# STABILITY IN THE ISOPERIMETRIC PROBLEM

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We shall discuss the question of stability of the solution of the classical isoperimetric problem in $\mathbb{R}^k$: if a domain $D$ in $\mathbb{R}^3$, say, of volume $V$ and surface area $S$ is such that

$$\frac{S}{4\pi}-\left(\frac{3V}{4\pi}\right)^{2/3}$$

is close to its minimal value 0, must then $D$ be 'close' to being a ball? We obtain a positive answer to this vaguely formulated question in the case of domains $D$ that are assumed from the outset to differ from suitable balls by at most 5%, in a certain sense: see §2, Theorem. Some such additional hypothesis is necessary for stability in dimension $k \geqslant 3$; see the remark at the end.

In the planar case $k = 2$, however, there is unrestricted stability. This is expressed by a number of inequalities due to Bonnesen [**1**]; see also his monograph [**2**, no. 43]. Alternatively, the stability can be read off immediately from Hurwitz' elegant proof [**3**; **4**, no. 6] that the disk is the unique solution of the isoperimetric problem in the plane; see §1. It seems, however, difficult to extend Hurwitz' method to higher dimensions (see [**4**, no. 8]); hence our alternative approach in §2. For *convex* bodies, stability can be inferred from Bonnesen [**2**, p. 135] (in uniform norm, only; see below).

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## 1.

Hurwitz' proof – as refined by Lebesgue [**5**] in order to avoid differentiability assumptions – runs as follows (in complex notation). Consider a simple, closed, positively oriented, rectifiable curve $\gamma$ in $\mathbb{C}$ of length $L = 2\pi$, represented parametrically with arc length $s \in [-\pi, \pi]$ as parameter:

$$\gamma: s \mapsto z(s), \quad |z'(s)| = 1 \text{ a.e.}, \tag{1}$$

the function $z$ being clearly absolutely continuous with $z(\pi) = z(-\pi)$. In terms of the (uniformly) convergent Fourier expansion

$$z(s) = \sum_{n\in\mathbb{Z}} c_n e^{ins}, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} z(s)e^{-ins}ds,$$

the area $A$ of the domain $D$ enclosed by $\gamma$ is

$$A = \tfrac{1}{2}\int_\gamma xdy - ydx = \tfrac{1}{2}\operatorname{Im}\int_{-\pi}^{\pi} \bar{z}(s)z'(s)\,ds = \pi \sum_{n\in\mathbb{Z}} n|c_n|^2. \tag{2}$$

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On the other hand, from $|z'(s)| = 1$ a.e. it follows that

$$1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |z'(s)|^2 ds = \sum_{n \in \mathbb{Z}} n^2 |c_n|^2. \tag{3}$$

Since $n^2 > n$ for $n \in \mathbb{Z} \setminus \{0, 1\}$, Hurwitz deduces from (2), (3) the isoperimetric inequality $A \leqslant \pi$, whereby equality holds for a circle only, viz: $c_n = 0$ for all $n \neq 0, 1$, that is

$$z(s) = c_0 + c_1 e^{is}. \tag{4}$$

Let us now consider the deviation

$$w(s) = z(s) - (c_0 + c_1 e^{is})$$

of the general curve, or motion, $\gamma$ in (1) from the associated circular motion (4). Since

$$1 + n^2 \leqslant \tfrac{5}{2}(n^2 - n), \quad n \in \mathbb{Z} \setminus \{0, 1\},$$

we obtain

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} (|w|^2 + |w'|^2)\, ds &= \sum_{n \neq 0,1} (1 + n^2)|c_n|^2 \\ &\leqslant \tfrac{5}{2} \sum_{n \in \mathbb{Z}} (n^2 - n)|c_n|^2 \\ &= \tfrac{5}{2}(1 - A/\pi). \end{aligned} \tag{5}$$

This estimate exhibits again the circle as the unique solution of the isoperimetric problem in the plane, but it shows moreover that the solution is *stable* in Sobolev 1-norm. And from that it follows easily that there is stability also in the uniform norm $\max_s |w(s)|$: since $\int_{-\pi}^{\pi} w(s) ds = 0$, there exists $t$ such that $\operatorname{Re} w(t) = 0$, hence for any $s$ (taken from $[t - \pi, t + \pi]$)

$$|\operatorname{Re} w(s)|^2 = \left| \int_t^s \operatorname{Re} w'(\tau)\, d\tau \right|^2 \leqslant \pi \int_{-\pi}^{\pi} |\operatorname{Re} w'(\tau)|^2 d\tau,$$

and similarly for $\operatorname{Im} w$, whence from (5)

$$|w(s)|^2 \leqslant \pi \int_{-\pi}^{\pi} |w'(\tau)|^2 d\tau \leqslant 5\pi(\pi - A).$$

It follows immediately from this inequality (dropping the above normalisation $L = 2\pi$) that $\gamma$ is contained in a circular annulus of width $d$, where $d^2 \leqslant 5(L^2 - 4\pi A)$. This consequence of (5) is a weaker version of a result of Bonnesen [1] according to which the constant 5 here can be replaced by $1/(4\pi)$, which is best possible. On the other hand, (5) itself is in a sense stronger than Bonnesen's inequalities because the Sobolev norm is stronger than the uniform norm. I do not know the best possible constant in (5).

## 2.

As mentioned in the introduction, Hurwitz' method does not seem to extend to higher dimensions. Measuring the deviation of $D$ from balls in a different way we proceed to establish stability, again in Sobolev 1-norm and in uniform norm, of the ball as solution of the isoperimetric problem in $\mathbb{R}^k$ for any $k$ in the case of *nearly* *spherical domains*: see (7) below. Such domains are not necessarily convex. Our analysis will be carried out in dimension $k = 3$, but applies equally well in any dimension $k \geqslant 2$, save for the values of constants. We shall consider a bounded Lipschitz domain $D$ in $\mathbb{R}^3$, starshaped with respect to its barycentre, and normalised so as to have the same *volume* $V = 4\pi/3$ as the unit ball. Taking the barycentre of $D$ as origin, we represent the boundary of $D$ in polar coordinates $R$, $\xi$:

$$R = R(\xi), \quad \xi = (x, y, z) \in \Sigma,$$

where $\Sigma$ denotes the unit sphere in $\mathbb{R}^3$. Writing

$$R^3 = 1 + 3u \tag{6}$$

(hence $u \approx R-1$ for small $|u|$) we finally impose the essential restriction that $D$ be nearly spherical in the sense that

$$|u| \leqslant c, \quad |\nabla u| \leqslant c \quad \text{on } \Sigma \tag{7}$$

for a suitable constant $c \leqslant 1/20$, $\nabla$ denoting the gradient operator on $\Sigma$. (From $|\nabla u| \leqslant c$ it follows incidentally that $|u| \leqslant \pi c$ since it follows from (12) below that there exists $\eta \in \Sigma$ with $u(\eta) = 0$.)

Writing $d\sigma$ for the normalised surface measure on $\Sigma$, and $S$ for the total surface area of $\Gamma = \partial D$, we have

$$\frac{S}{4\pi} = \int_\Sigma R(R^2 + |\nabla R|^2)^{\frac{1}{2}}\, d\sigma, \tag{8}$$

$$\frac{3V}{4\pi} = \int_\Sigma R^3\, d\sigma = 1, \tag{9}$$

$$0 = \int_\Sigma R^4 \xi\, d\sigma, \tag{10}$$

where (10) expresses that the barycentre of $D$ is 0.

THEOREM. *With the above notations we have for a starshaped Lipschitz domain $D$ of volume $V = 4\pi/3$ and satisfying* (7) *with $c = 1/20$*

$$\int_\Sigma (u^2 + |\nabla u|^2)\, d\sigma \leqslant 8\left(\frac{S}{4\pi} - 1\right).$$

*In view of the restriction* (7) *this implies, for each $p > 2$, the following estimate of the uniform norm of $u$:*

$$\max_{\xi \in \Sigma} |u(\xi)| \leqslant K_p \left(\frac{S}{4\pi} - 1\right)^{1/p}$$

*where*

$$K_p = 8^{1/p} c^{1-2/p} \left(\frac{\frac{1}{2}\pi q}{\sin(\frac{1}{2}\pi q)}\right)^{1/q}$$

*with $1/q = 1 - 1/p$ and $c = 1/20$. (One may incidentally replace 8 by 6 in $K_p$.)*

REMARK. Using (6) we could of course obtain quite similar estimates (with slightly different constants) whereby $u(\xi)$ is replaced by $R(\xi) - 1$ and $\nabla u$ by $\nabla R$.

*Proof of Theorem.* To begin with we shall allow any value of $c \leqslant 1/20$. In order to estimate $S$ from below consider the integrand in (8) and insert (6):

$$R(R^2 + |\nabla R|^2)^{\frac{1}{2}} = (1+3u)^{2/3}(1 + (1+3u)^{-2}|\nabla u|^2)^{\frac{1}{2}}. \tag{11}$$

By Taylor expansion and estimation we obtain

$$(1+3u)^{2/3} = 1+\sum_{n=1}^{\infty}\frac{1}{n!}\frac{2}{3}(-\tfrac{1}{3})(-\tfrac{4}{3})\dots(-(n-\tfrac{5}{3}))(3u)^n$$

$$\geqslant 1+\tfrac{2}{3}(3u)+\frac{1}{2!}\frac{2}{3}(-\tfrac{1}{3})(3u)^2-\frac{1}{3!}\frac{2}{3}\frac{1}{3}\frac{4}{3}\sum_{n=3}^{\infty}|3u|^n$$

because $n-5/3<n$. In view of (7) we thus have

$$(1+3u)^{2/3}\geqslant 1+2u-u^2-\frac{4}{3}\frac{c}{1-3c}u^2$$

$$\geqslant 1+2u-u^2-1.57\,cu^2\,(\geqslant 1-2.054\,|u|),$$

$$(1+(1+3u)^{-2}|\nabla u|^2)^{\frac{1}{2}}\geqslant (1+(1-6c)\,|\nabla u|^2)^{\frac{1}{2}}$$

$$\geqslant 1+(\tfrac{1}{2}-3c)\,|\nabla u|^2-\tfrac{1}{8}\,|\nabla u|^4$$

$$\geqslant 1+\left(\tfrac{1}{2}-3.01\,c\right)|\nabla u|^2.$$

Inserting these estimates in (11) we obtain, by further use of (7),

$$R(R^2+|\nabla R|^2)^{\frac{1}{2}}\geqslant 1+2u-u^2-1.57cu^2+(1-2.054|u|)\,(\tfrac{1}{2}-3.01\,c)\,|\nabla u|^2$$

$$\geqslant 1+2u-u^2+\tfrac{1}{2}\,|\nabla u|^2-c(1.57u^2+3.01|\nabla u|^2+1.027|u|\,|\nabla u|)$$

$$\geqslant 1+2u-u^2+\tfrac{1}{2}\,|\nabla u|^2-3.18c(u^2+|\nabla u|^2).$$

In view of (8) we obtain by integration, invoking also the following consequence of (6) and (9):

$$\int_\Sigma u\,d\sigma = 0, \tag{12}$$

the following key inequality

$$\frac{S}{4\pi}\geqslant\int_\Sigma (1-u^2+\tfrac{1}{2}\,|\nabla u|^2-3.18c(u^2+|\nabla u|^2))\,d\sigma.$$

In terms of the inner product $\langle f,g\rangle=\int_\Sigma fg\,d\sigma$

and the associated norm $\|\,.\,\|=\|\,.\,\|_{L^2}$, this estimate leads to

$$\frac{S}{4\pi}-1\geqslant\tfrac{1}{2}\langle u,-\Delta u\rangle-\|u\|^2-3.18c(\langle u,-\Delta u\rangle+\|u\|^2), \tag{13}$$

where $\Delta$ denotes the Laplace–Beltrami operator on $\Sigma$, satisfying $\|\nabla u\|^2=\langle u,-\Delta u\rangle$.

Now expand $u$ in normalised spherical harmonics $Y_n$ (each $Y_n$ belonging to the $(2n+1)$-dimensional eigenspace for $-\Delta$ corresponding to the eigenvalue $n(n+1)$):

$$u=\sum_{n=0}^{\infty}a_nY_n,\quad a_n=\langle u,Y_n\rangle,$$

$$\langle Y_m,Y_n\rangle=\delta_{mn},\quad -\Delta Y_n=n(n+1)Y_n.$$

In particular, $Y_0=1$, and so

$$a_0=\int_\Sigma u\,d\sigma=0 \tag{14}$$

by (12). Moreover, $Y_1$ has the form

$$Y_1(\xi)=\alpha x+\beta y+\gamma z \tag{15}$$

with $\alpha^2+\beta^2+\gamma^2 = 3$ and hence

$$|Y_1(\xi)| \geqslant \sqrt{3}, \quad \xi \in \Sigma. \tag{16}$$

In fact, $\|x\|^2 = \|y\|^2 = \|z\|^2 = \frac{1}{3}$, while $\langle x, y\rangle = 0$, etc., by symmetry.

We proceed to deduce from (10) that $a_1^2$ is very small compared to

$$\|u\|^2 = \sum_1^\infty a_n^2; \tag{17}$$

see (14). Expanding $R^4 = (1+3u)^{4/3}$ (see (6)) yields

$$|R^4-1-4u| \leqslant \frac{2}{1-3c}u^2 \leqslant 2.4\,u^2. \tag{18}$$

From (10), (15) it follows that $\int_\Sigma R^4 Y_1\,d\sigma = 0$, and since $\int_\Sigma Y_1\,d\sigma = 0$ by symmetry, we obtain in view of (18), (16)

$$\left|\int_\Sigma u Y_1\,d\sigma\right| = \tfrac{1}{4}\left|\int_\Sigma (R^4-1-4u)Y_1\,d\sigma\right| \leqslant 0.6\sqrt{3}\int_\Sigma u^2\,d\sigma \leqslant 0.6\sqrt{3}\,c\|u\|$$

because $\|u\| \leqslant c$ by (7). Hence

$$a_1^2 = \langle u, Y_1\rangle^2 \leqslant 1.08\,c^2\|u\|^2 \leqslant 0.054c\sum_1^\infty a_n^2 \tag{19}$$

on account of (17).

Inserting in (13) the expansion $u = \sum_1^\infty a_n Y_n$ (see (14)) we find

$$\begin{aligned}\frac{S}{4\pi}-1 &\geqslant \sum_2^\infty (\tfrac{1}{2}n(n+1)-1)a_n^2 - 3.18c\sum_1^\infty (n(n+1)+1)a_n^2\\ &\geqslant -\tfrac{6}{7}a_1^2+(\tfrac{2}{7}-3.18c)\sum_1^\infty (n(n+1)+1)a_n^2\end{aligned} \tag{20}$$

because $\frac{1}{2}n(n+1)-1 \geqslant \frac{2}{7}(n(n+1)+1)$ for $n \geqslant 2$. From (19) it follows that

$$a_1^2 \leqslant 0.018c\sum_1^\infty (n(n+1)+1)\,a_n^2,$$

and (20) yields

$$\frac{S}{4\pi}-1 \geqslant (\tfrac{2}{7}-3.2c)\sum_1^\infty (n(n+1)+1)\,a_n^2 = (\tfrac{2}{7}-3.2c)\,(\|\nabla u\|^2+\|u\|^2)$$

(under the hypothesis (7) with $c \leqslant 1/20$).

Taking from now on $c = 1/20$ we have thus obtained the desired estimate in Sobolev 1-norm

$$\|u\|^2+\|\nabla u\|^2 \leqslant 8\left(\frac{S}{4\pi}-1\right) \tag{21}$$

under the hypothesis that $|u|, |\nabla u| \leqslant 1/20$ pointwise on $\Sigma$. The estimate of the uniform norm $\max|u(\xi)|$ stated in the theorem follows from (21) in view of (12) by application of the following lemma (which is more or less known).

LEMMA. *For any Lipschitz function $u$ on $\Sigma$ and any number $p > 2$ we have*

$$\left\|u-\int_\Sigma u\,d\sigma\right\|_{L^\infty} \leqslant \left(\frac{\frac{1}{2}\pi q}{\sin(\frac{1}{2}\pi q)}\right)^{1/q}\|\nabla u\|_{L^\infty}^{1-2/p}\,\|\nabla u\|_{L^2}^{2/p},$$

*where* $1/p+1/q = 1$.

*Proof.* Let $\xi \in \Sigma$ be given. For any $\eta \in \Sigma$ we have

$$u(\xi)-u(\eta) = \int_\eta^\xi \nabla u(\zeta)\cdot d\zeta,$$

$$u(\xi)-\int_\Sigma u(\eta)\,d\sigma(\eta) = \int_\Sigma d\sigma(\eta)\int_\eta^\xi \nabla u(\zeta)\cdot d\zeta.$$

In terms of spherical coordinates $\phi\in[0,\pi]$, $\theta\in[0,2\pi]$, where $\phi = 0$ at $\xi$, this implies

$$\begin{aligned}\left|u(\xi)-\int_\Sigma u\,d\sigma\right| &\leqslant \frac{1}{4\pi}\int_0^{2\pi} d\theta\int_0^\pi \sin\phi\,d\phi\int_0^\phi |\nabla u(t,\theta)|\,dt \\ &= \frac{1}{4\pi}\int_0^{2\pi} d\theta\int_0^\pi (1+\cos\phi)\,|\nabla u(\phi,\theta)|\,d\phi \\ &= \int_\Sigma \cot\tfrac{1}{2}\phi\,|\nabla u|\,d\sigma \\ &\leqslant \|\cot\tfrac{1}{2}\phi\|_{L^q}\,\|\nabla u\|_{L^p} \\ &\leqslant \left(\frac{\tfrac{1}{2}\pi q}{\sin(\tfrac{1}{2}\pi q)}\right)^{1/q} \|\nabla u\|_{L^\infty}^{1-2/p}\,\|\nabla u\|_{L^2}^{2/p}\end{aligned}$$

by evaluation of $\|\cot\tfrac{1}{2}\phi\|_{L^q}$ and application of Hölder's inequality.

REMARK. The following simple example shows that the hypothesis (7) cannot be removed, nor can it be replaced by the weaker requirement

$$\|u\|^2+\|\nabla u\|^2 \leqslant 2c^2, \tag{22}$$

no matter how small the constant $c$ is taken. (The norms are $L^2$-norms as above.) Fix two antipodal points $\pm\xi_0$ of $\Sigma$. For suitably small $\varepsilon,\delta > 0$ define $v$ on $\Sigma$ by

$$v(\xi) = \delta(1-|\xi-\xi_0|/\varepsilon)^+ + \delta(1-|\xi+\xi_0|/\varepsilon)^+,$$

where $a^+$ denotes $\max(a,0)$, and write

$$u(\xi) = v(\xi) - \int_\Sigma v\,d\sigma$$

to achieve $\int u\,d\sigma = 0$. With $R$ defined by (6), the volume condition (9) and the barycentre condition (10) are clearly fulfilled. Since $\int v\,d\sigma \approx \tfrac{1}{6}\varepsilon^2\delta$ is negligible we have, if $\varepsilon/\delta$ is small,

$$\frac{S}{4\pi}-1 \approx \tfrac{1}{2}\varepsilon\delta,$$

$$\|u\|^2+\|\nabla u\|^2 \approx 0+\|\nabla v\|^2 \approx \tfrac{1}{2}\delta^2,$$

which is $\leqslant 2c^2$ if $\delta$ is fixed, $\delta \leqslant 2c$; but $\|u\|^2+\|\nabla u\|^2$ does not approach 0 as $\varepsilon\to 0$ and hence $(S/4\pi)-1 \to 0$. Likewise $\max|u(\xi)| \approx \delta$ does not approach 0.

It is worth noticing that the same type of example in the planar case, now with $\varepsilon = (\delta/c)^2$, shows that even then one cannot drop a condition like (7), or replace it by one like (22) above, if one wants stability in Sobolev 1-norm in the setup of the present section. This does not contradict the result obtained in § 1 by Hurwitz' method, where no condition like (7) was needed; in fact, the measures underlying the Sobolev norms are not the same in the two approaches. As to the uniform norm $\max|R(\xi)-1|$, there is stability in the case of planar domains (supposed starshaped with respect to their barycentre in order to make the representation in polar coordinates meaningful); this follows easily from the stability result in uniform norm established in § 1, or from Bonnesen's inequality.

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