Urban equilibria and displacement convexity

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What I will speak about

A number of agents must choose where to live in a urban region $\Omega \subset \mathbb{R}^d$. We denote by ρ their density over Ω ($\rho \geq 0$, $\int_\Omega \rho(x)dx = 1$, i.e. $\rho \in \mathcal{P}(\Omega)$).

Agents are supposed to be identical, to have the same preferences, and to be individually negligible.

Several criteria affect the choice of each agent. We look for a simple mathematical model describing the conditions on ρ so as to have an equilibrium, and we compare the notion of equilibrium density with that of "optimal" density".

M.J. Beckmann. Spatial equilibrium and the dispersed city, *Mathematical Land Use Theory*, 1976.

M. Fujita and J. F. Thisse. *Economics of Agglomeration : Cities, Industrial Location, and Regional Growth*. 2002.

The "cost" for each agent

Suppose that every agent chooses his own location $x \in \Omega$ in order to minimize the sum of three costs :

- an exogenous cost, depending on the amenities of x only : $V(x)$ (distance to the points of interest...) ;
- an interaction cost, depending on the distances with all the other individuals ; people living at x "pay" a cost of the form $\int W(x-y)\rho(y)\,dy$ where W is usually an increasing function of the distance ;
- a residential cost, which is an increasing function of the density at x ; the individuals living at x "pay" a function of the form $h(\rho(x))$, for $h : \mathbb{R}_+ \to \mathbb{R}$ increasing ; this takes into account the fact that where more people live, the price of land is higher (or that, for the same price, they have less space).

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The total cost that we consider is $f_\rho(x) := V(x) + (W * \rho)(x) + h(\rho(x))$.

About the residential cost

Suppose that agents have a certain budget to be divided into land consumption and money consumption, and that they have a concave and increasing utility function U for land. This means they solve a problem of the form

$$\max\{U(L) + m \,:\, pL + m \leq 0\},$$

where p represents the price for land, L is the land consumption, m is the left-over of the money, and the budget constraint has been set to 0 for simplicity. The optimal land consumption will be such that $U'(L) = p$. The optimal utility is $U(L) - U'(L)L$ (relation between L and utility). The land consumption is the reciprocal of the density, hence $L = \frac{1}{\rho}$, and the residential cost $h(\rho)$, which is the opposite of the utility, is

$$h(\rho) = \frac{1}{\rho} U'\left(\frac{1}{\rho}\right) - U\left(\frac{1}{\rho}\right).$$

Remark that $t \mapsto \frac{1}{t}U'(\frac{1}{t}) - U(\frac{1}{t})$ is the derivative of $-tU(\frac{1}{t})$, hence $h = H'$ with $H(t) = -tU(\frac{1}{t})$.

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Equilibria

We look for an equilibrium configuration, i.e. a density ρ such that, for every x_0, there is no reason for people at x_0 to move to another location, since the function f_ρ is minimal at x_0, in the spirit of Nash equilibria.

Nash equilibria

Several players $i = 1, \dots, n$ must choose a strategy among a set of possibilities S_i; the pay-off of each player is given by a function $f_i : S_1 \times \cdots \times S_n \to \mathbb{R}$.

A configuration $(s_1, \dots, s_n)$ ($s_i \in S_i$) is said to be a *Nash equilibrium* if, for every i, s_i optimizes $S_i \ni s \mapsto f_i(s_1, \dots, s_{i-1}, s, s_{i+1}, \dots, s_n)$ (i.e. s_i is optimal for player i under the assumption that the other players freeze their choice).

This can be extended to a continuum of identical players where each one is negligible compared to the others (*non-atomic games*). We have a common space S of possible strategies and we look for a density ρ on S. This density induces a payoff function $f_\rho : S \to \mathbb{R}$ and we want : there exists $C \in \mathbb{R}$ such that $f_\rho(x) = C$ on $\mathrm{spt}(\rho)$ and $f_\rho(x) \geq C$ everywhere.

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Equilibrium and optimality conditions

The equilibrium condition that we consider is the following

$$\exists C \text{ s.t. } f_\rho(x) \geq C \text{ for every } x \text{ and } f_\rho(x) = C \text{ if } \rho(x) > 0.$$

Consider the following quantity

$$F(\rho) := \int_\Omega V(x)\rho(x)dx + \frac{1}{2}\int_\Omega\int_\Omega W(x-y)\rho(x)\rho(y)dxdy + \int_\Omega H(\rho(x))dx,$$

where H is defined through $H' = h$.
Suppose that ρ minimizes F in $\mathcal{P}(\Omega)$ (i.e. among densities $\rho \geq 0$ with $\int_\Omega \rho(x)dx = 1$) : then ρ is an equilibrium.

Warning : the energy F is not the total cost for all the agents, which should be $\int_\Omega f_\rho(x)\rho(x)dx$.

Games where the equilibria are found by minimizing a global energy F are called *potential games*.

D. Monderer and L. S. Shapley. Potential games. *Games and Economic Behavior*, 1996.

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Where convexity comes into play

We can say that the equilibrium condition corresponds to $F'(\rho) = 0$. Is this equivalent to the minimization of F? This depends on convexity.

- $\rho \mapsto \int_\Omega V(x)\rho(x)dx$ is linear, hence convex.
- $\rho \mapsto \int_\Omega H(\rho(x))dx$, is convex, since H is convex ($h = H'$ was increasing).
- unfortunately, $\rho \mapsto \frac{1}{2}\int_\Omega\int_\Omega W(x-y)\rho(x)\rho(y)dxdy$ is not convex in general...

Example : take $W(x-y) = |x-y|^2$ and compute

$$\int_\Omega\int_\Omega |x-y|^2\rho(x)\rho(y)dxdy$$
$$= \int_\Omega\int_\Omega |x|^2\rho(x)\rho(y)dxdy + \int_\Omega\int_\Omega |y|^2\rho(x)\rho(y)dxdy - 2\int_\Omega\int_\Omega x\cdot y\,\rho(x)\rho(y)dxdy$$
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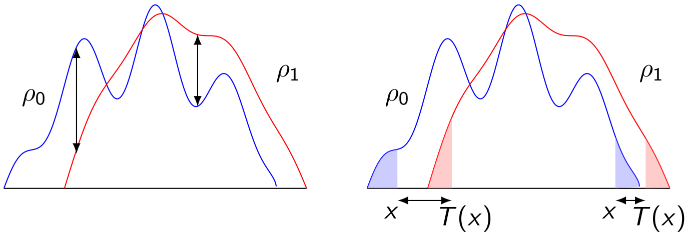
\end{aligned}$$

Vertical and horizontal distances

Given ρ_0, ρ_1 two densities on Ω, define

$$W_2(\rho_0, \rho_1) := \min \left\{ \sqrt{\int_\Omega |T(x) - x|^2 \rho_0(x) dx} \ : \ T_\# \rho_0 = \rho_1 \right\},$$

where the symbol $\#$ denotes the image measure : $\int \phi(T(x))\rho_0(x)dx = \int \phi(y)\rho_1(y)dy$ for every $\phi : \Omega \to \mathbb{R}$. This quantity, called **Wasserstein distance**, is a distance on probability densities in $\mathcal{P}(\Omega)$. It is somehow an "horizontal" distance, if compared to usual L^p distances



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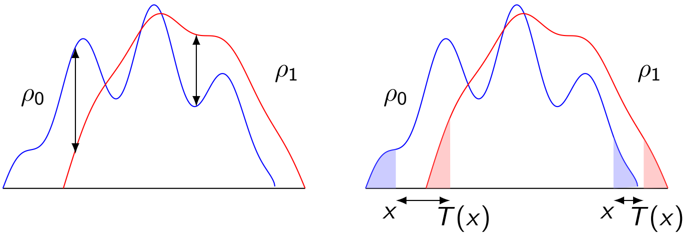

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BREAK

–

time for commercials

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Progress in Nonlinear Differential Equations and Their Applications

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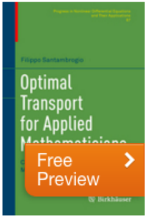

Optimal Transport for Applied Mathematicians

Calculus of Variations, PDEs, and Modeling

Authors: **Santambrogio**, Filippo

Birkhäuser

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Vertical and horizontal interpolations

Consider the optimal T in the minimization problem defining W_2. By the way, it exists, it is unique, and it is of the form $T = \nabla u$ for u convex (*Brenier Theorem*).

We can define ρ_t through $\rho_t = ((1-t)id + tT)_\#(\rho_0) \in \mathcal{P}(\Omega)$ (supposing Ω to be convex) This curve of densities is a **geodesic** for the distance W_2. It gives an "horizontal" interpolation between ρ_0 and ρ_1, different from the standard "vertical" one $(1-t)\rho_0 + t\rho_1$).
A functional $F : \mathcal{P}(\Omega) \to \mathbb{R}$ is said to be *displacement convex* if $t \mapsto F(\rho_t)$ is convex for every ρ_0, ρ_1.

Y. Brenier, Décomposition polaire et réarrangement monotone des champs de vecteurs. *C. R. A. S.*, 1987.

R. J. McCann A convexity principle for interacting gases. *Adv. Math.*, 1997.

Displacement convex energies

Fortunately, it can be proven (McCann) that

- $\rho \mapsto \int V(x)\rho(x)dx$ is displacement convex if V is convex,
- $\rho \mapsto \int\int W(x-y)\rho(x)\rho(y)dxdy$ is displacement convex if W is convex,
- $\rho \mapsto \int H(\rho(x))dx$ is displacement convex if H is convex and $t \mapsto t^d H(t^{-d})$ is convex and decreasing ($\Omega \subset \mathbb{R}^d$, where d is the dimension). **Examples :** $H(t) = t\log t, H(t) = t^p, p > 1\dots$

Moreover : if F is displacement convex, then every equilibrium is a minimizer, and if we have strict displacement convexity (if V is strictly convex) the equilibrium is unique. If only W and/or $t^d H(t^{-d})$ are strictly convex, it is unique up to translations.

Important : the assumption on H is easy to write in term of U. We need $t \mapsto U(t^d)$ to be increasing (which is fine) and concave.

A. Blanchet, P. Mossay and F. Santambrogio Existence and uniqueness of equilibrium for a spatial model of social interactions, *Int. Econ. Rev.*, 2015.

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- $\rho \mapsto \int\int W(x-y)\rho(x)\rho(y)dxdy$ is displacement convex if W is convex,
- $\rho \mapsto \int H(\rho(x))dx$ is displacement convex if H is convex and $t \mapsto t^d H(t^{-d})$ is convex and decreasing ($\Omega \subset \mathbb{R}^d$, where d is the dimension). **Examples :** $H(t) = t\log t, H(t) = t^p, p > 1\dots$

Moreover : if F is displacement convex, then every equilibrium is a minimizer, and if we have strict displacement convexity (if V is strictly convex) the equilibrium is unique. If only W and/or $t^d H(t^{-d})$ are strictly convex, it is unique up to translations.

Important : the assumption on H is easy to write in term of U. We need $t \mapsto U(t^d)$ to be increasing (which is fine) and concave.

A. Blanchet, P. Mossay and F. Santambrogio Existence and uniqueness of equilibrium for a spatial model of social interactions, *Int. Econ. Rev.*, 2015.

Displacement convex energies

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Example 1 : a Gaussian

In general, the equilibrium condition may be re-written as

$$h(\rho(x)) = \max\{C - V - (W * \rho), h(0)\}.$$

Take $U(t) = \log t$, hence $H(t) = t \log t$ and $h(t) = \log t + 1$. Take $V = 0$ and $W(x - y) = |x - y|^2$ and $\Omega = \mathbb{R}^d$. The equilibrium is unique up to translations. Moreover

$$(W * \rho)(x) = \int |x - y|^2 \rho(y) dy = |x|^2 - 2x \cdot \int y\rho(y)dy + \int |y|^2 \rho(y) dy$$
$$= |x - x_0|^2 + c,$$

where $x_0 = \int y\rho(y)dy$ and $c = \int |y|^2 \rho(y)dy - |x_0|^2$. The equilibrium condition reads

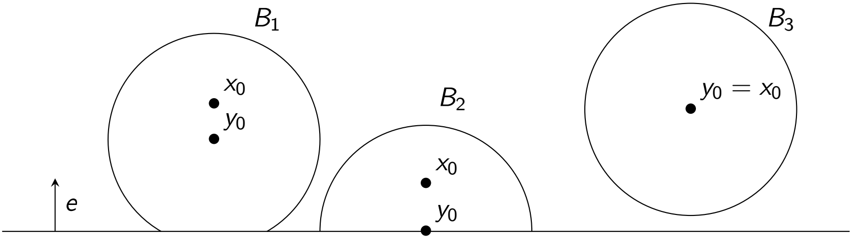
$$\log \rho(x) = C - |x - x_0|^2 \ \Rightarrow \ \rho(x) = ce^{-|x-x_0|^2}.$$

Example 2 : a sea-shore model

Take $U(t) = -\frac{1}{2t}$, hence $H(t) = \frac{1}{2}t^2$ and $h(t) = t$. Take $\Omega = \{x \in \mathbb{R}^2 : x \cdot e > 0\}$, $V(x) = x \cdot e$ and $W(x - y) = \frac{1}{2}|x - y|^2$. We have

$$\rho(x) = \left(C - \frac{1}{2}|x - x_0|^2 - x \cdot e\right)_+ = \left(C - \frac{1}{2}|x - (x_0 + e)|^2\right)_+ .$$

The spatial equilibrium distribution corresponds to a truncated paraboloid centered at $y_0 = x_0 - e$. The support of all possible spatial equilibria must intersect the boundary $e^\perp$ and that the distance from y_0 to that boundary must be fixed (the same for all equilibria).



A case without convexity - the model

Consider now $\Omega = \mathbb{S}^1 \approx [0, 2\pi]$, $W(x-y) = \tau d_{\mathbb{S}^1}(x, y)$ (where $d_{\mathbb{S}^1}(x, y) = \min\{|x - y + 2k\pi|,\ k \in \mathbb{Z}\}$), $V = 0$ and $h(t) = \beta t$, with $\tau, \beta > 0$.
We have

$$\rho(x) = \left(C - \delta^2\phi(x)\right)_+ ,$$

where $\delta^2 = \tau/\beta$ and

$$\phi(x) = \int_{\mathbb{S}^1} |x - y|\rho(y)dy - \frac{\pi}{2}.$$

Remark $\phi(x) + \phi(x + \pi) = 0$ and

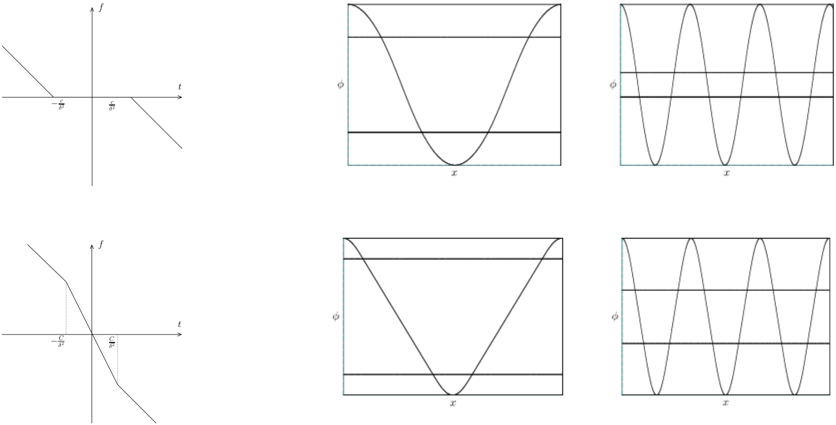
$$\phi''(x) = 2\rho(x) - 2\rho(x + \pi) = 2(C - \delta^2\phi(x))_+ - 2(C + \delta^2\phi(x))_+.$$

It is enough to solve $\phi'' = f(\phi)$ with $f(t) = 2(C - \delta^2 t)_+ - 2(C + \delta^2 t)_+$ and then find ρ.

P. Mossay and P. Picard. A spatial model of social interactions. *J. Econ. Theory*, 2011.

A case without convexity - solution

From the form of the function f, the solution ϕ is composed of sinusoidal oscillations. We distinguish $C > 0$ and $C < 0$.



There are multiple solutions, with possibly disconnected “cities”. The number of oscillations is odd and can arrive up to $\sqrt{2}\delta$.

The End

Thanks for your attention