Asymptotical problems in optimal location

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Optimal location

Facilities and discretization issues

Optimal location of N points

Take a domain $\Omega \subset \mathbb{R}^d$, a probability density $f : \Omega \to \mathbb{R}$, $f \geq 0$, $\int f = 1$, and an exponent $p > 0$. We look for a finite set Σ so that we solve

$$\min \quad J(\Sigma) := \int_\Omega d(x, \Sigma)^p f(x) dx \quad : \quad \Sigma \subset \overline{\Omega}, \ \#\Sigma \leq N,$$

where $d(x, \Sigma) := \min_{y \in \Sigma} |x - y|$.

Interpretations : Σ stands for some facilities to be located so as to satisfy in the best possible way a population f ; other possibility : f is a continuous distribution to be discretized with a one concentrated on Σ. It is a classical problem, known as **Fermat-Weber** Problem, or $(N, p)-$**mean** problem.

A. Weber. *Über den Standort der Industrien, Erster Teil : Reine Theorie des Standortes*, 1909.
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Voronoi cells

Every $\Sigma = \{x_1, x_2, \ldots, x_N\}$ induces a partition of Ω into convex polyhedral cells, called voronoi cells

$$V_i := \{x \in \Omega \ : \ d(x, \Sigma) = |x - x_i|\} = \{x \in \Omega \ : \ |x - x_i| \leq |x - x_j| \ \forall j\}.$$

If Σ is optimal, then we have an extra property :

$$\forall i, \quad y \mapsto \int_{V_i} |y - x|^p f(x) dx \quad \text{is minimal for } y = x_i.$$

In particular, for $p = 2$, this means that the points x_i are the barycenters of each cell V_i. This is called a *centroidal Voronoi tesselation*.

Also, if we solve

$$\min W_p(\nu, f) \quad : \quad \# \operatorname{spt}(\nu) \leq N,$$

(where W_p is the $p-$th Wasserstein distance between probability measures) then we find $\nu = \sum_{i=1}^N a_i \delta_{x_i}$ with $a_i = \int_{V_i} f$.

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From finite dimension to asymptotics

Finding the optimal Σ is a finite-dimensional problem and is quite feasible if N is small. Yet, the problem is not convex and has plenty of local minima, which makes it difficult to solve for larger N.

The question here is to look at some asymptotical behavior for $N \to \infty$ (i.e. if N is VERY large), thus predicting some general features of the optimal Σ without computing it for $N = 10^6$ or 10^9.

One of the main questions is the "density" of the points of Σ : it is clear that, if N is very large, we will put more points where f is higher. How to make it quantitative : **will the number of points per unit area be**

proportional to f ? to f^2 ? to $\sqrt{f}$?

And, besides the density, what about the geometry ? should we put the points on a regular grid ?

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$\Gamma-$convergence

The limit of a sequence of minimization problems

Limits of optimization problems : $\Gamma-$convergence

On a metric space X let $F_n : X \to \mathbb{R} \cup \{+\infty\}$ be a sequence of functions. We define the two lower-semicontinuous functions F^- and F^+ (called $\Gamma - \liminf$ and $\Gamma - \limsup$ F^+ of this sequence, respectively) by

$$F^-(x) := \inf\{\liminf_{n\to\infty} F_n(x_n) : x_n \to x\}, F^+(x) := \inf\{\limsup_{n\to\infty} F_n(x_n) : x_n \to x\}.$$

If $F^- = F^+ = F$ coincide, then we say $F_n \xrightarrow{\Gamma} F$.

Among the properties of $\Gamma-$convergence we have the following :

- if there exists a compact set $K \subset X$ such that $\inf_X F_n = \inf_K F_n$ for any n, then F attains its minimum and $\inf F_n \to \min F$;
- if $(x_n)_n$ is a sequence of minimizers for F_n admitting a subsequence converging to x, then x minimizes F
- if F_n is a sequence $\Gamma-$converging to F, then $F_n + G$ will $\Gamma-$converge to $F + G$ for any continuous function $G : X \to \mathbb{R} \cup \{+\infty\}$.

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Limits of optimization problems : Γ−convergence

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Setting of the Γ-convergence problem

For each N, the optimization problem on Σ lives in the class of the sets with no more than N points. A way to set them all on a same space, and to address the density question at the same time is the following.
If $\#\Sigma = N$, let's define a probability μ_Σ through

$$\mu_\Sigma := \frac{1}{N}\sum_{y\in\Sigma}\delta_y.$$

Then define

$$F_N(\mu) := \begin{cases} J(\Sigma) & \text{if } \mu = \mu_\Sigma, \\ +\infty & \text{oherwise.} \end{cases}$$

Minimizing F_N over $\mathcal{P}(\overline{\Omega})$ is equivalent to minimizing J over sets Σ with prescribed cardinality.

Can we recover the asymptotics by looking at the Γ−limit of F_N ?

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Scaling

Actually, we can see that $\min F_N \to 0$ and we can also see that the $\Gamma-$limit of F_N is the functional 0.
We need to rescale, and guess the order of $\min F_N$.

For any reasonably uniform configuration Σ_N with N points we have $d(x, \Sigma) \approx N^{-1/d}$ and hence we can guess $\min F_N \approx N^{-p/d}$.

As a consequence, the new functionals F_N that we consider are $N^{p/d} F_N$. What is the $\Gamma-$limit of these functionals ?

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The result

First define a magical constant

$$\theta_{p,d} := \inf\left\{\liminf_N N^{p/d}\int_{[0,1]^d} d(x,\Sigma_N)dx \;:\; \Sigma_N\subset[0,1]^d,\ \#\Sigma_N = N\right\}$$

Then we can prove

Theorem

The functionals $N^{p/d}F_N$ has a $\Gamma-$limit in the space $\mathcal{P}(\overline{\Omega})$ endowed with the weak convergence, which is the functional F_∞ given by

$$F_\infty(\mu) = \theta_{p,d}\int_\Omega \frac{f(x)}{\mu^a(x)^{p/d}}dx,$$

where $\mu = \mu^a + \mu^s$ is the Radon-Nikodym decomposition of μ into an absolutely continuous and a singular part.

G. Bouchitté, C. Jimenez, M. Rajesh, Asymptotique d'un problème de positionnement optimal, *C. R. Acad. Sci. Paris* 2002.

The constant $\theta_{p,d}$ and the limits of minima and minimizers

If $\theta_{p,d} > 0$ the minimizer of F_∞ is

$$\mu = \mu^a + \mu^s, \quad \mu^s = 0, \quad \mu^a(x) = cf(x)^{d/(d+p)},$$

(with c such that $\int \mu^a = 1$) which gives the limit density of the sets Σ_N.

What about their geometric shapes ? they follow the optimal structure of the minimizers in the definition of $\theta_{p,d}$ (i.e. they do not depend neither on f nor on Ω).

For $d = 2$ and any p the optimal structure is that of a regular hexagonal tiling. For instance, $\theta_{1,2} = \frac{4+3\ln 3}{6\sqrt{2}\,3^{3/4}} \approx 0,377$.

In higher dimension we only know $0 < \theta_{p,d} < +\infty$ and some bounds on it, but the optimal structure is an open problem !

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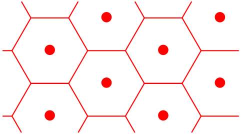

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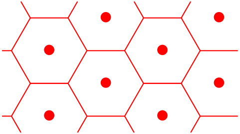

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Extensions

Non-positive f

Short and Long Term

Dropping the assumption $f \geq 0$

Some intriguing models could consider the distribution of some facilities which are loved by some agents and hated by others. Think at polluting industries, for instance.
This can be translated into the fact that f has not a fixed sign (some want to minimize the distance to Σ, some to maximize it, with possible different weights)

$$\min \int_\Omega d(x, \Sigma)[f^+(x) - f^-(x)]\, dx.$$

In this case some features of the optimization problem do not hold any more.

- It is no more true that $\min F_N \to 0$;
- The optimal sets Σ_N do not tend to fill Ω;
- By the way, the problem $\min J(\Sigma) \ : \ \Sigma \subset \overline{\Omega}$ (without constraints on the number of points) is not trivial.

Asymptotic results for non-positive f

The situation is the following. First solve

$$\min J(M) \, : \, M \text{ compact}, \; M \subset \overline{\Omega}.$$

Call M_0 an optimal set. It satisfies $f \geq 0$ a.e. on M_0.
Then, under some technical assumptions on the supports of the positive and negative parts of f :

Take a sequence of optimizers Σ_N such that $\Sigma_N \to M_0$ (in the Hausdorff topology ; any sequence of optimizers converge to one of these optimal sets). Then the behavior of Σ_N and of the measures μ_{Σ_N} is the same as what we see in the optimal sets when we replace f with $f I_{M_0} \geq 0$.

This is obtained by looking at the limits of

$$\min \; N^{p/d} \left(\int_\Omega d(x,\Sigma) f(x) dx - \int_\Omega d(x, M_0) f(x) dx \right) \; : \; \#\Sigma \leq N.$$

G. Buttazzo, F. Santambrogio, E. Stepanov Asymptotic optimal location of facilities in a competition between population and industries, *Ann. Sc. Norm. Sup.*, 2012

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Short and long term problems

Think at facilities location : in real life, we cannot in general choose the position of N points at a time. We are more likely to add one point at each step.
In this case we should solve this sequence of optimization problems :

$$(P_N) \quad \min J(\Sigma) \;:\; \Sigma \supset \Sigma_{N-1},\; \#\Sigma \leq \Sigma_{N-1} + 1$$

where we call Σ_N the (a) minimizer of (P_N).

This problem cannot unfortunately be attacked via Γ−convergence. But some questions can be considered as well :

- What about the minimal value of (P_N) ? does it decay as $N^{-p/d}$?
- how much does it cost to be "short term-minded" : what about the ration $(min\ of\ P_N)/\min F_n$ as $N \to \infty$?
- What is the asymptotical behavior of the solutions Σ_N, both in terms of the limit of μ_{Σ_N} and of the geometrical structure ?

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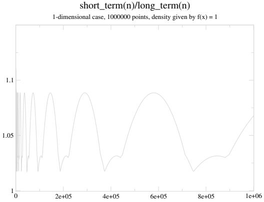
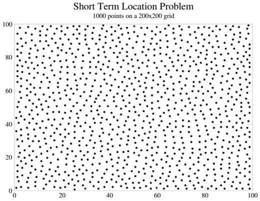
Very partial results

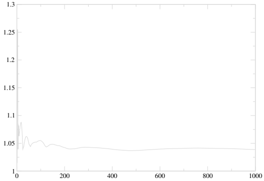
The case $d = 1$, $\Omega = [0,1]$, $f = 1$ can be solved explicitly. In this case

- The minimal value of (P_N) decays as N^{-p} but $N^p(\textit{min of } P_N)$ has no limit as $N \to \infty$
- The measures μ_{Σ_N} do not converge. They have converging subsequence, but none is converging to the uniform measure.

Almost all the other cases are open !







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Long-term planning versus short-term planning in the asymptotical location problem, *ESAIM :COCV*, 2009

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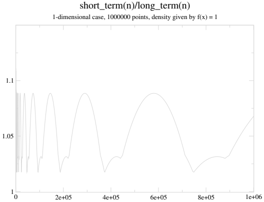

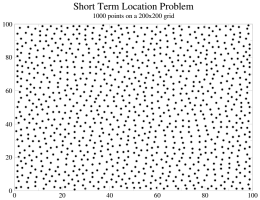

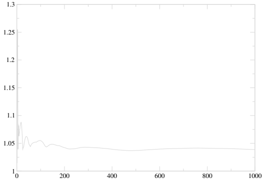

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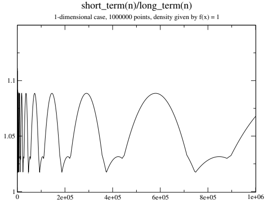
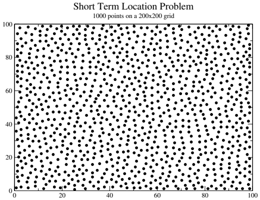
Very partial results

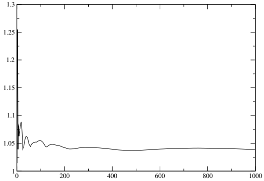
The case $d = 1$, $\Omega = [0, 1]$, $f = 1$ can be solved explicitly. In this case

- The minimal value of (P_N) decays as N^{-p} but $N^p(min\,of\,P_N)$ has no limit as $N \to \infty$
- The measures μ_{Σ_N} do not converge. They have converging subsequence, but none is converging to the uniform measure.

Almost all the other cases are open !







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The End

Thanks for your attention