \overline{\mathbb{R}}$ is defined through

$$\mathscr{G}(\rho) = \begin{cases} \int (V(x)\rho(x) + G(\rho(x))) \, \mathrm{d}x & \text{if } \rho \ll \mathscr{L}^d, \\ +\infty & \text{otherwise.} \end{cases}$$

The functional $\mathscr{G}$ is a typical local functional defined on measures (see [**?**]). It is lower-semicontinuous w.r.t. weak convergence of probability measures provided $\lim_{s\to\infty} G(s)/s = +\infty$ (which is the same as $\lim_{s\to\infty} g(s) = +\infty$), see Section 3.3.

In the Lagrangian language, it is possible to prove that the optimal $\bar{Q}$ satisfies the following optimality conditions: setting $\rho_t = (e_t)_\# \bar{Q}$ and $h(t,x) = g(\rho_t(x))$ (identifying as usual measures and densities), if we take take $\widetilde{Q}$ another competitor, we have

$$J_h(\widetilde{Q}) \geq J_h(\bar{Q}),$$

where J_h is the linear functional

$$J_h(Q) = \int K dQ + \int_0^T \int_\Omega h(t,x)(e_t)_\# Q + \int_\Omega \Psi d(e_T)_\# Q.$$

Since we only consider absolutely continuous $(e_t)_\# Q$, the functional J_h. It is well-defined for any $h \geq 0$ measurable. Formally, we can also write $\int_0^T \int_\Omega h(t,x)(e_t)_\# Q = \int_{\mathscr{C}} dQ \int_0^T h(t,\gamma(t))dt$ and hence we get that

$$Q \mapsto \int_{\mathscr{C}} dQ(\gamma) \left(K(\gamma) + \int_0^T h(t,\gamma(t))dt + \Psi(\gamma(T)) \right)$$

is minimal for $Q = \bar{Q}$. This corresponds to saying that $\bar{Q}$ is concentrated on curves minimizing $K_{\pi,h}$), hence it is a Lagrangian MFG equilibrium. Unfortunately, this argument is not rigorous at all because of two main difficulties: first, it has no meaning to integrate h on a path if h is a function which is only defined a.e.; second, in order to prove that $\bar{Q}$ is concentrated on optimal curves one needs to comapre it to a competitor $\widetilde{Q}$ which is indeed concentrated on optimal curves, but at the same time we need $(e_t)_\# \widetilde{Q} \ll \mathscr{L}^d$,and these two conditions can be incompatible with each other. These difficulties could be fixed by a density argument if h was continuous; another, more technical approach requires $h \in L^\infty$ by choosing a precise representative. This is explained for instance in [31] and is based on techniques from incompressible fluid mechanics (see [2]). In any case, it becomes crucial to prove regularity results on h or, equivalently, on ρ.

A general L^∞ regularity result is presented in [**?**] and is based on optimal transport techniques. Here we want, instead, to briefly explain that also this question can be attacked via the techniques for regularity via convex duality of Section 4.4. By the way, the first use of duality-based methods to prove regularity was eatly in time-dependent problems: first in [9] (later improved by [1]), in the study of variational models for the incompressible Euler Equation. This has been later adapted in [12] to density-constrained mean-field games.

We can then come back to the primal and the dual problems in variational MFG and do our usual computation taking arbitrary $(\rho, \mathbf{v})$ and (ϕ, p) admissible in the primal and dual problem. We compute

$$\mathscr{A}(\rho,\mathbf{v}) + \mathscr{B}(\phi,p) = \int_\Omega (\Psi - \phi_T)\rho_T + \int_0^T \int_\Omega (G(\rho) + G^*(p) - p\rho) + \frac{1}{2} \int_0^T \int_\Omega \rho |v + \nabla \phi|^2. \tag{5.12}$$

Notice $(G(\rho) + G^*(p) - p\rho) \geq \frac{\lambda}{2} |\rho - g^{-1}(p)|^2$ where $g = G'$ and $\lambda = \inf G''$. We suppose $\lambda > 0$ and, for simplicity, $\Omega = \mathbb{T}^d$. Using

$$\mathscr{A}(\rho,\mathbf{v}) + \mathscr{B}(\phi,p) \geq c \int_0^T \int_\Omega |\rho - g^{-1}(p)|^2$$

we can deduce, with the same technique as in Section 4.4, $\rho \in H^1$ (we can get both regularity in space and local in time). By the way, using the last term in (5.12), we can also get $\iint \rho |D^2 \phi|^2 < \infty$.

Chapter 6

Exercises

Exercise 6.1. Prove that the function $f : \mathbb{R}_+ \to \mathbb{R}$ given by $f(x) = x \log x$ admits a unique extension to $\mathbb{R}$ which is convex and l.s.c., and compute its Legendre transform.

Exercise 6.2. If $f : \mathbb{R}^n \to \mathbb{R}$ is given by $f(x) = |x| \log |x|$, compute f^* and f^{**}.

Exercise 6.3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Prove that f is $C^{1,1}$ if and only if f^* is elliptic (meaning that there exists $c > 0$ such that $f(x) - c|x|^2$ is convex).

Exercise 6.4. Prove that a convex and l.s.c. function $f : X \to \mathbb{R} \cup \{+\infty\}$ is 1-homogeneous (i.e. $f(tx) = tf(x)$ for every $t \geq 0$ and every $x \in X$) if and only if f^* takes its values in $\{0, +\infty\}$.

Exercise 6.5. Given $f : X \to \mathbb{R} \cup \{+\infty\}$ and $\xi_0 \in X'$, define $f_{\xi_0} : X \to \mathbb{R} \cup \{+\infty\}$ via $f_{\xi_0}(x) := f(x) - \langle \xi_0, x \rangle$. Prove that we have $f^*_{\xi_0}(\xi) = f^*(\xi + \xi_0)$ for every ξ.

Exercise 6.6. In dimension $N = 1$, prove the equivalence between these two facts:

1. f^* is strictly convex;
2. f satisfies the following property: it is C^1 in the interior Ω of its domain, and it coincides with $x \mapsto \sup_{x' \in \Omega} f(x') + \nabla f(x') \cdot (x - x')$.

Exercise 6.7. Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a convex and l.s.c. function. Prove that f is superlinear if and only if ∂f is surjective.

Exercise 6.8. Discuss what does not work in the following (wrong) counter-example to the statement of Exercise 6.7. Consider $X = L^2(\Omega)$ and $F(u) := \int |\nabla u|^2$ (a functional set to $+\infty$ if $u \notin H^1(\Omega)$). This functional is not superlinear because it vanishes on constant functions, but for every $u \in H^2(\Omega)$ we do have $\partial F(u) = \{-\Delta u\}$, which shows that any L^2 function (the elements of the dual of the Hilbert space on which the functional is defined) is a subgradient, so the subgradient is surjective.

Also discuss what would change if taking $F(u) := \int |\nabla u|^2$ if $u \in H^1_0(\Omega)$ (and $F = +\infty$ outside of $H^1_0(\Omega)$).

Exercise 6.9. Given a bounded, smooth and connected domain $\Omega \subset \mathbb{R}^d$, and $f \in L^2(\Omega)$, set $X(\Omega) = \{v \in L^2(\Omega;\mathbb{R}^d) \, : \, \nabla\cdot\mathbf{v} \in L^2(\Omega)\}$ and consider the minimization problems

$$(P) \quad := \quad \min\left\{F(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 + f(x)u\right) \mathrm{d}x \, : \, u \in H^1(\Omega)\right\}$$
$$(D) \quad := \quad \min\left\{G(v) := \int_\Omega \left(\frac{1}{2}|v|^2 + \frac{1}{2}|\nabla\cdot\mathbf{v} - f|^2\right) \mathrm{d}x \, : \, v \in X(\Omega)\right\},$$

1. Prove that (P) admits a unique solution;
2. Prove $\min(P) + \inf(D) \geq 0$;
3. Prove that there exist $v \in X(\Omega)$ and $u \in H^1(\Omega)$ such that $F(u) + G(v) = 0$;
4. Deduce that $\min(D)$ is attained and $\min(P) + \inf(D) = 0$;
5. Justify by a formal inf-sup exchange the duality $\min F(u) = \sup -G(v)$;
6. Prove $\min F(u) = \sup -G(v)$ via a duality proof based on convex analysis.

Exercise 6.10. Consider the problem

$$\min\{A(\mathbf{v}) := \int_{\mathbb{T}^d} H(\mathbf{v}(x))\,\mathrm{d}x \, : \, \mathbf{v} \in L^2, \nabla\cdot\mathbf{v} = f\}$$

for a function H which is elliptic. Prove that the problem has a solution, provided there exists at least an admissible $\mathbf{v}$ with $A(\mathbf{v}) < +\infty$. Prove that, if f is an H^1 function with zero mean, then the optimal $\mathbf{v}$ is also H^1.

Exercise 6.11. Let $H : \mathbb{R}^d \to \mathbb{R}$ be given by $H(z) = \sqrt{1+|z|^2+|z|^4}$ and Ω be the d-dimensional torus. Consider the equation

$$\nabla\cdot\left(\frac{(1+2|\nabla u|^2)\nabla u}{H(\nabla u)}\right) = f.$$

1. Given $f \in H^{-1}(\Omega)$ such that $\langle f, 1\rangle = 0$ (i.e. f has zero average), prove that there exists a solution $u \in H^1(\Omega)$ to this equation, which is unique up to additive constants.
2. If $f \in H^1$, prove that the solution u is H^2.
3. What can we say if $f \in L^2$?

Exercise 6.12. Let $f \in L^1([0,1])$ and $F \in W^{1,1}([0,1])$ be such that $F' = f$ and $F(0) = F(1) = 0$. Let $1 < p < \infty$ be a given exponent, and q be its conjugate exponent. Prove

$$\min\left\{\int_0^1 \frac{1}{p}|u'(t)|^p dt + \int_0^1 f(t)u(t)\,\mathrm{d}t \, : \, u \in W^{1,p}([0,1])\right\} = -\frac{1}{q}||F||^q_{L^q([0,1])}.$$

Exercise 6.13. Let $H : \mathbb{R}^n \to \mathbb{R}$ be given by

$$H(v) = \frac{(4|v|+1)^{3/2} - 6|v| - 1}{12}.$$

1. Prove that H is C^1 and strictly convex. Is it $C^{1,1}$? Is it elliptic?
2. Compute H^*. Is it C^1, strictly convex, $C^{1,1}$ and/or elliptic?
3. Consider the problem $\min\{\int H(\mathbf{v}) \ : \ \nabla\cdot\mathbf{v} = f\}$ (on the d-dimensional torus, for simplicity) and find its dual.
4. Supposing $f \in L^2$, prove that the optimal u in the dual problem is H^2.
5. Under the same assumption, prove that the optimal $\mathbf{v}$ in the primal problem belongs to $W^{1,p}$ for every $p < 2$ if $d = 2$, for $p = d/(d-1)$ if $3 \le d \le 5$, and for $p = 6/5$ if $d \ge 3$.

Exercise 6.14. Given a function $g \in L^2([0,L])$, consider the problem

$$\min\left\{\int_0^L \frac{1}{2}|u(t)-g(t)|^2\,\mathrm{d}t \ : \ u(0) = u(L) = 0, u \in \mathrm{Lip}([0,L]), |u'| \le 1 \text{ a.e.}\right\}.$$

1. Prove that this problem admits a solution.
2. Prove that the solution is unique.
3. Find the optimal solution in the case where g is the constant function $g = 1$ in the terms of the value of L, distinguishing $L > 2$ and $L \le 2$.
4. Computing the value of

$$\sup\left\{-\int_0^L (u(t)z'(t) + |z(t)|)\,\mathrm{d}t \ : \ z \in H^1([0,L])\right\}$$

   find the dual of the previous problem by means of a formal inf-sup exchange.
5. Assuming that the equality $\inf\sup = \sup\inf$ in the duality is satisfied, write the necessary and sufficient optimality conditions for the solutions of the primal and dual problem. Check that these conditions are satisfied by the solution found in the case $g = 1$.
6. Prove the the equality $\inf\sup = \sup\inf$.

Exercise 6.15. Given $u_0 \in C^1([0,1])$ consider the problem

$$\min\left\{\int_0^1 \frac{1}{2}|u-u_0|^2\,\mathrm{d}x \quad : \quad u' \ge 0\right\},$$

which consists in the projection of u_0 onto the set of monotone increasing functions (where the condition $u' \ge 0$ is intended in the weak sense).

1. Prove that this problem admits a unique solution.
2. Write the dual problem
3. Prove that the solution is actually the following: define U_0 through $U_0' = u_0$, set $U_1 := (U_0)^{**}$ to be the largest convex and l.s.c. function smaller than U_0, take $u = U_1'$.

Exercise 6.16. Given a space X let us fix a cost function $c : X \times X \to \mathbb{R}_+$ which is symmetric ($c(x,y) = c(y,x)$ for all x,y) and satisfies the triangle inequality $c(x,y) \le c(x,z) + c(z,y)$.For $\psi : X \to \mathbb{R}$ define $\psi^c : X \to \mathbb{R}$ via $\psi^c(x) := \inf_y c(x,y) - \psi(y)$. Prove that a function ϕ is of the form ψ^c if and only if it satisfies $|\phi(x) - \phi(y)| \le c(x,y)$ for all x,y. Also prove that for functions satisfying this condition we have $\phi^c = -\phi$.

Exercise 6.17. Prove the equality

$$\min\left\{\int K\,\mathrm{d}|\mathbf{v}| \ : \ \mathbf{v} \in \mathscr{M}^d(X);\ \nabla\cdot\mathbf{v} = \mu - \nu\right\} = \max\left\{\int \phi\,\mathrm{d}(\mu - \nu) \ : \ |\nabla\phi| \le K\right\}$$

using the duality methods inspired by those developed in Section 4.3.

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