Quasi-Minimizers & Hölder continuity

1) DIFFERENT DEFINITIONS OF QUASI MINIMIZERS
2) PROPERTIES OF HARMONIC FUNCTIONS
3) INTEGRAL CHARACTERIZATION OF HÖLDER FUNCTIONS
4) REGULARITY FOR QUASI-MINIMIZERS
5) WHEN min OF $\int L(x,u,Du)$ ARE QUASI-MINIMIZERS OF $\int |Du|^2$

1) GENERAL IDEA: SET $F(u,\Omega) := \int_\Omega L(x,u,Du)\,dx$

IF u MINIMIZES $F(u,\Omega)$ (WITH OR WITHOUT BOUNDARY CONDITIONS)

$\Rightarrow \forall \Omega' \subset\subset \Omega \quad F(u,\Omega') \le F(v,\Omega') \quad \forall v: v|_{\Omega\setminus\Omega'} = u|_{\Omega\setminus\Omega'}$

WHEN L IS QUADRATIC IN Du WE USE H^1 FUNCTIONS, AND WE SAY "$\forall v: v-u \in H^1_0(\Omega')$"

GIVEN $\alpha \ge 0$, WE CALL "QUASI-MINIMIZER" OF F A FUNCTION u S.T. $F(u, B_R) \le F(v, B_R)$ + AN ERROR OF ORDER R^α (IN SOME SENSE) $\forall B_R \subset \Omega \quad \forall v: v-u \in H^1_0(B_R)$.

THE IDEA IS TO STUDY QUASI-MINIMIZERS OF SIMPLE FUNCTIONALS (ESSENTIALLY $F(u,\Omega) = \int_\Omega |Du|^2$) AND THEN PROVE THAT min OF MORE GENERAL FUNCTIONALS ARE QUASI-min OF THE SIMPLE ONE. FROM NOW ON, WE WILL STICK TO THE CASE $\int |Du|^2$.

SET $\|f\|_r := \left(⨍_{B_r} |f|^2 \right)^{1/2}$ (NORMALIZED L^2 NORM ON B_r)

HERE ARE 3 POSSIBLE DEFINITIONS (WARNING: THEY ARE NOT UNIVERSAL)

u IS SAID TO BE A

- MULTIPLICATIVE QUASI-min (MQM) IF $\forall B_R \subset \Omega \quad \forall v: v-u \in H^1_0(B_R)$
$$⨍_{B_R} |Du|^2 \le (1 + CR^\alpha) ⨍_{B_R} |Dv|^2$$
- ADDITIVE QUASI-min (AQM) IF $⨍_{B_R} |Du|^2 \le ⨍_{B_R} |Dv|^2 + CR^{2\alpha}$
- PERTURBATIVE QUASI-min (PQM) IF
$$⨍_{B_R} |Du|^2 \le ⨍_{B_R} |Dv|^2 + CR^\alpha \| Du - Dv \|_R$$

THE IDEA IS, FOR EVERY $B_R \subset \Omega$, TO COMPARE u TO THE FUNCTION u_R^* CHOSEN SO THAT $u_R^* = \arg\min \left\{ \int_{B_R} |\nabla v|^2 : v - u \in H_0^1(B_R) \right\}$

I.E. u_R^* SOLVES $\begin{cases} \Delta u_R^* = 0 & B_R \\ u_R^* = u & \partial B_R \end{cases}$

AND TO TRANSFER SOME REGULARITY PROPERTIES OF HARMONIC FUNCTIONS TO u.

2) PROPERTIES OF HARMONIC FUNCTIONS.

SUPPOSE $\Delta u = 0$ IN Ω. THEN $u \in C^\infty(\Omega)$ (AND u IS EVEN ANALYTIC). MOREOVER WE HAVE

MEAN PROPERTY $\forall x$ $r \longmapsto \fint_{B_r(x)} u(y)\,dy$

AND $r \longmapsto \fint_{\partial B_r(x)} u(y)\,dy$ ARE CONSTANT FOR $r < R_0$ AND $B_{r_0}(x) \subset \Omega$

CACCIOPPOLI INEQUALITY TAKE $B_r \subset B_R \subset \Omega$. LET $\eta \in C^\infty(B_R)$ WITH $\eta = 1$ ON B_r $\eta = 0$ ON ∂B_R, WITH $|\nabla \eta| \le \frac{C}{R-r}$

$$0 = -\int \Delta u \,(u\eta^2) = \int \nabla u \cdot \nabla(u\eta^2) = \int |\nabla u|^2 \eta^2 + 2\int \eta u \nabla u \cdot \nabla \eta$$

$$\Rightarrow \int |\nabla u|^2 \eta^2 \le \left(\int \eta^2 |\nabla u|^2\right)^{1/2} \left(\int u^2 |\nabla \eta|^2\right)^{1/2}$$

$$\Rightarrow \int_{B_r} |\nabla u|^2 \le \int_{B_R} |\nabla u|^2 \eta^2 \le \int_{B_R} u^2 |\nabla \eta|^2 \le \frac{C}{(R-r)^2} \int_{B_R} u^2$$

WE ARE ALSO INTERESTED IN TWO PRECISE ESTIMATES

1) $r \longmapsto \fint_{B_r(x)} |u|^2(y)\,dy$ IS INCREASING

PROOF u^2 IS SUBHARMONIC: $\Delta(u^2) = 2u\Delta u + 2|\nabla u|^2 = 2|\nabla u|^2 \ge 0$

IF f IS S.T. $\Delta f \ge 0$ THEN $g(r) = \fint_{\partial B_r(x)} f$ IS INCREASING

INDEED $g(r) = \fint_{\partial B_1} f(x+ry)\,dy$ $\quad g'(r) = \fint_{\partial B_1} \nabla f(x+ry)\cdot y\,dy$

$$= \fint_{\partial B_r(x)} \frac{\partial f}{\partial n} = c\int_{B_r} \Delta f \ge 0$$

THEN, USE $h(r) = \fint_{B_r} f = c\int_0^1 t^{n-1}\, g(tr)\, dt$ TO GET h INCREASING.

b) SET $\bar{u} := \fint_{B_r} u$ ($\bar{u}$ DOES NOT DEPEND ON r)

$\exists$ UNIVERSAL C : $\fint_{B_r} |u-\bar{u}|^2 \le C\left(\frac{r}{R}\right)^2 \fint_{B_R} |u-\bar{u}|$ $\forall r<R$

PROOF FOR $r \ge \frac{R}{2}$ THIS IS TRIVIAL. FOR $r < R/2$ USE

$$\fint_{B_r} |u-\bar{u}|^2 \le r^2 \int_{B_r} |Du|^2 \le C\frac{r^2}{(R-r)^2} \int_{B_R} |u-\bar{u}|^2$$

POINCARÉ

CACCIOPPOLI APPLIED TO $u-\bar{u}$

$R-r \ge R/2$ AND CONCLUDE

THE IMPORTANT FACT IS THAT THE DECAY OF QUANTITIES SUCH AS $\fint_{B_r} |u-\bar{u}|^2$ CAN BE USED TO PROVE $u \in C^{0,\alpha}$

3) MORREY AND CAMPANATO SPACES

FIX $\lambda > 0$. DEFINE

$L^{p,\lambda}(\Omega) = \left\{ u \in L^p(\Omega) : \exists C \int_{\Omega\cap B_r(x)} |u|^p \le C r^\lambda \ \forall x\in\Omega\ \forall r < \mathrm{diam}(\Omega) \right\}$ (MORREY SPACES)

$\mathcal{L}^{p,\lambda}(\Omega) = \left\{ u \in L^p(\Omega) : \exists C : \int_{\Omega\cap B_r(x)} |u-\bar{u}(x,r)|^p \le C r^\lambda \ \forall x\in\Omega\ \forall r < \mathrm{diam}(\Omega) \right\}$ (CAMPANATO SPACES)

WHERE $\bar{u}(x,r) := \fint_{\Omega\cap B_r(x)} u$

WE WILL ONLY USE DOMAINS $\Omega \subset \mathbb{R}^d$ SATISFYING $\exists A > 0$

$|\Omega \cap B_r(x)| \ge A r^d$ (TYPE A DOMAINS: LIPSCHITZ DOMAINS ARE FINE)

THE PICTURE OF ALL THESE FUNCTIONAL SPACES IS THE FOLLOWING

	$\lambda = 0$	$0<\lambda<d$	$\lambda = d$	$d < \lambda \le d+p$	$\lambda > d+p$
$L^{p,\lambda}$	L^p	$L^{p,\lambda} = \mathcal{L}^{p,\lambda}$	L^∞	$\{0\}$	$\{0\}$
$\mathcal{L}^{p,\lambda}$	L^p		BMO	$C^{0,\alpha}$ WITH $\alpha = \frac{\lambda-d}{p}$	$\mathbb{R}$ (CONSTANTS)

WE ARE ESSENTIALLY INTERESTED IN $\mathcal{L}^{2,\lambda}$ SPACES

THE NORM ON THESE SPACES IS DEFINED AS

$$\|u\|_{\mathcal{L}^{p,\lambda}} := \|u\|_{L^p} + [u]_{p,\lambda} \qquad [u]_{p,\lambda} := \sqrt[p]{\sup_{\substack{x,r \\ r<\mathrm{diam}(\Omega)}} \frac{\int_{B_r(x)} |u-\bar{u}(x,r)|^p}{r^\lambda}}$$

THEOREM $\lambda > d$ $\quad u \in \mathcal{L}^{p,\lambda} \iff u$ ADMITS A $C^{0,\alpha}$ REPRESENTATIVE WITH $\alpha = \frac{\lambda-d}{p}$

MOREOVER $\|u\|_{\mathcal{L}^{p,\lambda}}$ AND $\|u\|_{C^{0,\alpha}}$ $\left(:= \|u\|_{L^\infty} + \sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \right)$

ARE EQUIVALENT NORMS

THE PROOF CAN BE FOUND IN GIAQUINTA-MARTINAZZI, THM 5.5

NOTE THAT WE CAN RECOVER THE INJECTION $W^{1,p} \hookrightarrow C^{0,\alpha}$ FOR $p>d$

INDEED, SUPPOSE $u \in W^{1,p}$ AND USE

$$\int_{B_r} |u-\bar{u}| \underset{\text{POINCARÉ}}{\leq} r \int |Du| \underset{\text{HÖLDER}}{\leq} r\, |B_r|^{\frac{1}{q}} \left(\int |Du|^p\right)^{1/p} \leq C\, r^{1+\frac{d}{q}}$$

$$\Rightarrow W^{1,p} \hookrightarrow \mathcal{L}^{1,1+\frac{d}{q}}$$

WE GET $C^{0,\alpha}$ FOR $\alpha = 1+\frac{d}{q} - d$, AND $\alpha > 0 \iff \frac{d}{q} > d-1 \iff d' > p'$
$\iff p > d$

4) REGULARITY OF QUASI-MIN

COMPARE u WITH u^*_R = HARMONIC IN B_R WITH $u-u^*_R \in H^1_0(B_R)$

$$\int_{B_R} |Du - Du^*_R|^2 = \int |Du|^2 - \int |Du^*_R|^2 + 2\underbrace{\int Du^*_R \cdot (Du^*_R - Du)}_{=0} \leq \begin{cases} CR^{2\alpha}\int |Du^*_R|^2 \leq CR^{2\alpha}\int_{B_R}|Du|^2 & (MQM) \\ CR^{2\alpha} & (AQM) \\ CR^\alpha \|Du-Du^*_R\|_R & (AQM) \end{cases}$$

FOR AQM & RQM WE GET $\|Du - Du^*_R\|_R \leq CR^\alpha$

FOR MQM IT'S THE SAME, PROVIDED $u \in Lip$.

PROP u MQM FOR $\alpha > 0 \Rightarrow u \in Lip$

PROOF $\|Du\|_r \leq \|Du - Du^*_R\|_r + \|Du^*_R\|_r$

$\|Du - Du^*_R\|_r \leq \left(\frac{R}{r}\right)^{d/2} \|Du - Du^*_R\|_R \leq CR^\alpha \|Du\|$

$\|Du^*_R\|_r \leq \|Du^*_R\|_R \leq \|Du\|_R$

USE $r_k = R_0 \cdot 2^{-k}$ $\quad \omega_k = \|Du\|_{r_k} \Rightarrow \omega_{k+1} \leq (1 + C(2^{-k})^\alpha)\omega_k \Rightarrow \omega_k$ IS BOUNDED

NOW WE KNOW THAT u ΔQM, MQM or PQM, $\alpha>0 \Rightarrow \|\nabla u - \nabla u_R^*\|_R \le CR^\alpha$

PROP u QUASI-MIN (ΔQM, MQM or PQM), $\alpha<1 \Rightarrow u \in C^{1,\alpha}$

PROOF $\|\nabla u - \overline{\nabla u}(r)\|_r \le \|\nabla u - \overline{\nabla u_R^*}\|_r \le \|\nabla u - \nabla u_R^*\|_r + \|\nabla u_R^* - \overline{\nabla u_R^*}\|_r$

$\min \int |f-c|^2$ IS REALIZED BY $c=\overline{f}$

$$\le \left(\frac{R}{r}\right)^{d/2} \underbrace{\|\nabla u - \nabla u_R^*\|_R}_{\le CR^\alpha} \qquad \le C\frac{r}{R}\|\nabla u_R^* - \overline{\nabla u_R^*}\|_R$$

USE $\overline{\nabla u_R^*} = \overline{\nabla u}(R)$

(THE AVERAGE OF ∇u ONLY DEPENDS ON $u|_{\partial B_R}$)

AND THE FACT THAT u_R^* MINIMIZES $\int_{B_R} |\nabla v|^2$ TO GET

$$\|\nabla u_R^* - \overline{\nabla u_R^*}\|_R \le \|\nabla u - \overline{\nabla u}(R)\|_R$$

SET $\omega(r) = \|\nabla u - \overline{\nabla u}(r)\|_r$. FOR $r=\tau R$, $\tau<1$, WE HAVE

$$\omega(\tau R) \le C(\tau)R^\alpha + C\tau\,\omega(R) \qquad *$$

LEMMA IF $\omega: \mathbb{R}_+ \to \mathbb{R}_+$ INCREASING (OR SUCH THAT $r^{d}\omega(r)\uparrow$) SATISFIES $* \Rightarrow \omega(r) \le Cr^\alpha$

PROOF SELECT τ SMALL S.T. $C\tau < \tau^\gamma$, $\gamma \in]\alpha,1[$, AND FIX $C=C(\tau)$.

BY INDUCTION $\omega(\tau^k R) \le \tau^{k\gamma}\omega(R) + CR^\alpha \tau^{(k-1)\alpha}\sum_{j=0}^{k-1}\tau^{(\gamma-\alpha)j}$

ONCE WE BELIEVE THIS FORMULA, WE GET

$$\omega(\tau^k R_0) \le C\left(\tau^{k\gamma} + \tau^{k\alpha}\right) \le C(\tau^k)^\alpha$$

$$\Rightarrow \omega(r) \le Cr^\alpha$$

5) EXAMPLES OF QUASI-MIN

$u \in \operatorname{argmin} \int |\nabla u|^2 + fu$, $f \in L^p$, $p>d$

$$\int_{B_R} |\nabla u|^2 + fu \le \int_{B_R} |\nabla v|^2 + fv \Rightarrow \int_{B_R} |\nabla u|^2 \le \int_{B_R} |\nabla v|^2 + \int f(v-u)$$

$$\int f(v-u) \le \left(\int_{B_R} f^2\right)^{1/2}\left(\int_{B_R}|v-u|^2\right)^{1/2}$$

$$\left(\int_{B_R} f^2\right)^{1/2} \le \left(\int_{B_R} f^p\right)^{1/p} R^{\frac{d}{2}-\frac{d}{p}} \quad \text{(HÖLDER)}, \qquad \left(\int_{B_R}|v-u|^2\right)^{1/2} \le R\left(\int|\nabla v - \nabla u|^2\right)^{1/2} \quad \text{(POINCARÉ)}$$

$$\int_{B_R} |Du|^2 \le \int_{B_R} |Dv|^2 + R^{1+\frac{d}{2}-\frac{d}{p}} \left(\int_{B_R} |Du-Dv|^2\right)^{1/2}$$

$$\Rightarrow \fint |Du|^2 \le \fint |Dv|^2 + R^{1-\frac{d}{p}} \|Du-Dv\|$$

$\Rightarrow$ u PQM for $\alpha = 1-\frac{d}{p}$

$u \in \arg\min \int a(x)|Du|^2 \qquad a \ge a_0,\ a \in C^{0,\beta}$

$$a(x_0)(1-CR^\beta)\int_{B_R}|Du|^2 \le \int_{B_R} a(x)|Du|^2 \le \int_{B_R} a(x)|Dv|^2 \le a(x_0)(1+CR^\beta)\int_{B_R}|Dv|^2$$

$\Rightarrow$ a MQM for $\alpha = \beta/2$

$u \in \arg\min \int |Du|^2 + L(x,u) \qquad L$ LIPSCHITZ IN u

$$\int_{B_R}|Du|^2 \le \int_{B_R}|Dv|^2 + C\int_{B_R}|u-v|$$

SAME ARGUMENT AS WITH $f(u-v)$ BUT $f \in L^\infty$

$\Rightarrow$ u PQM for $\alpha = 1$

$u \in \arg\min \int |Du|^2 + L(x,Du) \qquad L$ LIPSCHITZ IN Du

$\Rightarrow$ u PQM for $\alpha = 0$

$u \in \arg\min \int |Du|^2 + L(x,u,Du) \qquad L$ BOUNDED

$\Rightarrow$ u AQM for $\alpha = 0$

THE CASE $\alpha = 0$ FOR AQM AND PQM, WE EASILY GET, FOR $\omega(r) = \fint_{B_r}|Du|^2$

$$\omega(r) \le \omega(R) + C\left(\frac{R}{r}\right)^{d/2}, \text{ i.e. } \omega\left(\frac{R}{2}\right) \le \omega(R) + C$$

THIS IMPLIES $\omega(r) \le C+(|\ln(r)| \vee 1)$. USING $|\ln r| \le C r^{-\varepsilon} \vee 1$

WE GOT $\int_{B_r}|u-\bar{u}|^2 \le r^2\int_{B_r}|Du|^2 \le C r^2 r^d r^{-\varepsilon} \Rightarrow u \in \mathcal{L}^{2,d+2-\varepsilon}$

$\Rightarrow u \in C^{0,\alpha}\ \forall \alpha < 1$. ACTUALLY, IT'S ALSO POSSIBLE TO PROVE

$$|u(x)-u(y)| \le |x-y|\left(|\ln|x-y|| \vee 1\right).$$

FOR MQM WITH $\alpha = 0$ IT IS MORE COMPLICATED. $\int_{B_r}|Du|^2 \le (1+C)\int_{B_r}|Dv|^2$ IMPLIES $u \in C^{0,\alpha}$ IF C IS SMALL ENOUGH...