Crowd motion with density constraints and optimal transport

Filippo Santambrogio

Laboratoire de Mathématiques d'Orsay, Université Paris-Sud
http://www.math.u-psud.fr/∼santambr/

USTC, Hefei, 2012/04/24

1. Micro and macro models with constraints
    - Disks with no overlapping
    - Density $\rho \leq 1$
    - The continuity equation
2. Gradient-flow formulation
    - $x' = -\nabla F(x)$
    - The case of the Wasserstein spaces
    - Few words about optimal transport
    - Some examples of PDEs obtained via gradient-flows
3. The macroscopic model as a gradient-flow
    - The time-discretized problems
    - Existence of a solution with and without exit
    - Uniqueness questions
4. Perspectives
    - Strategies
    - Second-order models
    - Multi-populations

1. Micro and macro models with constraints
    - Disks with no overlapping
    - Density $\rho \leq 1$
    - The continuity equation
2. Gradient-flow formulation
    - $x' = -\nabla F(x)$
    - The case of the Wasserstein spaces
    - Few words about optimal transport
    - Some examples of PDEs obtained via gradient-flows
3. The macroscopic model as a gradient-flow
    - The time-discretized problems
    - Existence of a solution with and without exit
    - Uniqueness questions
4. Perspectives
    - Strategies
    - Second-order models
    - Multi-populations

1. Micro and macro models with constraints
   - Disks with no overlapping
   - Density $\rho \leq 1$
   - The continuity equation
2. Gradient-flow formulation
   - $x' = -\nabla F(x)$
   - The case of the Wasserstein spaces
   - Few words about optimal transport
   - Some examples of PDEs obtained via gradient-flows
3. The macroscopic model as a gradient-flow
   - The time-discretized problems
   - Existence of a solution with and without exit
   - Uniqueness questions
4. Perspectives
   - Strategies
   - Second-order models
   - Multi-populations

1. Micro and macro models with constraints
   - Disks with no overlapping
   - Density $\rho \leq 1$
   - The continuity equation
2. Gradient-flow formulation
   - $x' = -\nabla F(x)$
   - The case of the Wasserstein spaces
   - Few words about optimal transport
   - Some examples of PDEs obtained via gradient-flows
3. The macroscopic model as a gradient-flow
   - The time-discretized problems
   - Existence of a solution with and without exit
   - Uniqueness questions
4. Perspectives
   - Strategies
   - Second-order models
   - Multi-populations

Models

Micro and macro models with constraints

A general model

A particle population needs to move, and each particle - if alone - would follow its own velocity u (depend on time, position...) Yet, particles are rigid disks that cannot overlap, hence, the actual velocity v will not be u if u is too concentrating. Let us suppose $v = P_{adm(q)}(u)$, where q is the particle configuration, $adm(q)$ the set of velocities that do not induce overlapping, $P_{adm(q)}$ the projection on this set.

If every particle is a disk with radius R, located at q_i, we have

$$q \in K := \{q = (q_i)_i \in \Omega^N \ : \ |q_i - q_j| \geq 2R\}$$

$$adm(q) = \{v = (v_i)_i \ : \ (v_i - v_j) \cdot (q_i - q_j) \geq 0 \ \forall (i,j) \ : \ |q_i - q_j| = 2R\}$$

and we solve $q'(t) = P_{adm(q(t))}u(t)$ (with $q(0)$ given).

Typical case : $u_i(q) = -\nabla D(q_i)$, where $D(x) = d(x, \Gamma)$ and $\Gamma \subset \partial\Omega$; if Ω is not convex we consider as d the geodesic distance.

B. Maury, J. Venel, Handling of contacts in crowd motion simulations, *Traffic and Granular Flow*, 2007.

A general model

A particle population needs to move, and each particle - if alone - would follow its own velocity u (depend on time, position...) Yet, particles are rigid disks that cannot overlap, hence, the actual velocity v will not be u if u is too concentrating. Let us suppose $v = P_{adm(q)}(u)$, where q is the particle configuration, $adm(q)$ the set of velocities that do not induce overlapping, $P_{adm(q)}$ the projection on this set.
If every particle is a disk with radius R, located at q_i, we have

$$q \in K := \{q = (q_i)_i \in \Omega^N \ : \ |q_i - q_j| \geq 2R\}$$

$$adm(q) = \{v = (v_i)_i \ : \ (v_i - v_j) \cdot (q_i - q_j) \geq 0 \ \forall (i,j) \ : \ |q_i - q_j| = 2R\}$$

and we solve $q'(t) = P_{adm(q(t))}u(t)$ (with $q(0)$ given).

Typical case : $u_i(q) = -\nabla D(q_i)$, where $D(x) = d(x, \Gamma)$ and $\Gamma \subset \partial\Omega$; if Ω is not convex we consider as d the geodesic distance.

B. Maury, J. Venel, Handling of contacts in crowd motion simulations, *Traffic and Granular Flow*, 2007.

A general model

A particle population needs to move, and each particle - if alone - would follow its own velocity u (depend on time, position...) Yet, particles are rigid disks that cannot overlap, hence, the actual velocity v will not be u if u is too concentrating. Let us suppose $v = P_{adm(q)}(u)$, where q is the particle configuration, $adm(q)$ the set of velocities that do not induce overlapping, $P_{adm(q)}$ the projection on this set.

If every particle is a disk with radius R, located at q_i, we have

$$q \in K := \{q = (q_i)_i \in \Omega^N \; : \; |q_i - q_j| \geq 2R\}$$

$$adm(q) = \{v = (v_i)_i \; : \; (v_i - v_j) \cdot (q_i - q_j) \geq 0 \; \forall (i,j) \; : \; |q_i - q_j| = 2R\}$$

and we solve $q'(t) = P_{adm(q(t))} u(t)$ (with $q(0)$ given).

Typical case : $u_i(q) = -\nabla D(q_i)$, where $D(x) = d(x, \Gamma)$ and $\Gamma \subset \partial\Omega$; if Ω is not convex we consider as d the geodesic distance.

B. Maury, J. Venel, Handling of contacts in crowd motion simulations, *Traffic and Granular Flow*, 2007.

A general model

A particle population needs to move, and each particle - if alone - would follow its own velocity u (depend on time, position...) Yet, particles are rigid disks that cannot overlap, hence, the actual velocity v will not be u if u is too concentrating. Let us suppose $v = P_{adm(q)}(u)$, where q is the particle configuration, $adm(q)$ the set of velocities that do not induce overlapping, $P_{adm(q)}$ the projection on this set.
If every particle is a disk with radius R, located at q_i, we have

$$q \in K := \{q = (q_i)_i \in \Omega^N \ : \ |q_i - q_j| \geq 2R\}$$

$$adm(q) = \{v = (v_i)_i \ : \ (v_i - v_j) \cdot (q_i - q_j) \geq 0 \ \forall (i,j) \ : \ |q_i - q_j| = 2R\}$$

and we solve $q'(t) = P_{adm(q(t))} u(t)$ (with $q(0)$ given).

Typical case : $u_i(q) = -\nabla D(q_i)$, where $D(x) = d(x, \Gamma)$ and $\Gamma \subset \partial\Omega$; if Ω is not convex we consider as d the geodesic distance.

B. Maury, J. Venel, Handling of contacts in crowd motion simulations, *Traffic and Granular Flow*, 2007.

A general model

A particle population needs to move, and each particle - if alone - would follow its own velocity u (depend on time, position...) Yet, particles are rigid disks that cannot overlap, hence, the actual velocity v will not be u if u is too concentrating. Let us suppose $v = P_{adm(q)}(u)$, where q is the particle configuration, $adm(q)$ the set of velocities that do not induce overlapping, $P_{adm(q)}$ the projection on this set.
If every particle is a disk with radius R, located at q_i, we have

$$q \in K := \{q = (q_i)_i \in \Omega^N \; : \; |q_i - q_j| \geq 2R\}$$

$$adm(q) = \{v = (v_i)_i \; : \; (v_i - v_j) \cdot (q_i - q_j) \geq 0 \; \forall (i,j) \; : \; |q_i - q_j| = 2R\}$$

and we solve $q'(t) = P_{adm(q(t))}u(t)$ (with $q(0)$ given).

Typical case : $u_i(q) = -\nabla D(q_i)$, where $D(x) = d(x, \Gamma)$ and $\Gamma \subset \partial\Omega$; if Ω is not convex we consider as d the geodesic distance.

B. Maury, J. Venel, Handling of contacts in crowd motion simulations, *Traffic and Granular Flow*, 2007.

A general model

A particle population needs to move, and each particle - if alone - would follow its own velocity u (depend on time, position...) Yet, particles are rigid disks that cannot overlap, hence, the actual velocity v will not be u if u is too concentrating. Let us suppose $v = P_{adm(q)}(u)$, where q is the particle configuration, $adm(q)$ the set of velocities that do not induce overlapping, $P_{adm(q)}$ the projection on this set.
If every particle is a disk with radius R, located at q_i, we have

$$q \in K := \{q = (q_i)_i \in \Omega^N \ : \ |q_i - q_j| \geq 2R\}$$

$$adm(q) = \{v = (v_i)_i \ : \ (v_i - v_j) \cdot (q_i - q_j) \geq 0 \ \forall (i,j) \ : \ |q_i - q_j| = 2R\}$$

and we solve $q'(t) = P_{adm(q(t))}u(t)$ (with $q(0)$ given).

Typical case : $u_i(q) = -\nabla D(q_i)$, where $D(x) = d(x, \Gamma)$ and $\Gamma \subset \partial\Omega$; if Ω is not convex we consider as d the geodesic distance.

B. Maury, J. Venel, Handling of contacts in crowd motion simulations, *Traffic and Granular Flow*, 2007.

Continuous formulation

- The particles population will be described by a density $\rho \in \mathcal{P}(\Omega)$,
- the constraint by $K = \{\rho \in \mathcal{P}(\Omega) \, : \, \rho \leq 1\}$,
- $u : \Omega \to \mathbb{R}^d$ will be a vector field, possibly depending on time or on ρ,
- $adm(\rho) = \{v : \Omega \to \mathbb{R}^d \, : \, \nabla \cdot v \geq 0 \text{ on } \rho = 1\}$,
- P is the projection in $L^2(dx)$ or (which is the same) in $L^2(\rho)$,
- we'll solve $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t \left(P_{adm(\rho_t)} u_t\right)\right) = 0$.

The equation $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t v_t\right) = 0$ (*continuity equation*) is exactly the equation satisfied by the evolution of a density ρ when each particle follows the velocity field v (with $v \cdot n = 0$ on $\partial\Omega$, so that the density does not exit Ω).

Difficulty : $v = P_{adm(\rho_t)} u_t$ is not regular, neither depends regularly on ρ.

Continuous formulation

- The particles population will be described by a density $\rho \in \mathcal{P}(\Omega)$,
- the constraint by $K = \{\rho \in \mathcal{P}(\Omega) \,:\, \rho \leq 1\}$,
- $u : \Omega \to \mathbb{R}^d$ will be a vector field, possibly depending on time or on ρ,
- $adm(\rho) = \{v : \Omega \to \mathbb{R}^d \,:\, \nabla \cdot v \geq 0 \text{ on } \rho = 1\}$,
- P is the projection in $L^2(dx)$ or (which is the same) in $L^2(\rho)$,
- we'll solve $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t \left(P_{adm(\rho_t)} u_t\right)\right) = 0$.

The equation $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t v_t\right) = 0$ (*continuity equation*) is exactly the equation satisfied by the evolution of a density ρ when each particle follows the velocity field v (with $v \cdot n = 0$ on $\partial\Omega$, so that the density does not exit Ω).

Difficulty : $v = P_{adm(\rho_t)} u_t$ is not regular, neither depends regularly on ρ.

Continuous formulation

- The particles population will be described by a density $\rho \in \mathcal{P}(\Omega)$,
- the constraint by $K = \{\rho \in \mathcal{P}(\Omega) \,:\, \rho \leq 1\}$,
- $u : \Omega \to \mathbb{R}^d$ will be a vector field, possibly depending on time or on ρ,
- $adm(\rho) = \{v : \Omega \to \mathbb{R}^d \;:\; \nabla \cdot v \geq 0 \;\text{ on } \rho = 1\}$,
- P is the projection in $L^2(dx)$ or (which is the same) in $L^2(\rho)$,
- we'll solve $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t \left(P_{adm(\rho_t)} u_t\right)\right) = 0$.

The equation $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t v_t\right) = 0$ (*continuity equation*) is exactly the equation satisfied by the evolution of a density ρ when each particle follows the velocity field v (with $v \cdot n = 0$ on $\partial\Omega$, so that the density does not exit Ω).

Difficulty : $v = P_{adm(\rho_t)} u_t$ is not regular, neither depends regularly on ρ.

Continuous formulation

- The particles population will be described by a density $\rho \in \mathcal{P}(\Omega)$,
- the constraint by $K = \{\rho \in \mathcal{P}(\Omega) \, : \, \rho \leq 1\}$,
- $u : \Omega \to \mathbb{R}^d$ will be a vector field, possibly depending on time or on ρ,
- $adm(\rho) = \{v : \Omega \to \mathbb{R}^d \; : \; \nabla \cdot v \geq 0 \text{ on } \rho = 1\}$,
- P is the projection in $L^2(dx)$ or (which is the same) in $L^2(\rho)$,
- we'll solve $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t \left(P_{adm(\rho_t)} u_t\right)\right) = 0$.

The equation $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t v_t\right) = 0$ (*continuity equation*) is exactly the equation satisfied by the evolution of a density ρ when each particle follows the velocity field v (with $v \cdot n = 0$ on $\partial\Omega$, so that the density does not exit Ω).

Difficulty : $v = P_{adm(\rho_t)} u_t$ is not regular, neither depends regularly on ρ.

Continuous formulation

- The particles population will be described by a density $\rho \in \mathcal{P}(\Omega)$,
- the constraint by $K = \{\rho \in \mathcal{P}(\Omega) \, : \, \rho \leq 1\}$,
- $u : \Omega \to \mathbb{R}^d$ will be a vector field, possibly depending on time or on ρ,
- $adm(\rho) = \{v : \Omega \to \mathbb{R}^d \; : \; \nabla \cdot v \geq 0 \;\text{ on } \rho = 1\}$,
- P is the projection in $L^2(dx)$ or (which is the same) in $L^2(\rho)$,
- we'll solve $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t \left(P_{adm(\rho_t)} u_t\right)\right) = 0$.

The equation $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t v_t\right) = 0$ (*continuity equation*) is exactly the equation satisfied by the evolution of a density ρ when each particle follows the velocity field v (with $v \cdot n = 0$ on $\partial\Omega$, so that the density does not exit Ω).

Difficulty : $v = P_{adm(\rho_t)} u_t$ is not regular, neither depends regularly on ρ.

Continuous formulation

- The particles population will be described by a density $\rho \in \mathcal{P}(\Omega)$,
- the constraint by $K = \{\rho \in \mathcal{P}(\Omega) \,:\, \rho \leq 1\}$,
- $u : \Omega \to \mathbb{R}^d$ will be a vector field, possibly depending on time or on ρ,
- $adm(\rho) = \{v : \Omega \to \mathbb{R}^d \,:\, \nabla \cdot v \geq 0 \text{ on } \rho = 1\}$,
- P is the projection in $L^2(dx)$ or (which is the same) in $L^2(\rho)$,
- we'll solve $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t \left(P_{adm(\rho_t)} u_t\right)\right) = 0$.

The equation $\frac{\partial}{\partial t}\rho_t + \nabla \cdot (\rho_t v_t) = 0$ (*continuity equation*) is exactly the equation satisfied by the evolution of a density ρ when each particle follows the velocity field v (with $v \cdot n = 0$ on $\partial\Omega$, so that the density does not exit Ω).

Difficulty : $v = P_{adm(\rho_t)} u_t$ is not regular, neither depends regularly on ρ.

Continuous formulation

- The particles population will be described by a density $\rho \in \mathcal{P}(\Omega)$,
- the constraint by $K = \{\rho \in \mathcal{P}(\Omega) \, : \, \rho \leq 1\}$,
- $u : \Omega \to \mathbb{R}^d$ will be a vector field, possibly depending on time or on ρ,
- $adm(\rho) = \{v : \Omega \to \mathbb{R}^d \, : \, \nabla \cdot v \geq 0 \text{ on } \rho = 1\}$,
- P is the projection in $L^2(dx)$ or (which is the same) in $L^2(\rho)$,
- we'll solve $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t \left(P_{adm(\rho_t)} u_t\right)\right) = 0$.

The equation $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t v_t\right) = 0$ (*continuity equation*) is exactly the equation satisfied by the evolution of a density ρ when each particle follows the velocity field v (with $v \cdot n = 0$ on $\partial\Omega$, so that the density does not exit Ω).

Difficulty : $v = P_{adm(\rho_t)} u_t$ is not regular, neither depends regularly on ρ.

Continuous formulation

- The particles population will be described by a density $\rho \in \mathcal{P}(\Omega)$,
- the constraint by $K = \{\rho \in \mathcal{P}(\Omega) \,:\, \rho \le 1\}$,
- $u : \Omega \to \mathbb{R}^d$ will be a vector field, possibly depending on time or on ρ,
- $adm(\rho) = \{v : \Omega \to \mathbb{R}^d \;:\; \nabla \cdot v \ge 0 \text{ on } \rho = 1\}$,
- P is the projection in $L^2(dx)$ or (which is the same) in $L^2(\rho)$,
- we'll solve $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t \left(P_{adm(\rho_t)} u_t\right)\right) = 0$.

The equation $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t v_t\right) = 0$ (*continuity equation*) is exactly the equation satisfied by the evolution of a density ρ when each particle follows the velocity field v (with $v \cdot n = 0$ on $\partial\Omega$, so that the density does not exit Ω).

Difficulty : $v = P_{adm(\rho_t)} u_t$ is not regular, neither depends regularly on ρ.

Continuous formulation

- The particles population will be described by a density $\rho \in \mathcal{P}(\Omega)$,
- the constraint by $K = \{\rho \in \mathcal{P}(\Omega) \,:\, \rho \leq 1\}$,
- $u : \Omega \to \mathbb{R}^d$ will be a vector field, possibly depending on time or on ρ,
- $adm(\rho) = \{v : \Omega \to \mathbb{R}^d \,:\, \nabla \cdot v \geq 0 \text{ on } \rho = 1\}$,
- P is the projection in $L^2(dx)$ or (which is the same) in $L^2(\rho)$,
- we'll solve $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t \left(P_{adm(\rho_t)} u_t\right)\right) = 0$.

The equation $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t v_t\right) = 0$ (*continuity equation*) is exactly the equation satisfied by the evolution of a density ρ when each particle follows the velocity field v (with $v \cdot n = 0$ on $\partial\Omega$, so that the density does not exit Ω).

Difficulty : $v = P_{adm(\rho_t)} u_t$ is not regular, neither depends regularly on ρ.

Continuous formulation

- The particles population will be described by a density $\rho \in \mathcal{P}(\Omega)$,
- the constraint by $K = \{\rho \in \mathcal{P}(\Omega) \, : \, \rho \leq 1\}$,
- $u : \Omega \to \mathbb{R}^d$ will be a vector field, possibly depending on time or on ρ,
- $adm(\rho) = \{v : \Omega \to \mathbb{R}^d \; : \; \nabla \cdot v \geq 0 \text{ on } \rho = 1\}$,
- P is the projection in $L^2(dx)$ or (which is the same) in $L^2(\rho)$,
- we'll solve $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t \left(P_{adm(\rho_t)} u_t\right)\right) = 0$.

The equation $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t v_t\right) = 0$ (*continuity equation*) is exactly the equation satisfied by the evolution of a density ρ when each particle follows the velocity field v (with $v \cdot n = 0$ on $\partial\Omega$, so that the density does not exit Ω).

Difficulty : $v = P_{adm(\rho_t)} u_t$ is not regular, neither depends regularly on ρ.

Pressures and duality

The set $adm(\rho)$ may be better described by duality :

$$adm(\rho) = \{v \in L^2(\rho) \; : \; \int v \cdot \nabla p \leq 0 \quad \forall p \; : \; p \geq 0, \; p(1-\rho) = 0\}.$$

In this way we can characterize $v = P_{adm(\rho)}(u)$ through

$$u = v + \nabla p, \quad v \in adm(\rho), \quad \int v \cdot \nabla p = 0,$$

$$p \in press(\rho) := \{p \in H^1(\Omega), \; p \geq 0, \; p(1-\rho) = 0\}$$

This function p plays the role of a pressure affecting the movement.

$$\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t(u_t - \nabla p_t)\right) = 0$$

$$\rho_t \in K, \; p_t \in press(\rho_t)$$

Pressures and duality

The set $adm(\rho)$ may be better described by duality :

$$adm(\rho) = \{v \in L^2(\rho)\ :\ \int v \cdot \nabla p \leq 0 \quad \forall p\ :\ p \geq 0,\ p(1-\rho) = 0\}.$$

In this way we can characterize $v = P_{adm(\rho)}(u)$ through

$$u = v + \nabla p, \quad v \in adm(\rho), \quad \int v \cdot \nabla p = 0,$$

$$p \in press(\rho) := \{p \in H^1(\Omega),\ p \geq 0,\ p(1-\rho) = 0\}$$

This function p plays the role of a pressure affecting the movement.

$$\frac{\partial}{\partial t}\rho_t + \nabla \cdot (\rho_t(u_t - \nabla p_t)) = 0$$

$$\rho_t \in K,\ p_t \in press(\rho_t)$$

Pressures and duality

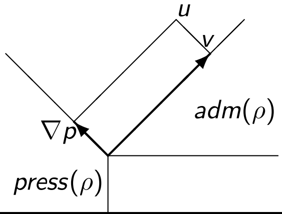
The set $adm(\rho)$ may be better described by duality :

$$adm(\rho) = \{v \in L^2(\rho) \ : \ \int v \cdot \nabla p \le 0 \quad \forall p \ : \ p \ge 0, \ p(1-\rho) = 0\}.$$

In this way we can characterize $v = P_{adm(\rho)}(u)$ through

$$u = v + \nabla p, \quad v \in adm(\rho), \quad \int v \cdot \nabla p = 0,$$
$$p \in press(\rho) := \{p \in H^1(\Omega), \ p \ge 0, \ p(1-\rho) = 0\}$$

This function p plays the role of a pressure affecting the movement.



$$\frac{\partial}{\partial t}\rho_t + \nabla \cdot (\rho_t(u_t - \nabla p_t)) = 0$$
$$\rho_t \in K, \ p_t \in press(\rho_t)$$

Pressures and duality

The set $adm(\rho)$ may be better described by duality :

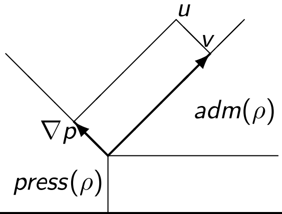
$$adm(\rho) = \{v \in L^2(\rho) \ : \ \int v \cdot \nabla p \leq 0 \quad \forall p \ : \ p \geq 0, \ p(1-\rho) = 0\}.$$

In this way we can characterize $v = P_{adm(\rho)}(u)$ through

$$u = v + \nabla p, \quad v \in adm(\rho), \quad \int v \cdot \nabla p = 0,$$

$$p \in press(\rho) := \{p \in H^1(\Omega), \ p \geq 0, \ p(1-\rho) = 0\}$$

This function p plays the role of a pressure affecting the movement.



$$\frac{\partial}{\partial t}\rho_t + \nabla \cdot \big(\rho_t(u_t - \nabla p_t)\big) = 0$$

$$\rho_t \in K, \ \ p_t \in press(\rho_t)$$

Gradient-flow

Gradient-flow formulation

What is a gradient-flow ?

A gradient-flow in $\mathbb{R}^n$ is an evolution equation of the kind

$$x'(t) = -\nabla F(x(t))$$

(we follow the steepest descent lines of a function F). In order to discretize in time such an equation we can recursively solve

$$x_{k+1}^\tau \in \operatorname{argmin}_x F(x) + \frac{1}{2\tau}|x - x_k^\tau|^2, \quad \tau > 0 \text{ fixed.}$$

Actually, the optimal x_{k+1}^τ satisfies

$$\frac{x_{k+1}^\tau - x_k^\tau}{\tau} + \nabla F(x_{k+1}^\tau) = 0$$

which corresponds to an implicit Euler scheme to solve $x' = -\nabla F(x)$, the solution being found as a limit $\tau \to 0$.
This formulation may easily be adapted to a general metric space... it allows to study evolution problems for a density ρ when we use the space $\mathcal{P}(\Omega)$ endowed with a suitable distance.

What is a gradient-flow ?

A gradient-flow in $\mathbb{R}^n$ is an evolution equation of the kind

$$x'(t) = -\nabla F(x(t))$$

(we follow the steepest descent lines of a function F). In order to discretize in time such an equation we can recursively solve

$$x^\tau_{k+1} \in \operatorname{argmin}_x F(x) + \frac{1}{2\tau}|x - x^\tau_k|^2, \quad \tau > 0 \text{ fixed.}$$

Actually, the optimal x^τ_{k+1} satisfies

$$\frac{x^\tau_{k+1} - x^\tau_k}{\tau} + \nabla F(x^\tau_{k+1}) = 0$$

which corresponds to an implicit Euler scheme to solve $x' = -\nabla F(x)$, the solution being found as a limit $\tau \to 0$.

This formulation may easily be adapted to a general metric space... it allows to study evolution problems for a density ρ when we use the space $\mathcal{P}(\Omega)$ endowed with a suitable distance.

What is a gradient-flow ?

A gradient-flow in $\mathbb{R}^n$ is an evolution equation of the kind

$$x'(t) = -\nabla F(x(t))$$

(we follow the steepest descent lines of a function F). In order to discretize in time such an equation we can recursively solve

$$x_{k+1}^\tau \in \operatorname{argmin}_x F(x) + \frac{1}{2\tau}|x - x_k^\tau|^2, \quad \tau > 0 \text{ fixed.}$$

Actually, the optimal x_{k+1}^τ satisfies

$$\frac{x_{k+1}^\tau - x_k^\tau}{\tau} + \nabla F(x_{k+1}^\tau) = 0$$

which corresponds to an implicit Euler scheme to solve $x' = -\nabla F(x)$, the solution being found as a limit $\tau \to 0$.

This formulation may easily be adapted to a general metric space... it allows to study evolution problems for a density ρ when we use the space $\mathcal{P}(\Omega)$ endowed with a suitable distance.

What is a gradient-flow ?

A gradient-flow in $\mathbb{R}^n$ is an evolution equation of the kind

$$x'(t) = -\nabla F(x(t))$$

(we follow the steepest descent lines of a function F). In order to discretize in time such an equation we can recursively solve

$$x^\tau_{k+1} \in \operatorname{argmin}_x F(x) + \frac{1}{2\tau}|x - x^\tau_k|^2, \quad \tau > 0 \text{ fixed.}$$

Actually, the optimal x^τ_{k+1} satisfies

$$\frac{x^\tau_{k+1} - x^\tau_k}{\tau} + \nabla F(x^\tau_{k+1}) = 0$$

which corresponds to an implicit Euler scheme to solve $x' = -\nabla F(x)$, the solution being found as a limit $\tau \to 0$.

This formulation may easily be adapted to a general metric space...

it allows to study evolution problems for a density ρ when we use the space $\mathcal{P}(\Omega)$ endowed with a suitable distance.

What is a gradient-flow ?

A gradient-flow in $\mathbb{R}^n$ is an evolution equation of the kind

$$x'(t) = -\nabla F(x(t))$$

(we follow the steepest descent lines of a function F). In order to discretize in time such an equation we can recursively solve

$$x^\tau_{k+1} \in \text{argmin}_x\, F(x) + \frac{1}{2\tau}|x - x^\tau_k|^2, \quad \tau > 0 \text{ fixed}.$$

Actually, the optimal x^τ_{k+1} satisfies

$$\frac{x^\tau_{k+1} - x^\tau_k}{\tau} + \nabla F(x^\tau_{k+1}) = 0$$

which corresponds to an implicit Euler scheme to solve $x' = -\nabla F(x)$, the solution being found as a limit $\tau \to 0$.

This formulation may easily be adapted to a general metric space... it allows to study evolution problems for a density ρ when we use the space $\mathcal{P}(\Omega)$ endowed with a suitable distance.

Optimal transport and Wasserstein distances

If two probabilities $\mu, \nu \in \mathcal{P}(\Omega)$ are given on a compact domain, the Monge-Kantorovitch problem reads

$$W_2^2(\mu,\nu) = \inf\left\{ \int |x - T(x)|^2 d\mu \; : \; T : \Omega \to \Omega, \; T_\# \mu = \nu \right\}$$

$W_2(\mu,\nu)$, the square root of the minimal value, is a distance on $\mathcal{P}(\Omega)$.

Kantorovitch potential : for every (μ,ν) with $\mu << \mathcal{L}^d$ there exists a Lipschitz function $\phi : \Omega \to \mathbb{R}$ with the following properties

- there exists an optimal T and $T(x) = x - \nabla\phi(x)$,
- we have $\frac{d}{d\varepsilon}\frac{1}{2}W_2^2(\mu+\varepsilon\chi,\nu)_{|\varepsilon=0} = \int \phi d\chi$

Notation : for $G : \mathcal{P}(\Omega) \to \mathbb{R}$ we call $\frac{\delta G}{\delta\rho}(\rho)$, if it exists, the only function such that $\frac{d}{d\varepsilon}G(\rho+\varepsilon\chi)_{|\varepsilon=0} = \int \frac{\delta G}{\delta \rho}(\rho) d\chi$. Hence, we say $\frac{\delta\left(\frac{1}{2}W_2^2(\cdot,\nu)\right)}{\delta\rho} = \phi$

Optimal transport and Wasserstein distances

If two probabilities $\mu, \nu \in \mathcal{P}(\Omega)$ are given on a compact domain, the Monge-Kantorovitch problem reads

$$W_2^2(\mu,\nu) = \inf \left\{ \int |x - T(x)|^2 d\mu \;:\; T : \Omega \to \Omega,\; T_\# \mu = \nu \right\}$$

$W_2(\mu,\nu)$, the square root of the minimal value, is a distance on $\mathcal{P}(\Omega)$.

Kantorovitch potential : for every (μ,ν) with $\mu << \mathcal{L}^d$ there exists a Lipschitz function $\phi : \Omega \to \mathbb{R}$ with the following properties

- there exists an optimal T and $T(x) = x - \nabla\phi(x)$,
- we have $\frac{d}{d\varepsilon}\frac{1}{2}W_2^2(\mu + \varepsilon\chi, \nu)_{|\varepsilon=0} = \int \phi d\chi$

Notation : for $G : \mathcal{P}(\Omega) \to \mathbb{R}$ we call $\frac{\delta G}{\delta \rho}(\rho)$, if it exists, the only function such that $\frac{d}{d\varepsilon}G(\rho + \varepsilon\chi)_{|\varepsilon=0} = \int \frac{\delta G}{\delta\rho}(\rho)d\chi$. Hence, we say $\frac{\delta\left(\frac{1}{2}W_2^2(\cdot,\nu)\right)}{\delta\rho} = \phi$

Optimal transport and Wasserstein distances

If two probabilities $\mu, \nu \in \mathcal{P}(\Omega)$ are given on a compact domain, the Monge-Kantorovich problem reads

$$W_2^2(\mu,\nu) = \inf \left\{ \int |x - T(x)|^2 d\mu \ : \ T : \Omega \to \Omega, \ T_{\#}\mu = \nu \right\}$$

$W_2(\mu,\nu)$, the square root of the minimal value, is a distance on $\mathcal{P}(\Omega)$.

Kantorovich potential : for every (μ,ν) with $\mu << \mathcal{L}^d$ there exists a Lipschitz function $\phi : \Omega \to \mathbb{R}$ with the following properties

- there exists an optimal T and $T(x) = x - \nabla\phi(x)$,
- we have $\frac{d}{d\varepsilon}\frac{1}{2}W_2^2(\mu + \varepsilon\chi, \nu)_{|\varepsilon=0} = \int \phi d\chi$

Notation : for $G : \mathcal{P}(\Omega) \to \mathbb{R}$ we call $\frac{\delta G}{\delta\rho}(\rho)$, if it exists, the only function such that $\frac{d}{d\varepsilon}G(\rho + \varepsilon\chi)_{|\varepsilon=0} = \int \frac{\delta G}{\delta\rho}(\rho)d\chi$. Hence, we say $\frac{\delta\left(\frac{1}{2}W_2^2(\cdot,\nu)\right)}{\delta\rho} = \phi$

Gradient-flows in W_2

Let F be a functional over $\mathcal{P}(\Omega)$ endowed with the W_2 distance. Let us solve

$$\rho^\tau_{k+1} \in \mathrm{argmin}_\rho \, F(\rho) + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau}$$

Discrete optimality conditions :

$$\frac{\delta F}{\delta \rho}(\rho^\tau_{k+1}) + \frac{\phi}{\tau} = const$$

which implies

$$-v(x) := \frac{T(x) - x}{\tau} = -\frac{\nabla \phi(x)}{\tau} = \nabla\big(\frac{\delta F}{\delta \rho}(\rho)\big)$$

and, since v represents the discrete velocity (displacement / time step), at the limit the continuity equation $\partial\rho/\partial t + \nabla \cdot (\rho v) = 0$ gives

$$\frac{\partial \rho}{\partial t} - \nabla \cdot \big(\rho \nabla\big(\frac{\delta F}{\delta \rho}(\rho)\big)\big) = 0.$$

Gradient-flows in W_2

Let F be a functional over $\mathcal{P}(\Omega)$ endowed with the W_2 distance. Let us solve

$$\rho^\tau_{k+1} \in \operatorname{argmin}_\rho F(\rho) + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau}$$

Discrete optimality conditions :

$$\frac{\delta F}{\delta \rho}(\rho^\tau_{k+1}) + \frac{\phi}{\tau} = const$$

which implies

$$-v(x) := \frac{T(x) - x}{\tau} = -\frac{\nabla \phi(x)}{\tau} = \nabla\left(\frac{\delta F}{\delta \rho}(\rho)\right)$$

and, since v represents the discrete velocity (displacement / time step), at the limit the continuity equation $\partial\rho/\partial t + \nabla \cdot (\rho v) = 0$ gives

$$\frac{\partial \rho}{\partial t} - \nabla \cdot \left(\rho \nabla\left(\frac{\delta F}{\delta \rho}(\rho)\right)\right) = 0.$$

Gradient-flows in W_2

Let F be a functional over $\mathcal{P}(\Omega)$ endowed with the W_2 distance. Let us solve

$$\rho^\tau_{k+1} \in \operatorname{argmin}_\rho F(\rho) + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau}$$

Discrete optimality conditions :

$$\frac{\delta F}{\delta \rho}(\rho^\tau_{k+1}) + \frac{\phi}{\tau} = const$$

which implies

$$-v(x) := \frac{T(x) - x}{\tau} = -\frac{\nabla\phi(x)}{\tau} = \nabla\Big(\frac{\delta F}{\delta \rho}(\rho)\Big)$$

and, since v represents the discrete velocity (displacement / time step), at the limit the continuity equation $\partial\rho/\partial t + \nabla\cdot(\rho v) = 0$ gives

$$\frac{\partial \rho}{\partial t} - \nabla\cdot\Big(\rho\,\nabla\Big(\frac{\delta F}{\delta\rho}(\rho)\Big)\Big) = 0.$$

Gradient-flows in W_2

Let F be a functional over $\mathcal{P}(\Omega)$ endowed with the W_2 distance. Let us solve

$$\rho^\tau_{k+1} \in \operatorname{argmin}_\rho F(\rho) + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau}$$

Discrete optimality conditions :

$$\frac{\delta F}{\delta \rho}(\rho^\tau_{k+1}) + \frac{\phi}{\tau} = const$$

which implies

$$-v(x) := \frac{T(x) - x}{\tau} = -\frac{\nabla \phi(x)}{\tau} = \nabla\left(\frac{\delta F}{\delta \rho}(\rho)\right)$$

and, since v represents the discrete velocity (displacement / time step), at the limit the continuity equation $\partial \rho / \partial t + \nabla \cdot (\rho v) = 0$ gives

$$\frac{\partial \rho}{\partial t} - \nabla \cdot \left(\rho \nabla\left(\frac{\delta F}{\delta \rho}(\rho)\right)\right) = 0.$$

Gradient-flows in W_2

Let F be a functional over $\mathcal{P}(\Omega)$ endowed with the W_2 distance. Let us solve

$$\rho^\tau_{k+1} \in \text{argmin}_\rho \, F(\rho) + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau}$$

Discrete optimality conditions :

$$\frac{\delta F}{\delta \rho}(\rho^\tau_{k+1}) + \frac{\phi}{\tau} = const$$

which implies

$$-v(x) := \frac{T(x) - x}{\tau} = -\frac{\nabla \phi(x)}{\tau} = \nabla\big(\frac{\delta F}{\delta \rho}(\rho)\big)$$

and, since v represents the discrete velocity (displacement / time step), at the limit the continuity equation $\partial\rho/\partial t + \nabla \cdot (\rho v) = 0$ gives

$$\frac{\partial \rho}{\partial t} - \nabla \cdot \big(\rho \nabla\big(\frac{\delta F}{\delta \rho}(\rho)\big)\big) = 0.$$

Examples

Take $F(\rho) = \int f(\rho(x))dx$. Then $\frac{\delta F}{\delta \rho}(\rho) = f'(\rho)$. The equation becomes

$$\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho \nabla f'(\rho)) = 0.$$

For instance, for $f(t) = t \log t$ we get $\nabla f'(\rho) = \frac{\nabla \rho}{\rho}$, which gives the heat equation $\frac{\partial \rho}{\partial t} - \Delta \rho = 0$.

For $F(\rho) = \int V(x) d\rho$ we get $\frac{\delta F}{\delta \rho}(\rho) = V$. We can obtain the Fokker-Planck equation in the case $F(\rho) = \int V(x)d\rho + \int \rho \log(\rho) \dots$

Advantages of this formulation/interpretation : existence results and uniqueness (under some convexity assumptions on F) in a general framework, moreover, we also directly have a time-discretization algorithm.
Ffor all the details on this theory, see for instance

L. Ambrosio, N. Gigli, G. Savaré *Gradient Flows*, Birkäuser, 2005,

Examples

Take $F(\rho) = \int f(\rho(x))dx$. Then $\frac{\delta F}{\delta \rho}(\rho) = f'(\rho)$. The equation becomes

$$\frac{\partial \rho}{\partial t} - \nabla \cdot \big(\rho \nabla f'(\rho)\big) = 0.$$

For instance, for $f(t) = t \log t$ we get $\nabla f'(\rho) = \frac{\nabla \rho}{\rho}$, which gives the heat equation $\frac{\partial \rho}{\partial t} - \Delta \rho = 0$.

For $F(\rho) = \int V(x) d\rho$ we get $\frac{\delta F}{\delta \rho}(\rho) = V$. We can obtain the Fokker-Planck equation in the case $F(\rho) = \int V(x)d\rho + \int \rho \log(\rho) \dots$

Advantages of this formulation/interpretation : existence results and uniqueness (under some convexity assumptions on F) in a general framework, moreover, we also directly have a time-discretization algorithm.

Ffor all the details on this theory, see for instance

L. Ambrosio, N. Gigli, G. Savaré *Gradient Flows*, Birkäuser, 2005,

Examples

Take $F(\rho) = \int f(\rho(x))dx$. Then $\frac{\delta F}{\delta \rho}(\rho) = f'(\rho)$. The equation becomes

$$\frac{\partial \rho}{\partial t} - \nabla \cdot \left(\rho \nabla f'(\rho)\right) = 0.$$

For instance, for $f(t) = t \log t$ we get $\nabla f'(\rho) = \frac{\nabla \rho}{\rho}$, which gives the heat equation $\frac{\partial \rho}{\partial t} - \Delta \rho = 0$.

For $F(\rho) = \int V(x) d\rho$ we get $\frac{\delta F}{\delta \rho}(\rho) = V$. We can obtain the Fokker-Planck equation in the case $F(\rho) = \int V(x)d\rho + \int \rho \log(\rho)\ldots$

Advantages of this formulation/interpretation : existence results and uniqueness (under some convexity assumptions on F) in a general framework, moreover, we also directly have a time-discretization algorithm.
Ffor all the details on this theory, see for instance
L. Ambrosio, N. Gigli, G. Savaré *Gradient Flows*, Birkäuser, 2005,

Examples

Take $F(\rho) = \int f(\rho(x))dx$. Then $\frac{\delta F}{\delta \rho}(\rho) = f'(\rho)$. The equation becomes

$$\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho \nabla f'(\rho)) = 0.$$

For instance, for $f(t) = t \log t$ we get $\nabla f'(\rho) = \frac{\nabla \rho}{\rho}$, which gives the heat equation $\frac{\partial \rho}{\partial t} - \Delta \rho = 0$.

For $F(\rho) = \int V(x) d\rho$ we get $\frac{\delta F}{\delta \rho}(\rho) = V$. We can obtain the Fokker-Planck equation in the case $F(\rho) = \int V(x) d\rho + \int \rho \log(\rho) \dots$

Advantages of this formulation/interpretation : existence results and uniqueness (under some convexity assumptions on F) in a general framework, moreover, we also directly have a time-discretization algorithm.

Ffor all the details on this theory, see for instance

L. Ambrosio, N. Gigli, G. Savaré *Gradient Flows*, Birkäuser, 2005,

Examples

Take $F(\rho) = \int f(\rho(x))dx$. Then $\frac{\delta F}{\delta \rho}(\rho) = f'(\rho)$. The equation becomes

$$\frac{\partial \rho}{\partial t} - \nabla \cdot \left(\rho \nabla f'(\rho)\right) = 0.$$

For instance, for $f(t) = t \log t$ we get $\nabla f'(\rho) = \frac{\nabla \rho}{\rho}$, which gives the heat equation $\frac{\partial \rho}{\partial t} - \Delta \rho = 0$.

For $F(\rho) = \int V(x) d\rho$ we get $\frac{\delta F}{\delta \rho}(\rho) = V$. We can obtain the Fokker-Planck equation in the case $F(\rho) = \int V(x) d\rho + \int \rho \log(\rho)$...

Advantages of this formulation/interpretation : existence results and uniqueness (under some convexity assumptions on F) in a general framework, moreover, we also directly have a time-discretization algorithm.

Ffor all the details on this theory, see for instance

L. Ambrosio, N. Gigli, G. Savaré *Gradient Flows*, Birkäuser, 2005,

Examples

Take $F(\rho) = \int f(\rho(x))dx$. Then $\frac{\delta F}{\delta \rho}(\rho) = f'(\rho)$. The equation becomes

$$\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho \nabla f'(\rho)) = 0.$$

For instance, for $f(t) = t \log t$ we get $\nabla f'(\rho) = \frac{\nabla \rho}{\rho}$, which gives the heat equation $\frac{\partial \rho}{\partial t} - \Delta \rho = 0$.

For $F(\rho) = \int V(x) d\rho$ we get $\frac{\delta F}{\delta \rho}(\rho) = V$. We can obtain the Fokker-Planck equation in the case $F(\rho) = \int V(x) d\rho + \int \rho \log(\rho) \dots$

Advantages of this formulation/interpretation : existence results and uniqueness (under some convexity assumptions on F) in a general framework, moreover, we also directly have a time-discretization algorithm.

Ffor all the details on this theory, see for instance

L. Ambrosio, N. Gigli, G. Savaré *Gradient Flows*, Birkäuser, 2005,

Examples

Take $F(\rho) = \int f(\rho(x))dx$. Then $\frac{\delta F}{\delta \rho}(\rho) = f'(\rho)$. The equation becomes

$$\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho \nabla f'(\rho)) = 0.$$

For instance, for $f(t) = t \log t$ we get $\nabla f'(\rho) = \frac{\nabla \rho}{\rho}$, which gives the heat equation $\frac{\partial \rho}{\partial t} - \Delta \rho = 0$.

For $F(\rho) = \int V(x) d\rho$ we get $\frac{\delta F}{\delta \rho}(\rho) = V$. We can obtain the Fokker-Planck equation in the case $F(\rho) = \int V(x) d\rho + \int \rho \log(\rho) \dots$

Advantages of this formulation/interpretation : existence results and uniqueness (under some convexity assumptions on F) in a general framework, moreover, we also directly have a time-discretization algorithm.

Ffor all the details on this theory, see for instance

L. Ambrosio, N. Gigli, G. Savaré *Gradient Flows*, Birkäuser, 2005,

Examples

Take $F(\rho) = \int f(\rho(x))dx$. Then $\frac{\delta F}{\delta \rho}(\rho) = f'(\rho)$. The equation becomes

$$\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho \nabla f'(\rho)) = 0.$$

For instance, for $f(t) = t \log t$ we get $\nabla f'(\rho) = \frac{\nabla \rho}{\rho}$, which gives the heat equation $\frac{\partial \rho}{\partial t} - \Delta \rho = 0$.

For $F(\rho) = \int V(x) d\rho$ we get $\frac{\delta F}{\delta \rho}(\rho) = V$. We can obtain the Fokker-Planck equation in the case $F(\rho) = \int V(x)d\rho + \int \rho \log(\rho) \dots$

Advantages of this formulation/interpretation : existence results and uniqueness (under some convexity assumptions on F) in a general framework, moreover, we also directly have a time-discretization algorithm.
Ffor all the details on this theory, see for instance
L. Ambrosio, N. Gigli, G. Savaré *Gradient Flows*, Birkäuser, 2005,

Our PDE

The macroscopic model as a gradient-flow

The functional

Let's come back to the case we are interested in, with $u = -\nabla D$ et $\rho \in K = \{\rho \in \mathcal{P}(\Omega) \,:\, \rho \leq 1\}$. Let's take F defined by

$$F(\rho) = \begin{cases} \int D(x)d\rho & \text{if } \rho \in K, \\ +\infty & \text{if } \rho \notin K. \end{cases}$$

The discrete iterative method is, as usual,

$$\rho^\tau_{k+1} \in \operatorname{argmin}_\rho F(\rho) + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau} = \operatorname{argmin}_{\rho \in K} \int D(x)d\rho + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau}$$

It happens that the limit of these trajectories for $\tau \to 0$, hence the gradient-flow of F, exactly provides the PDE we are looking at.

Why ? come back to the discrete optimality condition, which would be (if the constraint $\rho \in K$ is ignored) $D + \frac{\phi}{\tau} = const$.

How did we get it ? Set $\psi = D + \frac{\phi}{\tau}$ and $\chi = \tilde{\rho} - \rho$: we would have $\int \psi d\chi \geq 0$ and hence $\int \psi d\tilde{\rho} \geq \int \psi d\rho \quad \forall \tilde{\rho} \in \mathcal{P}(\Omega)$. This implies that ρ is concentrated on $\operatorname{argmin} \psi$ and hence $\psi = const \ \rho-$a.e.

But here we need to take into account $\rho \in K$!

The functional

Let's come back to the case we are interested in, with $u = -\nabla D$ et $\rho \in K = \{\rho \in \mathcal{P}(\Omega) \,:\, \rho \leq 1\}$. Let's take F defined by

$$F(\rho) = \begin{cases} \int D(x)d\rho & \text{if } \rho \in K, \\ +\infty & \text{if } \rho \notin K. \end{cases}$$

The discrete iterative method is, as usual,

$$\rho^\tau_{k+1} \in \operatorname{argmin}_\rho F(\rho) + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau} = \operatorname{argmin}_{\rho \in K} \int D(x)d\rho + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau}$$

It happens that the limit of these trajectories for $\tau \to 0$, hence the gradient-flow of F, exactly provides the PDE we are looking at.

Why ? come back to the discrete optimality condition, which would be (if the constraint $\rho \in K$ is ignored) $D + \frac{\phi}{\tau} = const$.

How did we get it ? Set $\psi = D + \frac{\phi}{\tau}$ and $\chi = \tilde{\rho} - \rho$: we would have $\int \psi d\chi \geq 0$ and hence $\int \psi d\tilde{\rho} \geq \int \psi d\rho \quad \forall \tilde{\rho} \in \mathcal{P}(\Omega)$. This implies that ρ is concentrated on $\operatorname{argmin} \psi$ and hence $\psi = const$ $\rho-$a.e.

But here we need to take into account $\rho \in K$!

The functional

Let's come back to the case we are interested in, with $u = -\nabla D$ et $\rho \in K = \{\rho \in \mathcal{P}(\Omega) \,:\, \rho \leq 1\}$. Let's take F defined by

$$F(\rho) = \begin{cases} \int D(x) d\rho & \text{if } \rho \in K, \\ +\infty & \text{if } \rho \notin K. \end{cases}$$

The discrete iterative method is, as usual,

$$\rho^\tau_{k+1} \in \operatorname{argmin}_\rho F(\rho) + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau} = \operatorname{argmin}_{\rho \in K} \int D(x) d\rho + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau}$$

It happens that the limit of these trajectories for $\tau \to 0$, hence the gradient-flow of F, exactly provides the PDE we are looking at.

Why ? come back to the discrete optimality condition, which would be (if the constraint $\rho \in K$ is ignored) $D + \frac{\phi}{\tau} = const$.

How did we get it ? Set $\psi = D + \frac{\phi}{\tau}$ and $\chi = \tilde{\rho} - \rho$: we would have $\int \psi d\chi \geq 0$ and hence $\int \psi d\tilde{\rho} \geq \int \psi d\rho \quad \forall \tilde{\rho} \in \mathcal{P}(\Omega)$. This implies that ρ is concentrated on argmin ψ and hence $\psi = const$ ρ−a.e.

But here we need to take into account $\rho \in K$!

The functional

Let's come back to the case we are interested in, with $u = -\nabla D$ et $\rho \in K = \{\rho \in \mathcal{P}(\Omega) \,:\, \rho \leq 1\}$. Let's take F defined by

$$F(\rho) = \begin{cases} \int D(x) d\rho & \text{if } \rho \in K, \\ +\infty & \text{if } \rho \notin K. \end{cases}$$

The discrete iterative method is, as usual,

$$\rho^\tau_{k+1} \in \operatorname{argmin}_\rho F(\rho) + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau} = \operatorname{argmin}_{\rho \in K} \int D(x) d\rho + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau}$$

It happens that the limit of these trajectories for $\tau \to 0$, hence the gradient-flow of F, exactly provides the PDE we are looking at.

Why ? come back to the discrete optimality condition, which would be (if the constraint $\rho \in K$ is ignored) $D + \frac{\phi}{\tau} = const.$

How did we get it ? Set $\psi = D + \frac{\phi}{\tau}$ and $\chi = \tilde{\rho} - \rho$: we would have $\int \psi d\chi \geq 0$ and hence $\int \psi d\tilde{\rho} \geq \int \psi d\rho \quad \forall \tilde{\rho} \in \mathcal{P}(\Omega)$. This implies that ρ is concentrated on argmin ψ and hence $\psi = const \;\; \rho-$a.e.

But here we need to take into account $\rho \in K$!

The functional

Let's come back to the case we are interested in, with $u = -\nabla D$ et $\rho \in K = \{\rho \in \mathcal{P}(\Omega) \, : \, \rho \leq 1\}$. Let's take F defined by

$$F(\rho) = \begin{cases} \int D(x) d\rho & \text{if } \rho \in K, \\ +\infty & \text{if } \rho \notin K. \end{cases}$$

The discrete iterative method is, as usual,

$$\rho^\tau_{k+1} \in \operatorname{argmin}_\rho F(\rho) + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau} = \operatorname{argmin}_{\rho \in K} \int D(x) d\rho + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau}$$

It happens that the limit of these trajectories for $\tau \to 0$, hence the gradient-flow of F, exactly provides the PDE we are looking at.

Why ? come back to the discrete optimality condition, which would be (if the constraint $\rho \in K$ is ignored) $D + \frac{\phi}{\tau} = const$.

How did we get it ? Set $\psi = D + \frac{\phi}{\tau}$ and $\chi = \tilde{\rho} - \rho$: we would have $\int \psi d\chi \geq 0$ and hence $\int \psi d\tilde{\rho} \geq \int \psi d\rho \quad \forall \tilde{\rho} \in \mathcal{P}(\Omega)$. This implies that ρ is concentrated on argmin ψ and hence $\psi = const \;\; \rho-$a.e.

But here we need to take into account $\rho \in K$!

The functional

Let's come back to the case we are interested in, with $u = -\nabla D$ et $\rho \in K = \{\rho \in \mathcal{P}(\Omega) \,:\, \rho \leq 1\}$. Let's take F defined by

$$F(\rho) = \begin{cases} \int D(x)d\rho & \text{if } \rho \in K, \\ +\infty & \text{if } \rho \notin K. \end{cases}$$

The discrete iterative method is, as usual,

$$\rho^\tau_{k+1} \in \operatorname{argmin}_\rho F(\rho) + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau} = \operatorname{argmin}_{\rho \in K} \int D(x)d\rho + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau}$$

It happens that the limit of these trajectories for $\tau \to 0$, hence the gradient-flow of F, exactly provides the PDE we are looking at.

Why ? come back to the discrete optimality condition, which would be (if the constraint $\rho \in K$ is ignored) $D + \frac{\phi}{\tau} = const$.

How did we get it ? Set $\psi = D + \frac{\phi}{\tau}$ and $\chi = \tilde{\rho} - \rho$: we would have $\int \psi d\chi \geq 0$ and hence $\int \psi d\tilde{\rho} \geq \int \psi d\rho \quad \forall \tilde{\rho} \in \mathcal{P}(\Omega)$. This implies that ρ is concentrated on $\operatorname{argmin} \psi$ and hence $\psi = const$ $\rho-$a.e.

But here we need to take into account $\rho \in K$!

The functional

Let's come back to the case we are interested in, with $u = -\nabla D$ et $\rho \in K = \{\rho \in \mathcal{P}(\Omega) : \rho \leq 1\}$. Let's take F defined by

$$F(\rho) = \begin{cases} \int D(x) d\rho & \text{if } \rho \in K, \\ +\infty & \text{if } \rho \notin K. \end{cases}$$

The discrete iterative method is, as usual,

$$\rho^\tau_{k+1} \in \operatorname{argmin}_\rho F(\rho) + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau} = \operatorname{argmin}_{\rho \in K} \int D(x) d\rho + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau}$$

It happens that the limit of these trajectories for $\tau \to 0$, hence the gradient-flow of F, exactly provides the PDE we are looking at.

Why ? come back to the discrete optimality condition, which would be (if the constraint $\rho \in K$ is ignored) $D + \frac{\phi}{\tau} = const$.

How did we get it ? Set $\psi = D + \frac{\phi}{\tau}$ and $\chi = \tilde{\rho} - \rho$: we would have $\int \psi d\chi \geq 0$ and hence $\int \psi d\tilde{\rho} \geq \int \psi d\rho \quad \forall \tilde{\rho} \in \mathcal{P}(\Omega)$. This implies that ρ is concentrated on $\operatorname{argmin} \psi$ and hence $\psi = const$ $\rho-$a.e.

But here we need to take into account $\rho \in K$!

The functional

Let's come back to the case we are interested in, with $u = -\nabla D$ et $\rho \in K = \{\rho \in \mathcal{P}(\Omega) \, : \, \rho \leq 1\}$. Let's take F defined by

$$F(\rho) = \begin{cases} \int D(x) d\rho & \text{if } \rho \in K, \\ +\infty & \text{if } \rho \notin K. \end{cases}$$

The discrete iterative method is, as usual,

$$\rho^\tau_{k+1} \in \operatorname{argmin}_\rho F(\rho) + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau} = \operatorname{argmin}_{\rho \in K} \int D(x) d\rho + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau}$$

It happens that the limit of these trajectories for $\tau \to 0$, hence the gradient-flow of F, exactly provides the PDE we are looking at.

Why ? come back to the discrete optimality condition, which would be (if the constraint $\rho \in K$ is ignored) $D + \frac{\phi}{\tau} = const$.

How did we get it ? Set $\psi = D + \frac{\phi}{\tau}$ and $\chi = \tilde{\rho} - \rho$: we would have $\int \psi d\chi \geq 0$ and hence $\int \psi d\tilde{\rho} \geq \int \psi d\rho \quad \forall \tilde{\rho} \in \mathcal{P}(\Omega)$. This implies that ρ is concentrated on $\operatorname{argmin} \psi$ and hence $\psi = const \;\; \rho-$a.e.

But here we need to take into account $\rho \in K$!

Optimality conditions, velocity and pressure

What changes here ? we only get $\int \psi d\tilde{\rho} \geq \int \psi d\rho \quad \forall \tilde{\rho} \in K$. This implies

$$\exists t \ : \ \rho = \begin{cases} 1 & \text{on } \psi < t, \\ 0 & \text{on } \psi > t, \\ \in [0,1] & \text{on } \psi = t \end{cases} \Rightarrow p := (t-\psi)_+ \geq 0, \quad p(1-\rho) = 0.$$

Passing to gradients we have

$$\rho - \text{a.e.} \quad \nabla p = -\nabla \psi = -\nabla D - \frac{\nabla \phi}{\tau} = u - v \Rightarrow v = u - \nabla p.$$

Hence, this discrete scheme may be used to build an approximation ρ^τ and, at the limit, a solution of the PDE.

B. Maury, A. Roudneff-Chupin and F. Santambrogio, A macroscopic crowd motion model of gradient flow type, *Math. Mod. Meth. Appl. Sci.*

Optimality conditions, velocity and pressure

What changes here ? we only get $\int \psi d\tilde{\rho} \geq \int \psi d\rho \quad \forall \tilde{\rho} \in K$. This implies

$$\exists t \; : \; \rho = \begin{cases} 1 & \text{on } \psi < t, \\ 0 & \text{on } \psi > t, \\ \in [0,1] & \text{on } \psi = t \end{cases} \Rightarrow p := (t-\psi)_+ \geq 0, \quad p(1-\rho) = 0.$$

Passing to gradients we have

$$\rho - \text{a.e.} \quad \nabla p = -\nabla \psi = -\nabla D - \frac{\nabla \phi}{\tau} = u - v \Rightarrow v = u - \nabla p.$$

Hence, this discrete scheme may be used to build an approximation ρ^τ and, at the limit, a solution of the PDE.

B. Maury, A. Roudneff-Chupin and F. Santambrogio, A macroscopic crowd motion model of gradient flow type, *Math. Mod. Meth. Appl. Sci.*

Optimality conditions, velocity and pressure

What changes here ? we only get $\int \psi d\tilde{\rho} \geq \int \psi d\rho \quad \forall \tilde{\rho} \in K$. This implies

$$\exists t \ : \ \rho = \begin{cases} 1 & \text{on } \psi < t, \\ 0 & \text{on } \psi > t, \\ \in [0,1] & \text{on } \psi = t \end{cases} \Rightarrow p := (t - \psi)_+ \geq 0, \quad p(1-\rho) = 0.$$

Passing to gradients we have

$$\rho - \text{a.e.} \quad \nabla p = -\nabla \psi = -\nabla D - \frac{\nabla \phi}{\tau} = u - v \Rightarrow v = u - \nabla p.$$

Hence, this discrete scheme may be used to build an approximation ρ^τ and, at the limit, a solution of the PDE.

B. Maury, A. Roudneff-Chupin and F. Santambrogio, A macroscopic crowd motion model of gradient flow type, *Math. Mod. Meth. Appl. Sci.*

Optimality conditions, velocity and pressure

What changes here ? we only get $\int \psi d\tilde{\rho} \geq \int \psi d\rho \quad \forall \tilde{\rho} \in K$. This implies

$$\exists t \ : \ \rho = \begin{cases} 1 & \text{on } \psi < t, \\ 0 & \text{on } \psi > t, \\ \in [0,1] & \text{on } \psi = t \end{cases} \Rightarrow p := (t - \psi)_+ \geq 0, \quad p(1-\rho) = 0.$$

Passing to gradients we have

$$\rho - \text{a.e.} \quad \nabla p = -\nabla \psi = -\nabla D - \frac{\nabla \phi}{\tau} = u - v \Rightarrow v = u - \nabla p.$$

Hence, this discrete scheme may be used to build an approximation ρ^τ and, at the limit, a solution of the PDE.

B. Maury, A. Roudneff-Chupin and F. Santambrogio, A macroscopic crowd motion model of gradient flow type, *Math. Mod. Meth. Appl. Sci.*

Optimality conditions, velocity and pressure

What changes here ? we only get $\int \psi d\tilde{\rho} \geq \int \psi d\rho \quad \forall \tilde{\rho} \in K$. This implies

$$\exists t \ : \ \rho = \begin{cases} 1 & \text{on } \psi < t, \\ 0 & \text{on } \psi > t, \\ \in [0,1] & \text{on } \psi = t \end{cases} \Rightarrow p := (t-\psi)_+ \geq 0, \ \ p(1-\rho) = 0.$$

Passing to gradients we have

$$\rho - \text{a.e.} \quad \nabla p = -\nabla \psi = -\nabla D - \frac{\nabla \phi}{\tau} = u - v \Rightarrow v = u - \nabla p.$$

Hence, this discrete scheme may be used to build an approximation ρ^τ and, at the limit, a solution of the PDE.

B. Maury, A. Roudneff-Chupin and F. Santambrogio, A macroscopic crowd motion model of gradient flow type, *Math. Mod. Meth. Appl. Sci.*

Warning : in this model the exit is closed

Let's see the model as a **crowd leaving a panic area :** $D(x) = d(x, \Gamma)$ where $\Gamma \subset \partial\Omega$ is the door.

The density evolves by minimizing this mean distance to the door...but never leaves Ω !
For $t \to \infty$ it fills a neighborhood of the door with density $\rho = 1$. This is the configuration that minimizes F. (If the particles are people trying to get to the door so as to escape from a fire, they will all die).
Modification of the model : new definition of K

$$K := \{\rho \in \mathcal{P}(\Omega) \; : \; \rho = \rho_\Gamma + \rho_\Omega, \; \rho_\Omega \leq 1 \, , \mathrm{supp}(\rho_\Gamma) \subset \Gamma\}$$

Interpretation : as soon as a particle reaches Γ, it is safe ($D = 0$) ; we leave it on Γ for simplicity, but this only means that we are no longer concerned with what happens to it.

Mathematically : analyzing the scheme with this new set K is more complicated but manage anyway to prove existence of a solution. Yet, uniqueness is out of reach for the moment.

Warning : in this model the exit is closed

Let's see the model as a **crowd leaving a panic area :** $D(x) = d(x, \Gamma)$ where $\Gamma \subset \partial\Omega$ is the door.

The density evolves by minimizing this mean distance to the door...but never leaves Ω !

For $t \to \infty$ it fills a neighborhood of the door with density $\rho = 1$. This is the configuration that minimizes F. (If the particles are people trying to get to the door so as to escape from a fire, they will all die).

Modification of the model : new definition of K

$$K := \{\rho \in \mathcal{P}(\Omega) \, : \, \rho = \rho_\Gamma + \rho_\Omega, \; \rho_\Omega \leq 1 \, , \mathrm{supp}(\rho_\Gamma) \subset \Gamma\}$$

Interpretation : as soon as a particle reaches Γ, it is safe ($D = 0$) ; we leave it on Γ for simplicity, but this only means that we are no longer concerned with what happens to it.

Mathematically : analyzing the scheme with this new set K is more complicated but manage anyway to prove existence of a solution. Yet, uniqueness is out of reach for the moment.

Warning : in this model the exit is closed

Let's see the model as a **crowd leaving a panic area :** $D(x) = d(x, \Gamma)$ where $\Gamma \subset \partial\Omega$ is the door.

The density evolves by minimizing this mean distance to the door. . .but never leaves Ω !
For $t \to \infty$ it fills a neighborhood of the door with density $\rho = 1$. This is the configuration that minimizes F. (If the particles are people trying to get to the door so as to escape from a fire, they will all die).
Modification of the model : new definition of K

$$K := \{\rho \in \mathcal{P}(\Omega) \, : \, \rho = \rho_\Gamma + \rho_\Omega, \; \rho_\Omega \leq 1 \, , \mathrm{supp}(\rho_\Gamma) \subset \Gamma\}$$

Interpretation : as soon as a particle reaches Γ, it is safe ($D = 0$) ; we leave it on Γ for simplicity, but this only means that we are no longer concerned with what happens to it.

Mathematically : analyzing the scheme with this new set K is more complicated but manage anyway to prove existence of a solution. Yet, uniqueness is out of reach for the moment.

Warning : in this model the exit is closed

Let's see the model as a **crowd leaving a panic area :** $D(x) = d(x, \Gamma)$ where $\Gamma \subset \partial\Omega$ is the door.

The density evolves by minimizing this mean distance to the door. . .but never leaves Ω !
For $t \to \infty$ it fills a neighborhood of the door with density $\rho = 1$. This is the configuration that minimizes F. (If the particles are people trying to get to the door so as to escape from a fire, they will all die).
Modification of the model : new definition of K

$$K := \{\rho \in \mathcal{P}(\Omega) \,:\, \rho = \rho_\Gamma + \rho_\Omega,\ \rho_\Omega \leq 1\,, \mathrm{supp}(\rho_\Gamma) \subset \Gamma\}$$

Interpretation : as soon as a particle reaches Γ, it is safe ($D = 0$) ; we leave it on Γ for simplicity, but this only means that we are no longer concerned with what happens to it.

Mathematically : analyzing the scheme with this new set K is more complicated but manage anyway to prove existence of a solution. Yet, uniqueness is out of reach for the moment.

Warning : in this model the exit is closed

Let's see the model as a **crowd leaving a panic area :** $D(x) = d(x, \Gamma)$ where $\Gamma \subset \partial\Omega$ is the door.

The density evolves by minimizing this mean distance to the door...but never leaves Ω !
For $t \to \infty$ it fills a neighborhood of the door with density $\rho = 1$. This is the configuration that minimizes F. (If the particles are people trying to get to the door so as to escape from a fire, they will all die).
Modification of the model : new definition of K

$$K := \{\rho \in \mathcal{P}(\Omega) \,:\, \rho = \rho_\Gamma + \rho_\Omega,\ \rho_\Omega \leq 1\,, \mathrm{supp}(\rho_\Gamma) \subset \Gamma\}$$

Interpretation : as soon as a particle reaches Γ, it is safe ($D = 0$) ; we leave it on Γ for simplicity, but this only means that we are no longer concerned with what happens to it.

Mathematically : analyzing the scheme with this new set K is more complicated but manage anyway to prove existence of a solution. Yet, uniqueness is out of reach for the moment.

Warning : in this model the exit is closed

Let's see the model as a **crowd leaving a panic area :** $D(x) = d(x, \Gamma)$ where $\Gamma \subset \partial\Omega$ is the door.

The density evolves by minimizing this mean distance to the door. . .but never leaves Ω !
For $t \to \infty$ it fills a neighborhood of the door with density $\rho = 1$. This is the configuration that minimizes F. (If the particles are people trying to get to the door so as to escape from a fire, they will all die).
Modification of the model : new definition of K

$$K := \{\rho \in \mathcal{P}(\Omega) \,:\, \rho = \rho_\Gamma + \rho_\Omega,\ \rho_\Omega \leq 1\,, \mathrm{supp}(\rho_\Gamma) \subset \Gamma\}$$

Interpretation : as soon as a particle reaches Γ, it is safe ($D = 0$) ; we leave it on Γ for simplicity, but this only means that we are no longer concerned with what happens to it.

Mathematically : analyzing the scheme with this new set K is more complicated but manage anyway to prove existence of a solution. Yet, uniqueness is out of reach for the moment.

Warning : in this model the exit is closed

Let's see the model as a **crowd leaving a panic area :** $D(x) = d(x, \Gamma)$ where $\Gamma \subset \partial\Omega$ is the door.

The density evolves by minimizing this mean distance to the door...but never leaves Ω !
For $t \to \infty$ it fills a neighborhood of the door with density $\rho = 1$. This is the configuration that minimizes F. (If the particles are people trying to get to the door so as to escape from a fire, they will all die).
Modification of the model : new definition of K

$$K := \{\rho \in \mathcal{P}(\Omega) \; : \; \rho = \rho_\Gamma + \rho_\Omega, \; \rho_\Omega \leq 1\, , \mathrm{supp}(\rho_\Gamma) \subset \Gamma\}$$

Interpretation : as soon as a particle reaches Γ, it is safe $(D = 0)$; we leave it on Γ for simplicity, but this only means that we are no longer concerned with what happens to it.

Mathematically : analyzing the scheme with this new set K is more complicated but manage anyway to prove existence of a solution. Yet, uniqueness is out of reach for the moment.

Warning : in this model the exit is closed

Let's see the model as a **crowd leaving a panic area :** $D(x) = d(x, \Gamma)$ where $\Gamma \subset \partial\Omega$ is the door.

The density evolves by minimizing this mean distance to the door...but never leaves Ω !
For $t \to \infty$ it fills a neighborhood of the door with density $\rho = 1$. This is the configuration that minimizes F. (If the particles are people trying to get to the door so as to escape from a fire, they will all die).
Modification of the model : new definition of K

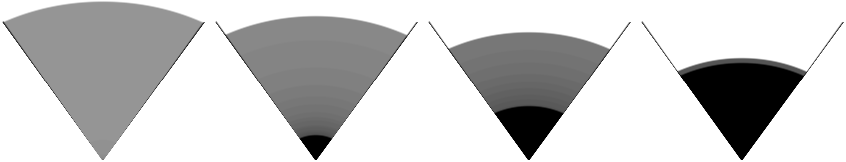
$$K := \{\rho \in \mathcal{P}(\Omega) \,:\, \rho = \rho_\Gamma + \rho_\Omega,\ \rho_\Omega \leq 1\,, \mathrm{supp}(\rho_\Gamma) \subset \Gamma\}$$

Interpretation : as soon as a particle reaches Γ, it is safe $(D = 0)$; we leave it on Γ for simplicity, but this only means that we are no longer concerned with what happens to it.

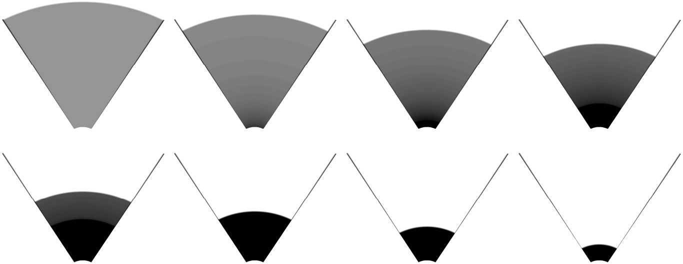
Mathematically : analyzing the scheme with this new set K is more complicated but manage anyway to prove existence of a solution. Yet, uniqueness is out of reach for the moment.

An example

Closed door



Open door



Perspectives

MFG, multiple populations, fluid mechanics

Adapting $P_{adm(\rho)}$ to other problems – Mean Field Games

Better models for **crowd motions :** one should insert the possibility for the agents to choose their strategy (for instance following the ideas of the so-called *mean-field games*, by J.-M. Lasry and P.-L. Lions).

$$\begin{cases} \partial_t \varphi + \frac{|\nabla\varphi|^2}{2} - \nabla\varphi\cdot\nabla p = 0, \\ \partial_t \rho + \nabla\cdot(\rho(\nabla\varphi - \nabla p)) = 0, \\ p \geq 0,\ p(1-\rho) = 0, \\ \varphi(T,x) = \Phi(x), \quad \rho(0,x) = \rho_0(x). \end{cases}$$

These equations describe a game where every player wants to maximize

$$\max \quad -\int_t^T \frac{|\alpha(s)|^2}{2} ds + \Phi(y(T)),\ :\ y'(s) = \alpha(s) - \nabla p_s(y(s)),\ y(t) = x.$$

subject to the pressure given by the density of others and the constraint. The function ϕ is the value function of the control problem for each player.

F. Santambrogio, A Modest Proposal for MFG with Density Constraints, to appear in a special issue of *Net. Het. Media*, 2012.

Adapting $P_{adm(\rho)}$ to other problems – Mean Field Games

Better models for **crowd motions :** one should insert the possibility for the agents to choose their strategy (for instance following the ideas of the so-called *mean-field games*, by J.-M. Lasry and P.-L. Lions).

$$\begin{cases} \partial_t \varphi + \frac{|\nabla \varphi|^2}{2} - \nabla \varphi \cdot \nabla p = 0, \\ \partial_t \rho + \nabla \cdot (\rho(\nabla \varphi - \nabla p)) = 0, \\ p \geq 0,\ p(1-\rho) = 0, \\ \varphi(T, x) = \Phi(x), \quad \rho(0, x) = \rho_0(x). \end{cases}$$

These equations describe a game where every player wants to maximize

$$\max \quad -\int_t^T \frac{|\alpha(s)|^2}{2} ds + \Phi(y(T)),\ :\ y'(s) = \alpha(s) - \nabla p_s(y(s)),\ y(t) = x.$$

subject to the pressure given by the density of others and the constraint. The function ϕ is the value function of the control problem for each player.

F. Santambrogio, A Modest Proposal for MFG with Density Constraints, to appear in a special issue of *Net. Het. Media*, 2012.

Adapting $P_{adm(\rho)}$ to other problems – Mean Field Games

Better models for **crowd motions :** one should insert the possibility for the agents to choose their strategy (for instance following the ideas of the so-called *mean-field games*, by J.-M. Lasry and P.-L. Lions).

$$\begin{cases} \partial_t\varphi + \frac{|\nabla\varphi|^2}{2} - \nabla\varphi\cdot\nabla p = 0, \\ \partial_t\rho + \nabla\cdot(\rho(\nabla\varphi - \nabla p)) = 0, \\ p \geq 0,\ p(1-\rho) = 0, \\ \varphi(T,x) = \Phi(x), \quad \rho(0,x) = \rho_0(x). \end{cases}$$

These equations describe a game where every player wants to maximize

$$\max \quad -\int_t^T \frac{|\alpha(s)|^2}{2}ds + \Phi(y(T)),\ :\ y'(s) = \alpha(s) - \nabla p_s(y(s)),\ y(t) = x.$$

subject to the pressure given by the density of others and the constraint. The function ϕ is the value function of the control problem for each player.

F. Santambrogio, A Modest Proposal for MFG with Density Constraints, to appear in a special issue of *Net. Het. Media*, 2012.

Adapting $P_{adm(\rho)}$ to other problems – Two populations

Many models in biology or in social sciences consider the case of two populations (with or without density constraints) where each one evolves so as to minimize its own energy (possibly depending on the interactions with the other) :

$$\begin{cases} \frac{\partial \rho}{\partial t} - \nabla \cdot \left(\rho \nabla \left(\frac{\delta E}{\delta \rho}(\rho, \mu)\right)\right) = 0, \\ \frac{\partial \mu}{\partial t} - \nabla \cdot \left(\mu \nabla \left(\frac{\delta F}{\delta \mu}(\rho, \mu)\right)\right) = 0. \end{cases}$$

If $E \neq F$ we do not have in general a gradient flow.
Inserting density constraints is delicate. If the constraint is $\rho + \mu \leq 1$ there will be a unique pressure p. Mathematically, it is difficult to let terms such as $\rho \nabla p$ pass to the limit (since they are non-linear). When there was a unique population the crucial point was $\rho \nabla p = \nabla p$ (since $p = 0$ on $\{\rho \neq 1\}$).

J. Dambrine, B. Maury, N. Meunier, A. Roudneff-Chupin, A congestion Model for Cell migration, *Communications in Pure and Applied Analysis*, 2012.

Adapting $P_{adm(\rho)}$ to other problems – Two populations

Many models in biology or in social sciences consider the case of two populations (with or without density constraints) where each one evolves so as to minimize its own energy (possibly depending on the interactions with the other) :

$$\begin{cases} \frac{\partial \rho}{\partial t} - \nabla \cdot \left(\rho \nabla \left(\frac{\delta E}{\delta \rho}(\rho, \mu)\right)\right) = 0, \\ \frac{\partial \mu}{\partial t} - \nabla \cdot \left(\mu \nabla \left(\frac{\delta F}{\delta \mu}(\rho, \mu)\right)\right) = 0. \end{cases}$$

If $E \neq F$ we do not have in general a gradient flow.

Inserting density constraints is delicate. If the constraint is $\rho + \mu \leq 1$ there will be a unique pressure p. Mathematically, it is difficult to let terms such as $\rho \nabla p$ pass to the limit (since they are non-linear). When there was a unique population the crucial point was $\rho \nabla p = \nabla p$ (since $p = 0$ on $\{\rho \neq 1\}$).

J. Dambrine, B. Maury, N. Meunier, A. Roudneff-Chupin, A congestion Model for Cell migration, *Communications in Pure and Applied Analysis*, 2012.

Adapting $P_{adm(\rho)}$ to other problems – Two populations

Many models in biology or in social sciences consider the case of two populations (with or without density constraints) where each one evolves so as to minimize its own energy (possibly depending on the interactions with the other) :

$$\begin{cases} \frac{\partial \rho}{\partial t} - \nabla \cdot \left(\rho \nabla \left(\frac{\delta E}{\delta \rho}(\rho, \mu)\right)\right) = 0, \\ \frac{\partial \mu}{\partial t} - \nabla \cdot \left(\mu \nabla \left(\frac{\delta F}{\delta \mu}(\rho, \mu)\right)\right) = 0. \end{cases}$$

If $E \neq F$ we do not have in general a gradient flow.

Inserting density constraints is delicate. If the constraint is $\rho + \mu \leq 1$ there will be a unique pressure p. Mathematically, it is difficult to let terms such as $\rho \nabla p$ pass to the limit (since they are non-linear). When there was a unique population the crucial point was $\rho \nabla p = \nabla p$ (since $p = 0$ on $\{\rho \neq 1\}$).

J. Dambrine, B. Maury, N. Meunier, A. Roudneff-Chupin, A congestion Model for Cell migration, *Communications in Pure and Applied Analysis*, 2012.

Adapting $P_{adm(\rho)}$ to other problems – Two populations

Many models in biology or in social sciences consider the case of two populations (with or without density constraints) where each one evolves so as to minimize its own energy (possibly depending on the interactions with the other) :

$$\begin{cases} \frac{\partial \rho}{\partial t} - \nabla \cdot \left(\rho \nabla \left(\frac{\delta E}{\delta \rho}(\rho, \mu)\right) = 0, \\ \frac{\partial \mu}{\partial t} - \nabla \cdot \left(\mu \nabla \left(\frac{\delta F}{\delta \mu}(\rho, \mu)\right) = 0. \end{cases}$$

If $E \neq F$ we do not have in general a gradient flow.
Inserting density constraints is delicate. If the constraint is $\rho + \mu \leq 1$ there will be a unique pressure p. Mathematically, it is difficult to let terms such as $\rho \nabla p$ pass to the limit (since they are non-linear). When there was a unique population the crucial point was $\rho \nabla p = \nabla p$ (since $p = 0$ on $\{\rho \neq 1\}$).

J. Dambrine, B. Maury, N. Meunier, A. Roudneff-Chupin, A congestion Model for Cell migration, *Communications in Pure and Applied Analysis*, 2012.

Adapting $P_{adm(\rho)}$ to other problems – Two populations

Many models in biology or in social sciences consider the case of two populations (with or without density constraints) where each one evolves so as to minimize its own energy (possibly depending on the interactions with the other) :

$$\begin{cases} \frac{\partial \rho}{\partial t} - \nabla \cdot \left(\rho \nabla \left(\frac{\delta E}{\delta \rho}(\rho, \mu)\right) = 0, \\ \frac{\partial \mu}{\partial t} - \nabla \cdot \left(\mu \nabla \left(\frac{\delta F}{\delta \mu}(\rho, \mu)\right) = 0. \end{cases}$$

If $E \neq F$ we do not have in general a gradient flow.
Inserting density constraints is delicate. If the constraint is $\rho + \mu \leq 1$ there will be a unique pressure p. Mathematically, it is difficult to let terms such as $\rho \nabla p$ pass to the limit (since they are non-linear). When there was a unique population the crucial point was $\rho \nabla p = \nabla p$ (since $p = 0$ on $\{\rho \neq 1\}$).

J. Dambrine, B. Maury, N. Meunier, A. Roudneff-Chupin, A congestion Model for Cell migration, *Communications in Pure and Applied Analysis*, 2012.

Adapting $P_{adm(\rho)}$ to other problems – Two populations

For **incompressible fluids :** study *second order* models for pressureless gas dynamics, such as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \quad \text{with} \quad \frac{\partial (\rho v)}{\partial t} + \nabla \cdot (\rho v \otimes v) + \nabla p = 0,$$

où $\rho \leq 1$, $p \in Press(\rho)$ et $v(t^+) = P_{adm(\rho_t)}(v(t^-))$, generalizing a work by Bouchut, Brenier, Cortes, Ripoll in 1D.

The difficulty is the fact that here we are more interested in a Cauchy problem prescribing ρ_0 and v_0, leading to a non-variational structure.

F. Bouchut, Y. Brenier, J. Cortes, J.-F Ripoll, A hierarchy of models for two-phase flows, *J. nonlinear sci*, 2000.

Adapting $P_{adm(\rho)}$ to other problems – Two populations

For **incompressible fluids :** study *second order* models for pressureless gas dynamics, such as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \quad \text{with} \quad \frac{\partial (\rho v)}{\partial t} + \nabla \cdot (\rho v \otimes v) + \nabla p = 0,$$

où $\rho \leq 1$, $p \in Press(\rho)$ et $v(t^+) = P_{adm(\rho_t)}(v(t^-))$, generalizing a work by Bouchut, Brenier, Cortes, Ripoll in 1D.
The difficulty is the fact that here we are more interested in a Cauchy problem prescribing ρ_0 and v_0, leading to a non-variational structure.

F. Bouchut, Y. Brenier, J. Cortes, J.-F Ripoll, A hierarchy of models for two-phase flows, *J. nonlinear sci*, 2000.

The End

Thanks for your attention and for your warm welcome in Hefei

The End

Thanks for your attention and for your warm welcome in Hefei