5) Calcul par changement de variables

Proposition. Si f est une fonction continue et u une fonction dérivable alors :

$\int f(u(x))u'(x)dx = \int f(u)du = F(u(x)) + C$ où $F' = f$.

démonstration. $(F \circ u)' = F' \circ u.u' = f \circ u.u'$.

Exemples. $\int \frac{u'}{u} = \ln(|u|) + C$.

$\int \frac{x^3 dx}{2 + x^4} = \frac{1}{4}\ln(2 + x^4) + C$

$\int x^2 \underbrace{\sqrt{1 + x^3}}_{u = 1 + x^3} dx = \frac{1}{3}\int \sqrt{u}u' = \frac{1}{3} \times \frac{2}{3}(1 + x^3)^{\frac{3}{2}} + C$

6) $\int R(e^x)dx$ où R est une fraction rationnelle.

On pose $u = e^x$ et donc $du = e^x dx = u dx \Rightarrow dx = \frac{du}{u}$.

$(u = h(x) \Rightarrow du = h'(x)dx)$

Exemple. $\int \frac{dx}{2 + e^x} = \int \frac{\frac{du}{u}}{2 + u} = \int \frac{du}{u(2 + u)} = \frac{1}{2}\left(\int \frac{du}{u} - \int \frac{du}{2 + u}\right) = \frac{1}{2}\ln(|u|) - \frac{1}{2}\ln(|2 + u|) + C$

$= \frac{1}{2}\ln(e^x) - \frac{1}{2}\ln(2 + e^x) + C = \frac{x}{2} - \frac{1}{2}\ln(2 + e^x) + C$.

on pose $u = e^x$

Or $\frac{1}{u(2 + u)} = \frac{\frac{1}{2}}{u} + \frac{\frac{-1}{2}}{2 + u}$.

7) $\int \cos^m(x)\sin^n(x)dx$ où $m, n \epsilon \mathbb{N}$.

a) si m ou n impair

si m impair, on pose $u = \sin(x)$

si n impair, on pose $u = \cos(x)$.

Exemple. $\int \sin^5(x)d\ x = \int \sin^4(x)\underbrace{\sin(x)dx}_{-du} = -\int (1 - u^2)^2 du = -\int du + 2\int u^2 du - \int u^4 du = -\frac{u^2}{2} + 2\frac{u^3}{3} - \frac{u^5}{5} + C = -\frac{\cos^2(x)}{2} + 2\frac{\cos^3(x)}{3} - \frac{\cos^5(x)}{5} + C$

on pose $u = \cos(x) \Rightarrow du = -\sin(x)dx$, $\sin^4(x) = (\sin^2(x))^2 = (1 - \cos^2(x))^2$

b) sinon (si m, n pairs)

On linéarise ...(c-à-d : on exprime tout comme une combinaison linéaire de $\cos(kx)$, $k\epsilon\mathbb{N}$).

Exemple. $\int \cos^4(x)dx$

$\cos^4(x) = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^4 = \frac{1}{16}(e^{4ix} + 4e^{3ix}e^{-ix} + 6e^{2ix}e^{-2ix} + 4e^{-ix}e^{3ix} + e^{-4ix})$

Or $e^{4ix} + e^{-4ix} = 2\cos(4x)\ldots$

Donc $\cos^4(x) = \frac{1}{16}(2\cos(4x) + 8\cos(2x) + 6)$

D'où : $\int \cos^4(x)dx \underset{u=4x \Rightarrow du=4dx}{=} \frac{1}{8}\int \cos(4x)dx + \frac{1}{2}\int \cos(2x)dx + \frac{3}{8}\int dx = \frac{1}{32}\sin(4x) + \frac{1}{4}\sin(2x) + \frac{3}{8}x + C$.

Exercice. Calculer $\int \sin^4$ x d x

indication : $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

8) $\int R(\sin(x), \cos(x))dx$ où $R(X, Y)$ est une fraction rationnelle

Règles de Bioche.

si $R(\sin(x), \cos(x))dx$ est invariante par $x \leftrightarrow -x$	on pose $u = \cos(x)(\Rightarrow du = -\sin(x)dx)$
si $R(\sin(x), \cos(x))dx$ est invariante par $x \leftrightarrow \pi - x$	on pose $u = \sin(x)(\Rightarrow du = \cos(x)dx)$
si $R(\sin(x), \cos(x))dx$ est invariante par $x \leftrightarrow \pi + x$	on pose $u = \tan(x)(\Rightarrow du = (1 + u^2)dx)$
sinon	on pose $u = \tan\left(\frac{x}{2}\right)$

Remarques.

– on a : $\sin(-x) = -\sin(x), \cos(-x) = \cos(x), \sin(\pi - x) = \sin(x), \cos(\pi - x) = -\cos(x), \sin(\pi + x) = -\sin(x), \cos(\pi + x) = -\cos(x)$

– rappel si $t = \tan\left(\frac{x}{2}\right)$, alors $\cos(x) = \frac{1 - t^2}{1 + t^2}$ et $\sin(x) = \frac{2t}{1 + t^2}$

Exemples.

a) $\int \frac{\sin^3(x)dx}{1 + \cos^2(x)} = \int \frac{\sin^3(-x)d(-x)}{1 + \cos^2(-x)}$ donc on est invité à poser $u = \cos(x)$

$du = -\sin(x)dx$

$\int \frac{\sin^3(x)dx}{1+\cos^2(x)} = \int -\frac{(1-u^2)du}{1+u^2} = \int du - 2\int \frac{du}{1+u^2} = u - 2\arctan(u) + C$

$\boxed{=\cos(x) - 2\arctan(\cos(x)) + C}$.

$-\frac{1-u^2}{1+u^2} = \frac{u^2-1}{1+u^2} = \frac{u^2+1-2}{1+u^2} = 1 - \frac{2}{1+u^2}$

b) $\int \frac{\cos(x)dx}{\sin^2(x)+2\tan^2(x)} = \int \frac{\cos(\pi-x)d(\pi-x)}{\sin^2(\pi-x)+2\tan^2(\pi-x)}$

Donc on est invité à poser $u = \sin(x)$ et donc $du = \cos(x)dx$.

$\int \frac{\cos(x)dx}{\sin^2(x)+2\tan^2(x)} = \int \frac{du}{u^2+2\frac{u^2}{1-u^2}} = \int \frac{(1-u^2)du}{u^2-u^4+2u^2} = \int \frac{(1-u^2)du}{u^2(3-u^2)}$

Or $\frac{(1-u^2)}{u^2(3-u^2)} = \frac{\frac{1}{3}}{u^2} + \frac{0}{u} + \frac{\frac{-1}{3\sqrt{3}}}{\sqrt{3}-u} + \frac{\frac{-1}{3\sqrt{3}}}{\sqrt{3}+u}$

$\Rightarrow \int \frac{\cos(x)dx}{\sin^2(x)+2\tan^2(x)} = -\frac{1}{3u} - \frac{1}{3\sqrt{3}}(-\ln(|\sqrt{3}-u|) + \ln(|\sqrt{3}+u|)) + C$

$= -\frac{1}{3\sin(x)} - \frac{1}{3\sqrt{3}}\ln\left(\frac{\sqrt{3}+\sin(x)}{\sqrt{3}-\sin(x)}\right) + C$

c) $\int \frac{\sin(x)dx}{\cos^3(x)+\sin^3(x)} = \int \frac{\sin(\pi+x)d(\pi+x)}{\cos^3(\pi+x)+\sin^3(\pi+x)}$

On est donc invité à poser $u = \tan x$.

$\Rightarrow du = (1+\tan^2(x))dx = (1+u^2)dx$

$\int \frac{\sin(x)dx}{\cos^3(x)+\sin^3(x)} = \int \frac{\frac{du}{1+u^2}}{\frac{\cos^2(x)}{\tan(x)}+\sin^2(x)} = \int \frac{\frac{du}{1+u^2}}{\frac{1}{u(1+u^2)}+\frac{u^2}{1+u^2}} = \int \frac{du}{\frac{1}{u}+u^2} = \int \frac{udu}{u^3+1}$

or $\cos^2(x) = \frac{1}{1+\tan^2(x)}$ et $\sin^2(x) = \frac{\tan^2(x)}{1+\tan^2(x)}$

$\frac{u}{u^3+1} = \frac{u}{(u+1)(u^2-u+1)} = \frac{\frac{-1}{3}}{u+1} + \frac{au+b}{u^2-u+1} = \frac{\frac{-1}{3}}{u+1} + \frac{1}{3}\frac{u+1}{u^2-u+1}$

$\frac{u}{u^3+1} + \frac{\frac{1}{3}}{u+1} = \frac{u+\frac{1}{3}(u^2-u+1)}{u^3+1} = \frac{\frac{1}{3}(u+1)^2}{(u+1)(u^2-u+1)} = \frac{\frac{1}{3}(u+1)}{u^2-u+1}$

$\int \frac{\sin(x)dx}{\cos^3(x)+\sin^3(x)} = -\frac{1}{3}\int \frac{du}{u+1} + \frac{1}{3}\int \frac{(u+1)du}{u^2-u+1}$

$$=-\frac{1}{3}\int\frac{du}{u+1}+\frac{1}{3}\underbrace{\int\frac{\left(\frac{1}{2}(2u-1)+\frac{3}{2}\right)du}{u^2-u+1}}_{\frac{1}{2}\int\frac{2u-1}{u^2-u+1}du+\frac{3}{2}\int\frac{du}{u^2-u+1}}$$

$$=-\frac{1}{3}\ln(|u+1|)+\frac{1}{3}\times\frac{1}{2}\ln(u^2-u+1)+\frac{1}{3}\times\frac{3}{2}\int\frac{du}{u^2-u+1} \qquad \text{or, } u^2-u+1=(u-\tfrac{1}{2})^2+\tfrac{3}{4}=(u-\tfrac{1}{2})^2+\left(\tfrac{\sqrt{3}}{2}\right)^2$$

$$=-\frac{1}{3}\ln(|u+1|)+\frac{1}{6}\ln(u^2-u+1)+\frac{1}{2}\times\frac{2}{\sqrt{3}}\arctan\left(\frac{u-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C$$

$$=-\frac{1}{3}\ln(|\tan(x)+1|)+\frac{1}{6}\ln(\tan^2(x)-\tan(x)+1)+\frac{1}{2}\times\frac{2}{\sqrt{3}}\arctan\left(\frac{2\tan(x)-1}{\sqrt{3}}\right)+C$$