Inf-sup constant and stability for some inverse parameter problems and their discretizations

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Workshop spectral problems, inverse problems and more

ÉCOLE CENTRALE LYON

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An inverse problem

Find μ such that

$$-\nabla \cdot (\mu S) = \mathbf{f} \quad \text{in } \Omega,$$

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Find μ such that

$$-\nabla \cdot (\mu S) = \mathbf{f} \quad \text{in } \Omega,$$

- Ω is a Lipschitz domain of $\mathbb{R}^d$
- $S \in L^\infty(\Omega, \mathbb{R}^{d \times d})$ is given
- $\mathbf{f}$ is a given vector field (can be zero)
- μ is the unknown parameter function

1. Scientific context

2. Discretization of the Reverse Weak Formulation

3. *inf-sup* constant and *inf-sup* condition

4. Stability estimates

5. The honeycomb finite element pair

6. Numerical and experimental applications

Joint work with E. Bretin, P. Millien

Medical elastography

Goal

Measure the elastic parameters of soft biological tissues

Model: linear elasticity equation

$$\begin{cases} -\nabla\cdot(2\mu\mathcal{E}(\mathbf{u}))-\nabla(\lambda\nabla\cdot\mathbf{u})=\mathbf{f} & \Omega \\ BC & \partial\Omega \end{cases}$$

Interrest:

- High contrast (for the shear modulus)
- Good discrimination between pathological states

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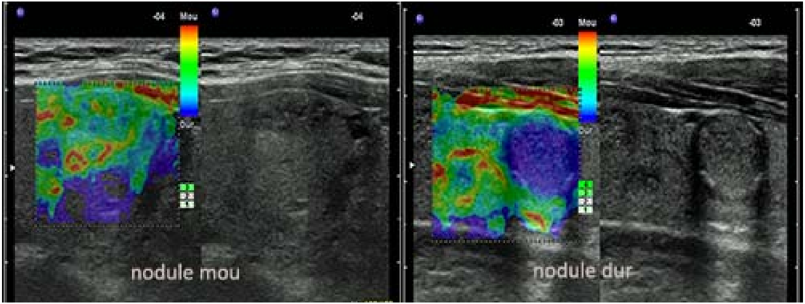

Figure: Thyroid nodules image by UF Ultrasound elatography (soft/hard)

Inversion step 2 : recover the shear modulus

Linear elasticity:

$$\begin{cases} -\nabla\cdot(2\mu\mathcal{E}(\mathbf{u})) - \nabla(\lambda\nabla\cdot\mathbf{u}) = \mathbf{f} & \Omega \\ BC & \partial\Omega \end{cases}$$

with $\mathbf{u}\in\mathbb{R}^d$ the displacement field, $\mathcal{E}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ and (λ,μ) are the Lamé coefficients.

Inverse problem

Recover (λ,μ) from the knowledge of $\mathbf{u}$ in Ω.

Remark

In soft tissues, $\lambda(x)\sim\lambda_0$ and assumed known.

Available inversion methods (1)

$$-\nabla \cdot (\mu \mathcal{E}(\mathbf{u})) = \mathbf{f}$$

Solving a first order transport equation in μ

INSTITUTE OF PHYSICS PUBLISHING INVERSE PROBLEMS

Inverse Problems **20** (2004) 1–24 PII: S0266-5611(04)62168-X

Recovery of the Lamé parameter μ in biological tissues

Lin Ji and Joyce McLaughlin

Department of Mathematics, Rensselaer Polytechnic Institute, Troy, NY 12180, USA

IOP Publishing Inverse Problems

Inverse Problems **30** (2014) 125004 (22pp) doi:10.1088/0266-5611/30/12/125004

Reconstruction of constitutive parameters in isotropic linear elasticity from noisy full-field measurements

Guillaume Bal[1], Cédric Bellis[2], Sébastien Imperiale[3] and François Monard[4]

Available inversion methods (1)

Transport : Assume that μ is smooth and known near $\partial\Omega$ and remark that

$$\nabla \cdot (\mu S) = S \nabla \mu + \mu \nabla \cdot S$$

assume that S is a.e. invertible μ is solution of the transport problem,

$$\nabla \mu + \mu (S)^{-1} \nabla \cdot S = -(S)^{-1} \mathbf{f}$$
$$\nabla \mu + \mu \mathbf{b} = -\tilde{\mathbf{f}}$$

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Provide a proof of uniqueness and stability with several measurements and strong smoothness hypothesis and boundary data

Available inversion methods (2)

Least squares : Assume knowledge of **g** the surface density of force outside of Ω and define

$$F : \mu \mapsto \mathbf{u}[\mu] : \begin{cases} -\nabla \cdot (\mu S) = \mathbf{f} & \text{in } \Omega, \\ \mu S \cdot \nu = \mathbf{g} & \text{on } \partial\Omega, \end{cases}$$

defining $F : L^\infty(\Omega, [\mu_0, +\infty)) \to H^1(\Omega, \mathbb{R}^d)$ fréchet differentiable. Then minimize

$$J[\mu] = \|F[\mu] - \mathbf{u}_{mes}\|^2_{H^1(\Omega)} + \text{ reg}(\mu)$$

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- Very slow (flat problem)
- needs knowledge of $\mathbf{g}$ and μ on the boundary

Available inversion methods (3)

Wave front traking : assuming that μ is piecewise constant,

$$\partial_{tt}\mathbf{u} - \mu\Delta\mathbf{u} \approx \mathbf{0}, \quad \textit{a.e.},$$

Then the wave speed is $c := \sqrt{\mu}$ (locally true,).

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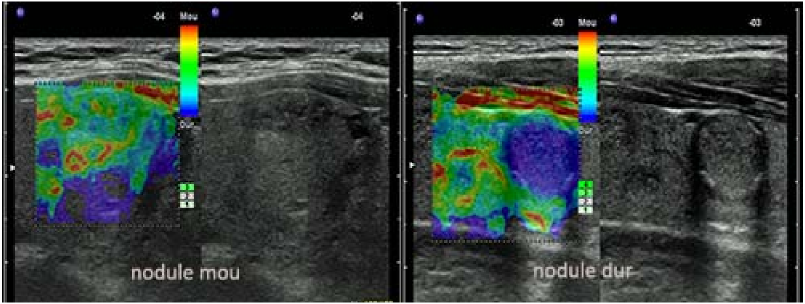

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Available inversion methods (4)

Algebraic inversion :

IOP Publishing — Physics in Medicine and Biology
Phys. Med. Biol. **52** (2007) 1577–1593 — doi:10.1088/0031-9155/52/6/003

Elastic modulus imaging: some exact solutions of the compressible elastography inverse problem

Paul E Barbone[1] and Assad A Oberai[2]

426 — IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control, vol. 49, no. 4, April 2002

Shear Modulus Imaging with 2-D Transient Elastography

Laurent Sandrin, Mickaël Tanter, Stefan Catheline, and Mathias Fink

Approximation for experimental applications: If μ is constant then

$$\mu \Delta \mathbf{u} = \rho \partial_{tt} \mathbf{u} \Longrightarrow \mu \approx \rho \frac{|\partial_{tt} \mathbf{u}|}{|\Delta \mathbf{u}|}$$

Current chalenges for medical elastography

- Increase the resolution

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- Increase the resolution
- Be more quantitative
- Be more stable
- Be more practical (quasi-static with acoustic probe ?)

A general equation

The problem takes the general form

Reduced elastography problem

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in the cases

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- in plane stress approximation (sliced 2D model)

But also for conductivity equation with two internal data:

$$-\nabla \cdot (\sigma[\nabla u_1 \ \nabla u_2]) = \mathbf{f}$$

And other problems...

The Reverse Weak Formulation

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$$a_{\mathbf{u}}(\mu, \mathbf{v}) := \langle T\mu, \mathbf{v} \rangle_{H^{-1}, H_0^1} := \int_\Omega \mu S : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^{d \times d})$$

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Reverse Weak formulation

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- No boundary data used

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- Only smoothness hypothesis: $S \in L^\infty(\Omega, \mathbb{R}^{d\times d})$

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- No boundary data used
- Only smoothness hypothesis: $S \in L^\infty(\Omega, \mathbb{R}^{d \times d})$
- "Easy" to discretize with finite element spaces

Reverse Weak Formulation: discretization

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Find $\mu_h \in M_h$ s.t.

$$\langle T_h\mu_h, \mathbf{v}_h \rangle_{H^{-1},H_0^1} = \langle \mathbf{f}_h, \mathbf{v}_h \rangle_{H^{-1},H_0^1} \quad \forall \mathbf{v} \in V_h$$

where

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where

- M_h approaches $(L^2(\Omega)$
- V_h approaches $H_0^1(\Omega, \mathbb{R}^d)$
- T_h approaches T
- $\mathbf{f}_h$ approaches $\mathbf{f}$

Approximation of the spaces

Let M be a Hilbert and $M_h \subset M$ a sub-Hilbert space and $\pi_h : M \to M_h$ the orthogonal projection.

Definition

The sequence $(M_h)_{h>0}$ approaches M if for any $\mu \in M$,

$$\lim_{h \to 0} \|\pi_h \mu - \mu\|_M = 0.$$

For any non zero $\mu \in M$, we define its relative error of interpolation onto M_h by

$$\varepsilon_h^{\text{int}}(\mu) := \frac{\|\pi_h \mu - \mu\|_M}{\|\mu\|_M}.$$

Approximation of the operator

The operator $T : L^2 \to H^{-1}$ given by

$$\langle T\mu, \mathbf{v}\rangle_{H^{-1},H_0^1} := \int_\Omega \mu S : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^{d\times d})$$

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is approached by $T_h : M_h \to V_h'$

$$\langle T_h\mu, \mathbf{v}\rangle_{V_h',V_h} := \int_\Omega \mu S_h : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in V_h.$$

Hence

Approximation of the operator

The operator $T : L^2 \to H^{-1}$ given by

$$\langle T\mu, \mathbf{v}\rangle_{H^{-1},H^1_0} := \int_\Omega \mu S : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in H^1_0(\Omega, \mathbb{R}^{d\times d})$$

is approached by $T_h : M_h \to V'_h$

$$\langle T_h\mu, \mathbf{v}\rangle_{V'_h,V_h} := \int_\Omega \mu S_h : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in V_h.$$

Hence

$$\begin{aligned}\langle (T_h - T)\mu, \mathbf{v}\rangle_{V'_h,V_h} &= \int_\Omega \mu(S_h - S) : \nabla \mathbf{v} \\ &\leq \|\mu\|_{L^\infty} \|S_h - S\|_{L^2(\Omega)} \|\mathbf{v}\|_{H^1_0}\end{aligned}$$

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The error $T_h - T$ is small for the $\mathcal{L}(L^\infty, V_h')$ topology weaker than the $\mathcal{L}(L^2, V_h')$) topology

Approximation of the operator

Definition

The interpolation error $\varepsilon_h^{\text{op}}$ between T and T_h is defined by

$$\varepsilon_h^{\text{op}} := \|T_h - T\|_{L^\infty, V_h'} := \sup_{\mu \in M_h} \sup_{\mathbf{v} \in V_h} \frac{\langle (T_h - T)\mu, \mathbf{v} \rangle_{V_h', V_h}}{\|\mu\|_{L^\infty} \|\mathbf{v}\|_{H_0^1}}.$$

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- This particular norm does not allow us to use directly the sensitivity analysis and discretization analysis for the Moore-Penrose generalized inverse of T when T is a closed range operator

Approximation of the right-hand side

Definition

The relative error of interpolation $\varepsilon_h^{\text{rhs}}$ between $\mathbf{f} \neq \mathbf{0}$ and $\mathbf{f}_h$ is defined by

$$\varepsilon_h^{\text{rhs}} := \frac{1}{\|\mathbf{f}\|_{V'}} \sup_{\mathbf{v} \in V_h} \frac{\langle \mathbf{f}_h - \mathbf{f}, \mathbf{v} \rangle_{V_h', V_h}}{\|\mathbf{v}\|_V} := \frac{\|\mathbf{f}_h - \mathbf{f}\|_{V_h'}}{\|\mathbf{f}\|_{V'}}$$

Questions

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- Is the solution μ_h close to μ in $L^2(\Omega)$?

A model problem

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Proposition

If Ω is Lipschitz, then $\nabla : L^2 \to H^{-1}$ has closed range. i.e. there exists $C > 0$ s.t.

$$\|q\|_{L^2(\Omega)} \leq C \|\nabla q\|_{H^{-1}(\Omega)} \quad \forall q \in L^2(\Omega) \cap N(\nabla)^{\perp}, \tag{1}$$

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$$\beta := \inf_{q \in L^2(\Omega) \cap N(\nabla)^{\perp}} \sup_{\mathbf{v} \in H_0^1(\Omega, \mathbb{R}^d)} \frac{\int_\Omega q \nabla \cdot \mathbf{v}}{\|\mathbf{v}\|_{H_0^1(\Omega)} \|q\|_{L^2(\Omega)}} > 0 \tag{2}$$

Operator ∇ satisfies the inf-sup condition.

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Operator ∇ satisfies the inf-sup condition.
It has a continuous inverse in $N(\nabla)^{\perp}$.

A model problem: discretization

Problem: the constant β may not behave well in finite element spaces!
Take $M_h \subset L^2_0(\Omega)$ and $V_h \subset H^1_0(\Omega, \mathbb{R}^d)$ the discrete *inf-sup* constant

$$\beta_h := \inf_{q \in M_h} \sup_{\mathbf{v} \in V_h} \frac{\int_\Omega q \nabla \cdot \mathbf{v}}{\|\mathbf{v}\|_{H^1_0(\Omega)} \|q\|_{L^2(\Omega)}}$$

may not satisfy the discrete *inf-sup* condition (of *inf-sup* condition for Ladyzhenskaya-Babuska-Brezzi):

Discrete *inf-sup* condition

$$\forall h > 0, \;\; \beta_h \geq \beta^* > 0$$

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Discrete *inf-sup* condition

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Pairs of finite element spaces that satisfy the discrete *inf-sup* condition are known as *inf-sup* stable elements and play an important role in the stability of the Galerkin approximation for the Stokes problem.

Inf-sup constant for the operator T

Theoretical study of $T\mu := -\nabla \cdot (\mu S)$, with Ammari, Bretin and Millien (2020):

Theorem

If $S \in W^{1,p}$ $p > d$ and $|\det S(x)| \geq c > 0$ a.e, we have

- $dimN(T) \leq 1$
- if $dimN(T) = 1$, T has closed range.

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- At worst T is a "gradient type" operator

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- At worst T is a "gradient type" operator
- extension to S "piecewise" $W^{1,p}$

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If $S \in W^{1,p}$ $p > d$ and $|\det S(x)| \geq c > 0$ a.e, we have

- $dim N(T) \leq 1$
- if $dim N(T) = 1$, T has closed range.

- At worst T is a "gradient type" operator
- extension to S "piecewise" $W^{1,p}$
- minimal assumption on S to have closed range property is an open question (as far as we know)

Generalized inf-sup constant

$M, \; V$ two Hilbert spaces and $T \in \mathcal{L}(M, V')$,

Definition (classic constants)

$$\alpha(T) := \inf_{\mu \in M} \frac{\|T\mu\|_{V'}}{\|\mu\|_M} \quad \text{and} \quad \rho(T) := \sup_{\mu \in M} \frac{\|T\mu\|_{V'}}{\|\mu\|_M}.$$

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Correspondance

Proposition

If $N(T) \neq \{0\}$*, consider any* $z \in N(T)$ *such that* $\|z\|_M = 1$*. Then we have* $\beta(T) = \beta_z(T)$*.*

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Proposition

If there exists $z \in M$ such that $\|z\|_M = 1$ and $\|Tz\|_{V'} = \alpha(T)$, Then we have $\beta(T) = \beta_z(T)$.

True for any finite rank (and finite dimensional) operator

Upper semi-continuity of the inf-sup constant

What is the behavior of $\beta(T_h)$ with respect to $\beta(T)$?

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Theorem

If if the problem $Tz = \mathbf{0}$ *admits a non zero solution* $z \in L^\infty(\Omega)$ *and if* $\varepsilon_h^{op} \to 0$ *when* $h \to 0$ *then*

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- $\beta(T_h)$ is never asymptotically better than $\beta(T)$.
- It might be a possible way to show that T as closed range.

1. Scientific context

2. Discretization of the Reverse Weak Formulation

3. *inf-sup* constant and *inf-sup* condition

4. Stability estimates

5. The honeycomb finite element pair

6. Numerical and experimental applications

Discrete stability estimate (case $\mathbf{f} = \mathbf{0}$)

Let $z \in L^\infty(\Omega)$ be a solution of $T\,z = \mathbf{0}$ with $\|z\|_M = 1$. Consider $z_h \in M_h$ a solution of

$$\|T_h z_h\|_{V_h'} = \alpha(T_h) \quad \text{with} \quad \|z_h\|_M = 1 \quad \text{and} \quad \langle z_h, z\rangle_M \geq 0. \tag{3}$$

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Theorem (1)

If $\beta(T_h) > 0$ we have

$$\|z_h - \pi_h z\|_{L^2(\Omega)} \leq \frac{C}{\beta(T_h)} \Big(\|z\|_{L^\infty(\Omega)} \, \varepsilon_h^{op} + \rho(T)\varepsilon_h^{int}(z) \Big).$$

Moreover, if $\beta(T_h) \geq \beta^* > 0$ and if $\varepsilon_h^{op} \to 0$, then $z_h \to z$.

where $\varepsilon_h^{\text{int}}(z) := \|\pi_h z - z\|_M$.

Discrete stability estimate general case

Let $\mu \in L^\infty(\Omega)$ be a solution of $T\mu = \mathbf{f}$. Consider $z_h \in M_h$ a solution of

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Consider now $\mu_h \in M_h$ a solution of $\mu_h = \underset{\substack{m \in M_h \\ m \perp z_h}}{\arg\min} \|T_h m - \mathbf{f}_h\|_{V_h'}$.

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Theorem (2)

Fix $r > 0$ such that $\|\mu\|_{L^\infty} \leq r\,\|\mu\|_{L^2}$. If $\beta(T_h) > 0$, there exits $t \in \mathbb{R}$ such that $\mu_{h,t} := t\, z_h + \mu_h$ satisfies

$$\frac{\|\mu_{h,t} - \pi_h \mu\|_{L^2}}{\|\pi_h \mu\|_{L^2}} \leq \frac{4}{\beta(T_h)} \left[r\,\varepsilon_h^{op} + \rho(T)\left(\varepsilon_h^{rhs} + \varepsilon_h^{int}(\mu)\right) + \frac{\alpha(T_h)}{2} \right].$$

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honeycomb finite element

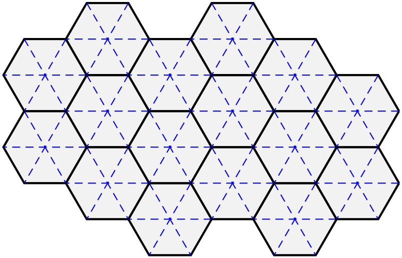


Figure: Honeycomb space discretization. In plain black, the hexagonal subdivision and in dashed blue, the triangular subdivision.

$$M_h := \mathbb{P}^0\left(\Omega_h^{\text{hex}}\right) = \left\{\mu \in L^2(\Omega_h) \mid \forall j \ \mu|_{\Omega_{h,j}^{\text{hex}}} \ \text{ is constant}\right\}.$$

$$V_h := \mathbb{P}_0^1\left(\Omega_h^{\text{tri}}, \mathbb{R}^2\right) = \left\{\mathbf{v} \in H_0^1(\Omega_h, \mathbb{R}^d) \mid \forall k \ \mathbf{v}|_{\Omega_{h,k}^{\text{tri}}} \ \text{ is linear}\right\}.$$

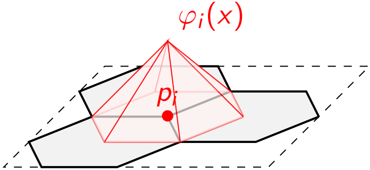
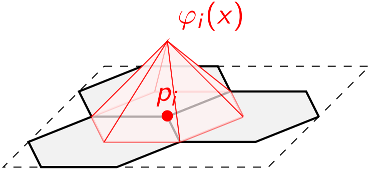


Figure: Support and graph of basis test function φ_i.

Why does it work ?



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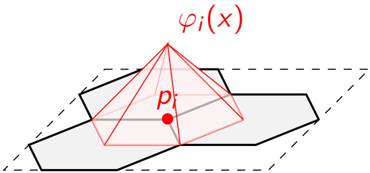

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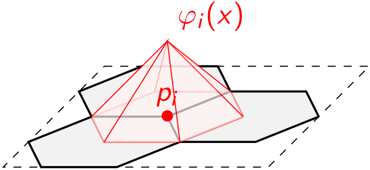
- Case $T = \nabla$: We show that this pair satisfies the *inf-sup* condition.
- General case: We show that for each internal node, we have a system of 2 independent equations for 3 values of the parameters.



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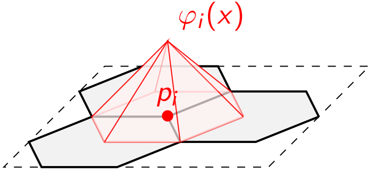

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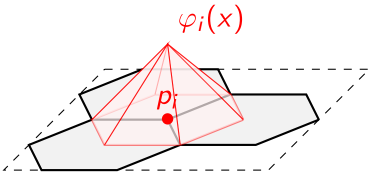

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One value is given $\Rightarrow$ all the other are fixed. $\Rightarrow$ null-space is at most of dimension 1 $\Rightarrow \beta(T_h) > 0$

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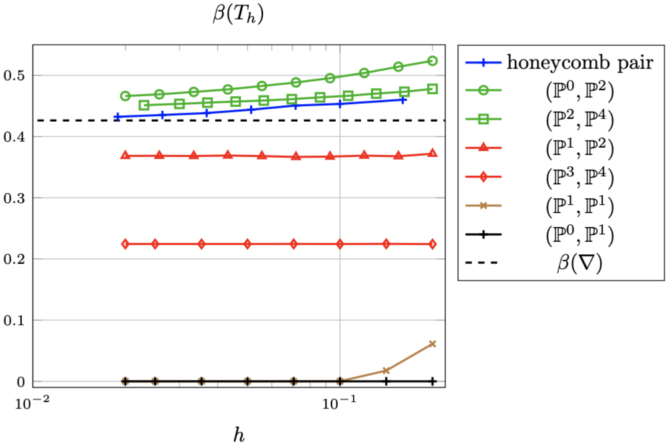
Inverse gradient problem

In $\Omega = (0,1)^2$ we approach $-\nabla \mu = \mathbf{f}$. Here $T_h := -\nabla|_{M_h}$ and then $\varepsilon_h^{\text{op}} = 0$. Moreover $\rho(\nabla) \leq 1$. In the absence of noise, the result of Theorem 2 reads,

$$\frac{\|\mu_h - \pi_h \mu\|_{L^2}}{\|\pi_h \mu\|_{L^2}} \leq \frac{4}{\beta\left(T_h\right)} \left(\frac{\|\mathbf{f} - \mathbf{f}_h\|_{V_h'}}{\|\mathbf{f}\|_{H^{-1}}} + \frac{\|\mu - \pi_h \mu\|_{L^2}}{\|\mu\|_{L^2}} \right).$$

Note that we know $\beta(\nabla) = \sqrt{1/2 - 1/\pi}$ as a conjecture.

Inverse gradient problem: behavior of $\beta(T_h)$



Inverse gradient problem: result

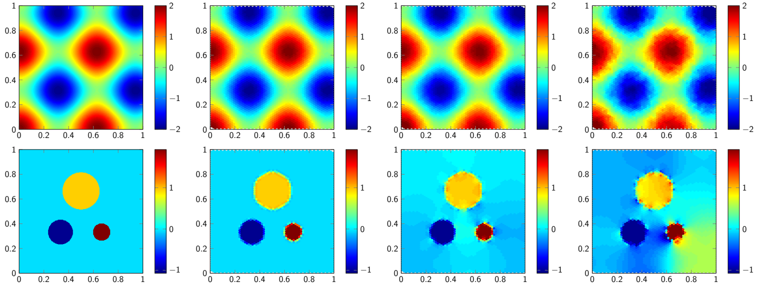


Figure: Numerical stability of the reconstruction of maps μ_1 and μ_2 using method given by Theorem 2 with resolution $h = 0.01$. From left to right: column 1: exact map to recover, 2. reconstruction with no noise, column 3: reconstruction with noise level $\sigma = 1$, column 4: reconstruction with noise level $\sigma = 2$.

Quasistatic elastography

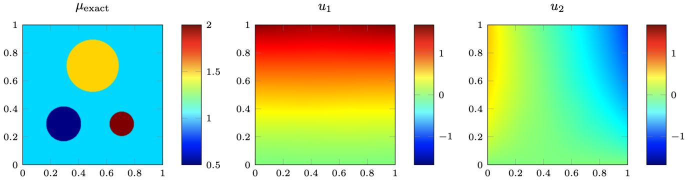


FIG. 5. *First line, from left to right: The exact map μ_{exact}, the two components of the data field $\boldsymbol{u} = (u_1, u_2)$ computed via (5.6), the only data used to inverse the problem.*

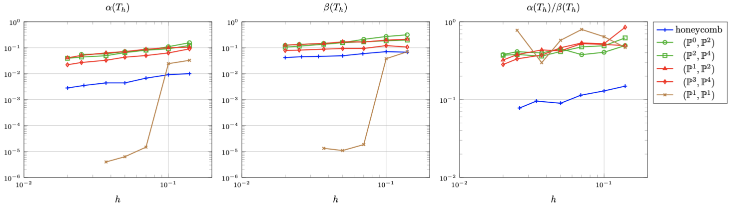


FIG. 6. *Behavior of the contants $\alpha(T_h)$, $\beta(T_h)$ and the ratio $\alpha(T_h)/\beta(T_h)$ for the inverse static elastography problem in the unit square $\Omega := (0,1)^2$, for various choices of pair of discretization spaces.*

Algorithm

Approach the solution of $T\mu = \mathbf{0}$:

Write T_h as a matrix in the basis of the chosen spaces (M_h, V_h). Define the matrix

$$\mathcal{M} := \mathcal{B}_V^{-1} T_h \mathcal{B}_M^{-1}$$

where $\mathcal{B}_M$ and $\mathcal{B}_V$ are the basis matrix of M_h and V_h. Then

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- μ is the first singular vector of $\mathcal{M}$.

Reconstruction for the honeycomb

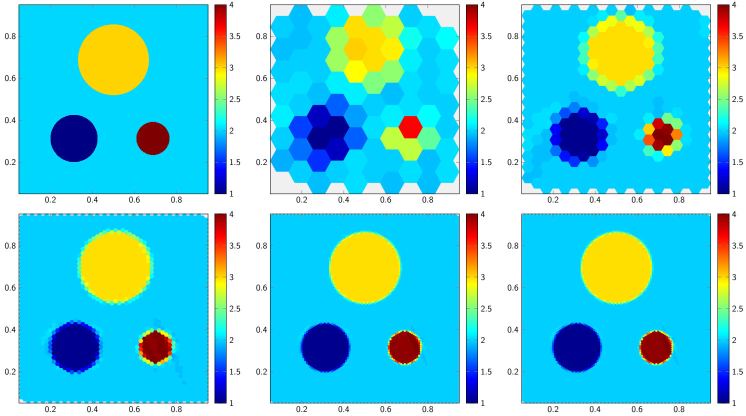

Figure: Reconstruction of the shear modulus map μ using the honeycomb pair.

Reconstruction for various pairs of spaces

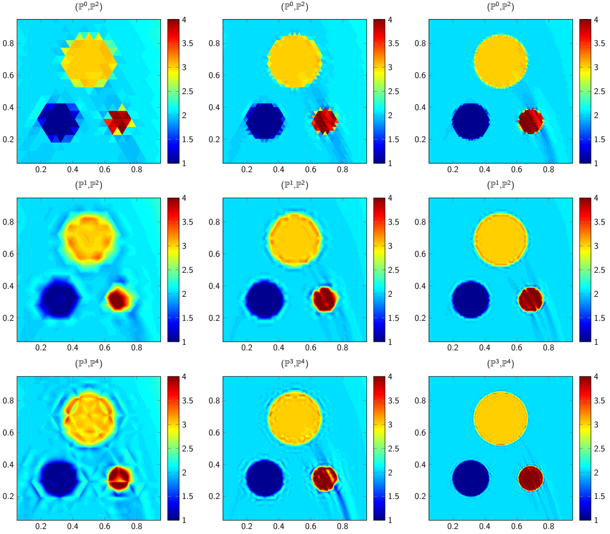

Figure: Reconstruction of the shear modulus map μ using various pairs of finite element spaces in the subdomain of interest $(0.1, 0.9)^2$.

Quasi-static elastography

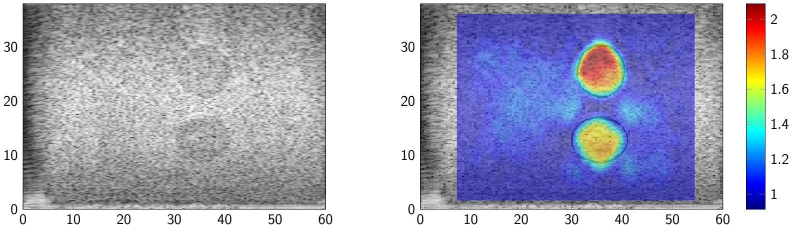

Figure: Shear modulus image of phantom from quasi-static data (data from E. Brusseau and L. Pretrusca - CREATIS/INSA)

In vivo quasistatic elastography

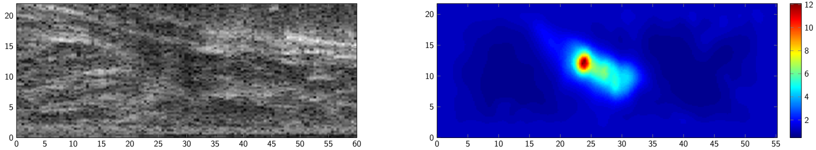

Figure: Reconstruction of the shear modulus of *in-vivo* malignant breast tumor from quasi-static elastography (data from E. Brusseau - INSA/CREATIS) $h = 0.7$ mm.

Thank you for your attention