**4.1**

$$210 = 48 \times 4 + 18$$
$$48 = 18 \times 2 + 12$$
$$18 = 12 \times 1 + \mathbf{6}$$
$$12 = 6 \times 2 + 0$$

Donc $pgcd(210, 48) = 6$.
De plus, on a :

$$\begin{aligned} 6 &= 18 - 12 \\ &= 18 - (48 - 18.2) = 18.3 - 48 \\ (210 - 48.4).3 - 48 &= 210.3 - 48.13 \ . \end{aligned}$$

**4.2**

$$35 = 8.4 + 3$$
$$8 = 3.2 + 2$$
$$3 = 2 + 1$$

Donc
$$1 = 3 - 2 = 3 - (8 - 3.2) = 3.3 - 8 = (35 - 8.4).3 - 8 = 35.3 - 8.13 \ .$$
Donc, $13.8 = 1 \bmod 35$.

**4.3**

$$237 = 81.2 + 75$$
$$81 = 75 + 6$$
$$75 = 6.12 + \mathbf{3}$$
$$6 = 3.2 + 0$$

donc $pgcd(237, 81) = 3$.
On a aussi une relation de Bézout :

$$\begin{aligned} 3 &= 75 - 6.12 = 75 - (81 - 75).12 = 75.13 - 81.12 = (237 - 81.2).13 - 81.12 \\ &= 237.13 - 81.38 \end{aligned}$$

En particulier, $237\mathbb{Z} + 81\mathbb{Z} = 3\mathbb{Z} \not\ni 1$. Donc

$$237x + 81y = 1$$

n'a pas de solution $x, y \in \mathbb{Z}$ !

On a $3|9$ donc on peut trouver des solutions de

$$237x + 81y = 9 \ .$$

$$\begin{aligned} 237x + 81y = 9 &\Leftrightarrow 79x + 27y = 3 \\ &\Leftrightarrow \begin{cases} 79x + 27y &= 3 \\ 79x &= 3 \bmod 27 \\ 27y &= 3 \bmod 79 \end{cases} \end{aligned}$$

or, $1 = 79.13 - 27.38$ donc $79.13 = 1 \bmod 27$ et $27.(-38) = 1 \bmod 79$.
Or :

$$79x = 3 \bmod 27 \Leftrightarrow x = 3.13 \bmod 27 = 39 \bmod 27 = 12 \bmod 27$$

et :

$$27y = 3 \bmod 79 \Leftrightarrow y = 3.(-38) \bmod 79 = -35 \bmod 79 \ .$$

Donc

$$237x + 81y = 9 \Leftrightarrow \begin{cases} 79x + 27y &= 3 \\ x &= 12 \bmod 27 \\ y &= -35 \bmod 79 \end{cases} .$$

Or si $k, l \in \mathbb{Z}$,

$$79(12 + 27k) + 27(-35 + 79l) = 3 \Leftrightarrow 3 + 27.79(k + l) = 3 \Leftrightarrow 27.79(k + l) = 0 \Leftrightarrow l = -k \ .$$

Donc

$$237x + 81y = 9 \Leftrightarrow 79x + 27y = 3 \Leftrightarrow \exists \, k \in \mathbb{Z}, \, x = 12 + 27k \text{ et } y = -35 - 79k \ .$$

**4.4**

(i)

$$x^2 = x \bmod 6 \Leftrightarrow x(x-1) = 0 \bmod 6 \Leftrightarrow x(x-1) = 0 \bmod 2 \text{ et } x(x-1) = 0 \bmod 3 \Leftrightarrow x(x-1) = 0 \bmod 3$$

$$\Leftrightarrow x = 0 \bmod 3 \text{ ou } x = 1 \bmod 3 \ .$$

(ii)

$$12x + 14 = 0 \bmod 8 \Leftrightarrow 6x + 7 = 0 \bmod 4 \Leftrightarrow 6x = 1 \bmod 4$$

**impossible** car $6x$ est pair et les éléments de $1 + 4\mathbb{Z}$ sont impairs !

(iii)

$$12x + 14 = 0 \bmod 37 \Leftrightarrow (-3).12x = 42 \bmod 37$$

car 12 est inversible d'inverse $(-3)$ modulo 37.
Donc

$$12x + 14 = 0 \bmod 37 \Leftrightarrow x = 5 \bmod 37 \ .$$

**4.5**

(i)

$$\left\{\begin{array}{lll} x & = & 13 \bmod 19 \\ x & = & 6 \bmod 12 \end{array}\right. \Leftrightarrow \left\{\begin{array}{lll} x & = & 13+19k \\ 13+19k & = & 6 \bmod 12 \end{array}\right.$$

Or,

$$\begin{aligned} 13+19k = 6 \bmod 12 &\Leftrightarrow 19k = -7 \bmod 12 \\ &\Leftrightarrow -5k = 5 \bmod 12 \ . \end{aligned}$$

Or on a $-5.7 = 1 \bmod 12$. Donc :

$$-5k = 5 \bmod 12 \Leftrightarrow k = 7 \bmod 12 \ .$$

Donc

$$\begin{aligned} \left\{\begin{array}{lll} x & = & 13 \bmod 19 \\ x & = & 6 \bmod 12 \end{array}\right. &\Leftrightarrow \exists\, n \in \mathbb{Z},\ x = 13 + 19(6+12n) = 13 + 114 + 228n \\ &\Leftrightarrow \exists\, n \in \mathbb{Z},\ x = 127 + 228n \ . \end{aligned}$$