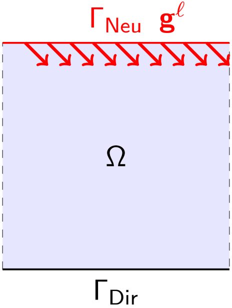
Direct inversion method for quasi-static medical elastography: stability and discretization

Elie Bretin, Pierre Millien and Laurent Seppecher

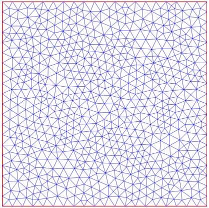
May 27, 2022

IPMS2022, Malta

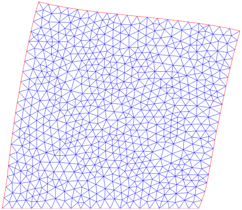
Elastography from internal data



Original elastic object



Deformed elastic object



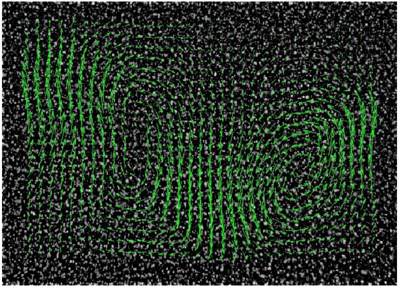
Inverse problem in two steps

- step 1: Record the displacement field $\mathbf{u}(x)$ inside the domain
- step 2: Reconstruct the elastic properties of the medium

Inversion step 1: recover the displacement

Methods used :

- Speckle correlation



- Optimal transport: minimise

$$J(\mathbf{u}) = \|I(x, t + dt) - I(x + \mathbf{u}(x), t)\|_2^2 + R(\mathbf{u})$$

where R is a regularization cost. (Optical flow method)

Inversion step 2 : recover the shear modulus

Linear elasticity:

$$\begin{cases} -\nabla \cdot (2\mu\mathcal{E}(\mathbf{u})) - \nabla(\lambda\nabla\cdot\mathbf{u}) = 0 \quad \Omega \subset \mathbb{R}^3 \\ BC \quad \partial\Omega \end{cases}$$

with $\mathbf{u} \in \mathbb{R}^d$ the displacement field, $\mathcal{E}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ and (λ, μ) are the Lamé coefficients.

Inversion step 2 : recover the shear modulus

Linear elasticity:

$$\begin{cases} -\nabla\cdot(2\mu\mathcal{E}(\mathbf{u}))-\nabla(\lambda\nabla\cdot\mathbf{u})=0 \quad \Omega\subset\mathbb{R}^3 \\ BC \quad \partial\Omega \end{cases}$$

with $\mathbf{u}\in\mathbb{R}^d$ the displacement field, $\mathcal{E}(\mathbf{u})=\frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^T)$ and (λ,μ) are the Lamé coefficients.

Inverse problem

Recover (λ,μ) from the knowledge of $\mathbf{u}$ in Ω.

Inversion step 2 : recover the shear modulus

Linear elasticity:

$$\begin{cases} -\nabla\cdot(2\mu\mathcal{E}(\mathbf{u})) - \nabla(\lambda\nabla\cdot\mathbf{u}) = 0 & \Omega\subset\mathbb{R}^3 \\ BC & \partial\Omega \end{cases}$$

with $\mathbf{u}\in\mathbb{R}^d$ the displacement field, $\mathcal{E}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ and (λ,μ) are the Lamé coefficients.

Inverse problem

Recover (λ,μ) from the knowledge of $\mathbf{u}$ in Ω.

Remark (Plane stress approximation)

The equivalent 2D elastic model reads $\lambda_{2D} = 2\mu$ *and* $\mu_{2D} = \mu$.

Least squares approach

Assume knowledge of $\mathbf{g}$ the surface density of force outside of Ω and define $\mathbf{u}[\mu]$ solution of

$$\begin{cases} -\nabla \cdot (\mu S(\mathbf{u})) & = 0 \ , \quad \text{in } \Omega, \\ \mu \nabla^s \mathbf{u} \cdot \nu & = \mathbf{g} \quad \text{on } \partial\Omega, \end{cases}$$

Where $S(\mathbf{u}) = \mathcal{E}(\mathbf{u}) + (\nabla \cdot \mathbf{u})I$. Then find μ by minimizing

$$J[\mu] = \|\mathbf{u}[\mu] - \mathbf{u}_{mes}\|^2_{H^1(\Omega)} + \text{ reg. term}$$

- Very slow (flat problem)
- needs knowledge of $\mathbf{g}$ and μ on the boundary

The Reverse Weak Formulation

A direct method : Define the operator

$$T : L^\infty(\Omega) \subset L^2(\Omega) \to H^{-1}(\Omega, \mathbb{R}^d)$$
$$\mu \mapsto -\nabla \cdot (\mu S)$$

or by the equivalent variational formulation

$$a(\mu, \mathbf{v}) := \langle T\mu, \mathbf{v} \rangle_{H^{-1}, H_0^1} := \int_\Omega \mu S : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^{d \times d})$$

The Reverse Weak Formulation

A direct method : Define the operator

$$T : L^\infty(\Omega) \subset L^2(\Omega) \to H^{-1}(\Omega, \mathbb{R}^d)$$
$$\mu \mapsto -\nabla \cdot (\mu S)$$

or by the equivalent variational formulation

$$a(\mu, \mathbf{v}) := \langle T\mu, \mathbf{v}\rangle_{H^{-1}, H_0^1} := \int_\Omega \mu S : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^{d\times d})$$

the problem takes the form

Find $\mu \in L^2(\Omega)$ s.t.

$$a(\mu, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^{d\times d})$$

The Reverse Weak Formulation

A direct method : Define the operator

$$T : L^{\infty}(\Omega) \subset L^2(\Omega) \to H^{-1}(\Omega, \mathbb{R}^d)$$
$$\mu \mapsto -\nabla \cdot (\mu S)$$

or by the equivalent variational formulation

$$a(\mu, \mathbf{v}) := \langle T\mu, \mathbf{v} \rangle_{H^{-1}, H_0^1} := \int_\Omega \mu S : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^{d \times d})$$

the problem takes the form

Find $\mu \in L^2(\Omega)$ s.t.

$$a(\mu, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^{d \times d})$$

- No boundary data used

The Reverse Weak Formulation

A direct method : Define the operator

$$T : L^\infty(\Omega) \subset L^2(\Omega) \to H^{-1}(\Omega, \mathbb{R}^d)$$
$$\mu \mapsto -\nabla \cdot (\mu S)$$

or by the equivalent variational formulation

$$a(\mu, \mathbf{v}) := \langle T\mu, \mathbf{v} \rangle_{H^{-1}, H_0^1} := \int_\Omega \mu S : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^{d\times d})$$

the problem takes the form

Find $\mu \in L^2(\Omega)$ s.t.

$$a(\mu, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^{d\times d})$$

- No boundary data used
- Only smoothness hypothesis: $S \in L^\infty(\Omega, \mathbb{R}^{d\times d})$

The Reverse Weak Formulation

A direct method : Define the operator

$$T : L^\infty(\Omega) \subset L^2(\Omega) \to H^{-1}(\Omega, \mathbb{R}^d)$$
$$\mu \mapsto -\nabla \cdot (\mu S)$$

or by the equivalent variational formulation

$$a(\mu, \mathbf{v}) := \langle T\mu, \mathbf{v} \rangle_{H^{-1}, H_0^1} := \int_\Omega \mu S : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^{d \times d})$$

the problem takes the form

Find $\mu \in L^2(\Omega)$ s.t.

$$a(\mu, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^{d \times d})$$

- No boundary data used
- Only smoothness hypothesis: $S \in L^\infty(\Omega, \mathbb{R}^{d \times d})$
- "Easy" to discretize through the Galerkin approximation

Reverse Weak Formulation: discretization

Find $\mu \in L^2(\Omega)$ s.t.

$$\langle T\mu, \mathbf{v} \rangle_{H^{-1},H_0^1} = 0 \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^d)$$

becomes

Find $\mu_h \in M_h$ s.t.

$$\langle T_h\mu_h, \mathbf{v}_h \rangle_{H^{-1},H_0^1} = 0 \quad \forall \mathbf{v} \in V_h$$

where

- (M_h, V_h) approaches $(M, V) := (L^2(\Omega), H_0^1(\Omega, \mathbb{R}^d))$
- T_h approaches T

Reverse Weak Formulation: discretization

Find $\mu \in L^2(\Omega)$ s.t.

$$\langle T\mu, \mathbf{v}\rangle_{H^{-1},H_0^1} = 0 \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^d)$$

becomes

Find $\mu_h \in M_h$ s.t.

$$\langle T_h\mu_h, \mathbf{v}_h\rangle_{H^{-1},H_0^1} = 0 \quad \forall \mathbf{v} \in V_h$$

where

- (M_h, V_h) approaches $(M, V) := (L^2(\Omega), H_0^1(\Omega, \mathbb{R}^d))$
- T_h approaches T

In what sense?

Questions

It is a null space determination problem.

- Continuous case: $T\mu = \mathbf{0}$ with $\|\mu\| = 1$
  Well posed problem ?

Questions

It is a null space determination problem.

- Continuous case: $T\mu = \mathbf{0}$ with $\|\mu\| = 1$
  Well posed problem ?
- Discrete case : $T_h\mu_h = \mathbf{0}$ with $\|\mu_h\| = 1$.
  Null space of T_h ?

Questions

It is a null space determination problem.

- Continuous case: $T\mu = \mathbf{0}$ with $\|\mu\| = 1$
  Well posed problem ?

- Discrete case : $T_h\mu_h = \mathbf{0}$ with $\|\mu_h\| = 1$.
  Null space of T_h ?

  what conditions on M_h, V_h and T_h ?

Questions

It is a null space determination problem.

- Continuous case: $T\mu = \mathbf{0}$ with $\|\mu\| = 1$
  Well posed problem ?

- Discrete case : $T_h\mu_h = \mathbf{0}$ with $\|\mu_h\| = 1$.
  Null space of T_h ?

  what conditions on M_h, V_h and T_h ?

- Is the solution μ_h close to μ in $L^2(\Omega)$?

Stability in continuous case

Theoretical study of $T\mu := -\nabla \cdot (\mu S)$, with Ammari, Bretin and Millien (2020):

Theorem

If $S \in W^{1,p}$ $p > d$ and $|\det S(x)| \geq c > 0$ a.e, we have

- $dim N(T) \leq 1$
- *T has closed range.*

- At worst T is a "gradient type" operator

Stability in continuous case

Theoretical study of $T\mu := -\nabla \cdot (\mu S)$, with Ammari, Bretin and Millien (2020):

Theorem

If $S \in W^{1,p}$ $p > d$ and $|\det S(x)| \geq c > 0$ a.e, we have

- $dim N(T) \leq 1$
- *T has closed range.*

- At worst T is a "gradient type" operator
- works for S "piecewise" $W^{1,p}$

Stability in continuous case

Theoretical study of $T\mu := -\nabla \cdot (\mu S)$, with Ammari, Bretin and Millien (2020):

Theorem

If $S \in W^{1,p}$ $p > d$ and $|\det S(x)| \geq c > 0$ a.e, we have

- *$dim N(T) \leq 1$*
- *T has closed range.*

- At worst T is a "gradient type" operator
- works for S "piecewise" $W^{1,p}$
- minimal assumption on S to have closed range property is an open question (as far as we know)

First numerical experiments

Choice of finite element spaces : $(\mathbb{P}^0, \mathbb{P}^1)$ to approximate (M^h, V^h)

Figure: Shear modulus μ and simulated displacement fields $\mathbf{u}$

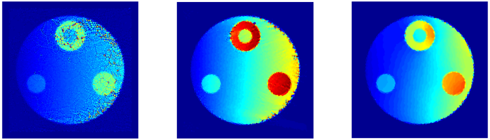


Figure: Shear modulus reconstruction μ^h using TV regularization

Stability problem : how to choose better finite element spaces?

Approximation of the spaces

Let M be a Hilbert and $M_h \subset M$ a sub-Hilbert space and $\pi_h : M \to M_h$ the orthogonal projection.

Definition

The sequence $(M_h)_{h>0}$ approaches M if for any $\mu \in M$,

$$\lim_{h \to 0} \|\pi_h \mu - \mu\|_M = 0.$$

For any non zero $\mu \in M$, we define its relative error of interpolation onto M_h by

$$\varepsilon_h^{\text{int}}(\mu) := \frac{\|\pi_h \mu - \mu\|_M}{\|\mu\|_M}.$$

Approximation of the operator

The operator $T : L^2 \to H^{-1}$ given by

$$\langle T\mu, \mathbf{v}\rangle_{H^{-1},H_0^1} := \int_\Omega \mu S : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^{d\times d})$$

is approached by $T_h : M_h \to V_h'$

$$\langle T_h\mu, \mathbf{v}\rangle_{V_h',V_h} := \int_\Omega \mu S_h : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in V_h.$$

Approximation of the operator

The operator $T : L^2 \to H^{-1}$ given by

$$\langle T\mu, \mathbf{v}\rangle_{H^{-1},H_0^1} := \int_\Omega \mu S : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^{d\times d})$$

is approached by $T_h : M_h \to V_h'$

$$\langle T_h\mu, \mathbf{v}\rangle_{V_h',V_h} := \int_\Omega \mu S_h : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in V_h.$$

Hence

$$\begin{aligned}\langle (T_h - T)\mu, \mathbf{v}\rangle_{V_h',V_h} &= \int_\Omega \mu(S_h - S) : \nabla \mathbf{v} \\ &\le \|\mu\|_{L^\infty} \|S_h - S\|_{L^2(\Omega)} \|\mathbf{v}\|_{H_0^1}\end{aligned}$$

Approximation of the operator

The operator $T : L^2 \to H^{-1}$ given by

$$\langle T\mu, \mathbf{v}\rangle_{H^{-1},H_0^1} := \int_\Omega \mu S : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^{d\times d})$$

is approached by $T_h : M_h \to V_h'$

$$\langle T_h\mu, \mathbf{v}\rangle_{V_h',V_h} := \int_\Omega \mu S_h : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in V_h.$$

Hence

$$\begin{aligned}\langle (T_h - T)\mu, \mathbf{v}\rangle_{V_h',V_h} &= \int_\Omega \mu(S_h - S) : \nabla \mathbf{v} \\ &\leq \|\mu\|_{L^\infty} \|S_h - S\|_{L^2(\Omega)} \|\mathbf{v}\|_{H_0^1}\end{aligned}$$

The error $T_h - T$ is small for the $\mathcal{L}(L^\infty, V_h')$ topology weaker than the $\mathcal{L}(L^2, V_h'))$ topology!

Approximation of the operator

Definition

The interpolation error $\varepsilon_h^{\mathrm{op}}$ between T and T_h is defined by

$$\varepsilon_h^{\mathrm{op}} := \| T_h - T \|_{L^\infty, V_h'} := \sup_{\mu \in M_h} \sup_{\mathbf{v} \in V_h} \frac{\langle (T_h - T)\mu, \mathbf{v} \rangle_{V_h', V_h}}{\|\mu\|_{L^\infty} \|\mathbf{v}\|_{H_0^1}}.$$

Approximation of the operator

Definition

The interpolation error $\varepsilon_h^{\text{op}}$ between T and T_h is defined by

$$\varepsilon_h^{\text{op}} := \| T_h - T \|_{L^\infty, V_h'} := \sup_{\mu \in M_h} \sup_{\mathbf{v} \in V_h} \frac{\langle (T_h - T)\mu, \mathbf{v} \rangle_{V_h', V_h}}{\|\mu\|_{L^\infty} \|\mathbf{v}\|_{H_0^1}}.$$

- This error contains both the data noise and the interpolation error over (M_h, V_h).

Approximation of the operator

Definition

The interpolation error $\varepsilon_h^{\text{op}}$ between T and T_h is defined by

$$\varepsilon_h^{\text{op}} := \|T_h - T\|_{L^\infty, V_h'} := \sup_{\mu \in M_h} \sup_{\mathbf{v} \in V_h} \frac{\langle (T_h - T)\mu, \mathbf{v} \rangle_{V_h', V_h}}{\|\mu\|_{L^\infty} \|\mathbf{v}\|_{H_0^1}}.$$

- This error contains both the data noise and the interpolation error over (M_h, V_h).
- This particular norm does not allow us to use directly the sensitivity analysis and discretization analysis for the Moore-Penrose generalized inverse of T when T is a closed range operator

Generalized inf-sup constant

$M, \; V$ two Hilbert spaces and $T \in \mathcal{L}(M, V')$,

Definition (classic constants)

$$\alpha(T) := \inf_{\mu \in M} \frac{\|T\mu\|_{V'}}{\|\mu\|_M} \quad \text{and} \quad \rho(T) := \sup_{\mu \in M} \frac{\|T\mu\|_{V'}}{\|\mu\|_M}.$$

Generalized inf-sup constant

M, V two Hilbert spaces and $T \in \mathcal{L}(M, V')$,

Definition (classic constants)

$$\alpha(T) := \inf_{\mu \in M} \frac{\|T\mu\|_{V'}}{\|\mu\|_M} \quad \text{and} \quad \rho(T) := \sup_{\mu \in M} \frac{\|T\mu\|_{V'}}{\|\mu\|_M}.$$

Definition (Generalized inf-sup constant)

The generalized *inf-sup* constant $\beta(T)$ is built as follows:

$$\beta(T) := \sup_{\substack{e \in M \\ \|e\|_M = 1}} \beta_e(T) \text{ where } \beta_e(T) := \inf_{\substack{\mu \in M \\ \mu \perp e}} \frac{\|T\mu\|_{V'}}{\|\mu\|_M}$$

Discrete inf-sup constant

Definition (Discrete inf-sup constant)

$$\beta(T_h) := \inf_{\substack{\mu \in M_h \\ \mu \perp z_h}} \sup_{\mathbf{v} \in V_h} \frac{\langle T_h \mu, \mathbf{v} \rangle_{V_h', V_h}}{\|\mu\|_M \|\mathbf{v}\|_V}.$$

where

$$z_h = \operatorname*{arg\,min}_{z \in M_h} \sup_{\mathbf{v} \in V_h} \frac{\langle T_h \mu, \mathbf{v} \rangle_{V_h', V_h}}{\|z\|_M \|\mathbf{v}\|_V}.$$

Discrete stability estimate

Theorem

Let $z \in L^\infty(\Omega)$ be a solution of $T\,z = \mathbf{0}$ with $\|z\|_M = 1$. Fix $r \geq \|z\|_\infty$ and consider $z_h \in M_h$ a solution of

$$\|T_h z_h\|_{V_h'} = \alpha(T_h) \quad \text{with} \quad \|z_h\|_M = 1 \quad \text{and} \quad \langle z_h, z\rangle_M \geq 0. \quad (1)$$

If $\beta(T_h) > 0$ we have

$$\|z_h - \pi_h z\|_{L^2(\Omega)} \leq \frac{C}{\beta(T_h)}\big(\, r\, \|T_h - T\|_{L^\infty, V_h'} + \|\pi_h z - z\|_M\,\big).$$

Moreover, if $\beta(T_h) \geq \beta^* > 0$ and if $\varepsilon_h^{op} \to 0$, then $z_h \to z$.

Algorithm

Write T_h as a matrix $\mathcal{T}$ in the basis of the chosen M_h and V_h. Define the matrix

$$\mathcal{M} := \mathcal{B}_V^{-1} \mathcal{T} \mathcal{B}_M^{-1}$$

where $\mathcal{B}_M$ and $\mathcal{B}_V$ are the basis matrix of M_h and V_h. Then

Algorithm

Write T_h as a matrix $\mathcal{T}$ in the basis of the chosen M_h and V_h. Define the matrix

$$\mathcal{M} := \mathcal{B}_V^{-1} \mathcal{T} \mathcal{B}_M^{-1}$$

where $\mathcal{B}_M$ and $\mathcal{B}_V$ are the basis matrix of M_h and V_h. Then

- $\alpha(T_h)$ is the smallest singular value of $\mathcal{M}$

Algorithm

Write T_h as a matrix $\mathcal{T}$ in the basis of the chosen M_h and V_h. Define the matrix

$$\mathcal{M} := \mathcal{B}_V^{-1}\mathcal{T}\mathcal{B}_M^{-1}$$

where $\mathcal{B}_M$ and $\mathcal{B}_V$ are the basis matrix of M_h and V_h. Then

- $\alpha(T_h)$ is the smallest singular value of $\mathcal{M}$
- $\beta(T_h)$ is the second smallest singular value of $\mathcal{M}$

Algorithm

Write T_h as a matrix $\mathcal{T}$ in the basis of the chosen M_h and V_h. Define the matrix

$$\mathcal{M} := \mathcal{B}_V^{-1} \mathcal{T} \mathcal{B}_M^{-1}$$

where $\mathcal{B}_M$ and $\mathcal{B}_V$ are the basis matrix of M_h and V_h. Then

- $\alpha(T_h)$ is the smallest singular value of $\mathcal{M}$
- $\beta(T_h)$ is the second smallest singular value of $\mathcal{M}$
- μ is the first singular vector of $\mathcal{M}$.

$\Rightarrow$ Main algorithm: partial svd of $\mathcal{M}$.

Quasistatic elastography

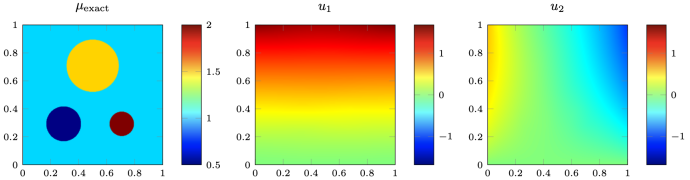


FIG. 5. *First line, from left to right: The exact map μ_{exact}, the two components of the data field $\boldsymbol{u} = (u_1, u_2)$ computed via (5.6), the only data used to inverse the problem.*

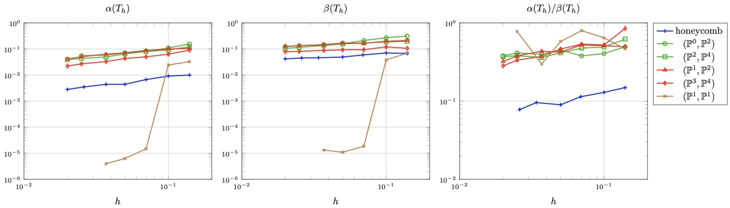


FIG. 6. *Behavior of the contants $\alpha(T_h)$, $\beta(T_h)$ and the ratio $\alpha(T_h)/\beta(T_h)$ for the inverse static elastography problem in the unit square $\Omega := (0,1)^2$, for various choices of pair of discretization spaces.*

First singular vector various pairs of spaces

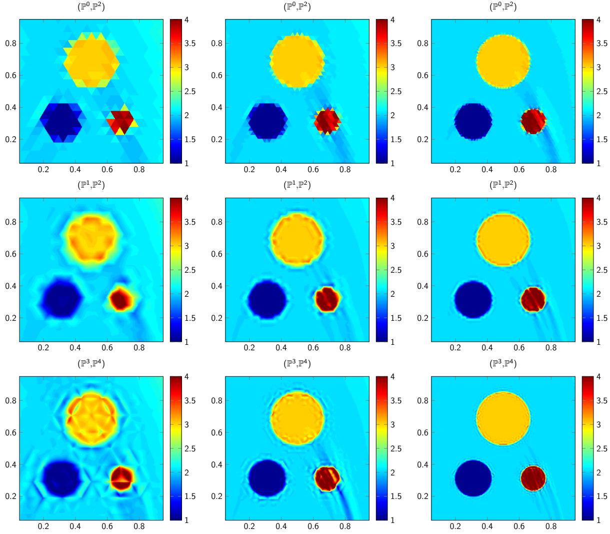


Figure: Reconstruction of the shear modulus map μ using various pairs of finite element spaces in the subdomain of interest $(0.1, 0.9)^2$.

honeycomb finite element

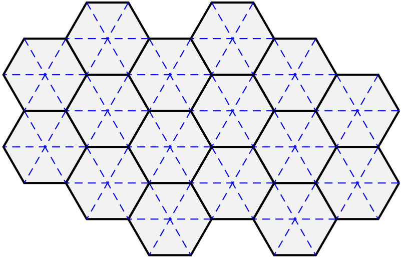


Figure: Honeycomb space discretization. In plain black, the hexagonal subdivision and in dashed blue, the triangular subdivision.

$$M_h := \mathbb{P}^0\left(\Omega_h^{\text{hex}}\right) = \left\{\mu \in L^2(\Omega_h) \mid \forall j \ \mu|_{\Omega_{h,j}^{\text{hex}}} \text{ is constant}\right\}.$$

honeycomb finite element

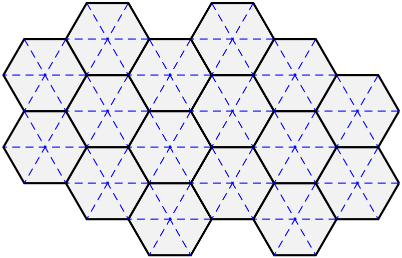


Figure: Honeycomb space discretization. In plain black, the hexagonal subdivision and in dashed blue, the triangular subdivision.

$$M_h := \mathbb{P}^0\left(\Omega_h^{\text{hex}}\right) = \left\{\mu \in L^2(\Omega_h) \mid \forall j\ \mu|_{\Omega_{h,j}^{\text{hex}}} \text{ is constant}\right\}.$$

$$V_h := \mathbb{P}_0^1\left(\Omega_h^{\text{tri}}, \mathbb{R}^2\right) = \left\{\mathbf{v} \in H_0^1(\Omega_h, \mathbb{R}^d) \mid \forall k\ \mathbf{v}|_{\Omega_{h,k}^{\text{tri}}} \text{ is linear}\right\}.$$

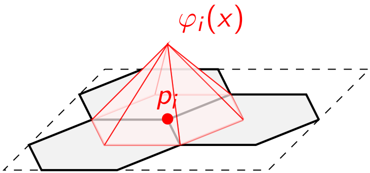


Figure: Support and graph of basis test function φ_i.

Why does it work ?

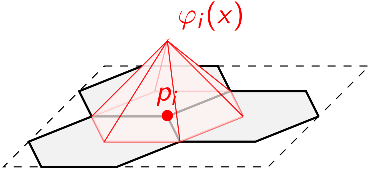


Figure: Support and graph of basis test function φ_i.

Why does it work ?

- Case $T = \nabla$: We show that this pair satisfies the so called *inf-sup* condition.

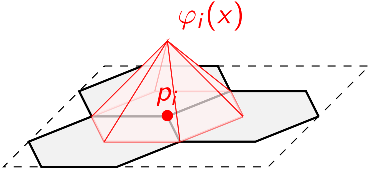


Figure: Support and graph of basis test function φ_i.

Why does it work ?

- Case $T = \nabla$: We show that this pair satisfies the so called *inf-sup* condition.
- General case: We show that for each internal node, we have a system of 2 independent equations for 3 values of the parameters.

One value is given $\Rightarrow$ all the other are fixed. $\Rightarrow$ null-space is at most of dimension 1 $\Rightarrow \beta(T_h) > 0$

Reconstruction for the honeycomb

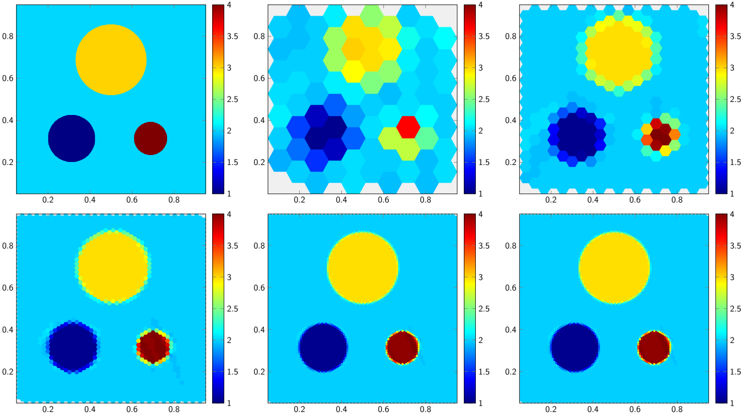


Figure: Reconstruction of the shear modulus map μ using the honeycomb pair.

Quasi-static elastography

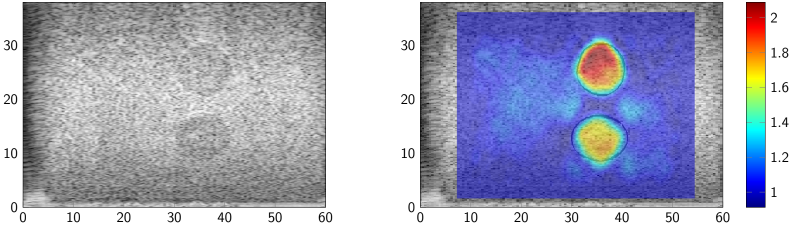


Figure: Shear modulus image of phantom from quasi-static data (data from E. Brusseau and L. Pretrusca - CREATIS/INSA)

In vivo quasistatic elastography

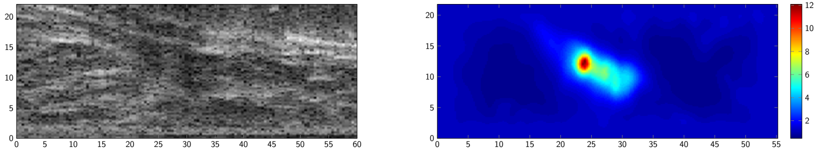


Figure: Reconstruction of the shear modulus of *in-vivo* malignant breast tumor from quasi-static elastography (data from E. Brusseau - INSA/CREATIS) $h = 0.7$ mm.

Open questions:

- Minimal conditions on S such that $\mu \mapsto \nabla \cdot (\mu S)$ has closed range.
- Behavior of $\beta(T_h)$ when $h \to 0$.
- Som sort of optimality of the honeycomb pair of spaces for this class of problems.

Thank you for your attention