Toroidal deformations and the homotopy type of Berkovich spaces

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Contents

Berkovich spaces

Non-Archimedean fields

- A non-Archimedean field is a field k endowed with an absolute value $|\cdot| : k^\times \to \mathbb{R}$ satisfying the ultrametric inequality:

$$|a+b| \leqslant \max\{|a|,|b|\}.$$

  We will always assume that $(k,|\cdot|)$ is complete.
- Morphisms are isometric.
- The closed unit ball $k^\circ = \{a \in k, |a| \leqslant 1\}$ is a local ring with fraction field k and residue field $\widetilde{k}$.

Examples:

(i) p-adic numbers: $k = \mathbb{Q}_p$, $k^\circ = \mathbb{Z}_p$ and $\widetilde{k} = \mathbb{F}_p$.

(ii) Laurent series: if F is a field, $k = \mathrm{F}((t))$, $k^\circ = \mathrm{F}[[t]]$ and $\widetilde{k} = \mathrm{F}$. For $\rho \in (0,1)$, set $|f| = \rho^{-\mathrm{ord}_0(f)}$.

(iii) Any field k, with the trivial absolute value: $|k^\times| = \{1\}$. Then $k = k^\circ = \widetilde{k}$.

Specific features

- Any point of a disc is a center. It follows that two discs are either disjoint or nested, and that closed discs with positive radius are open. Therefore, the metric topology on k is totally disconnected.
- Any non-Archimedean field k has (many) non-trivial non-Archimedean extensions.
  Example: the Gauß norm on $k[t]$, defined by

$$\left|\sum_n a_n t^n\right|_1 = \max_n |a_n|,$$

is multiplicative ($|fg|_1 = |f|_1 \cdot |g|_1$), hence induces an absolute value on $k(t)$ extending $|\cdot|$. The completion K of $(k(t), |.|_1)$ is a non-Archimedean extension of k with $|\mathrm{K}^\times| = |k^\times|$ and $\widetilde{\mathrm{K}} = \widetilde{k}(t)$.
Comparison: any Archimedean extension of $\mathbb{C}$ is trivial.

Non-Archimedean analytic geometry

- Since the topology is totally discontinuous, analycity is *not* a local property on k^n: there are too many locally analytic functions on Ω.
- J. TATE (60') introduced the notion of a rigid analytic function by restricting the class of open coverings used to check local analycity.
- V. BERKOVICH (80') had the idea to add (many) new points to k^n in order to obtain a better topological space.
- In BERKOVICH's approach, the underlying topological space of a k-analytic space X is always locally arcwise connected and locally compact. It carries a sheaf of Fréchet k-algebras satisfying some conditions.

Analytification of an algebraic variety

There exists an analytification functor $X \rightsquigarrow X^{an}$ from the category of k-schemes of finite type to the category of k-analytic spaces.

- A point of X^{an} can be described as a pair $x = (\xi, |\cdot|(x))$, where
  - ξ is a point of X;
  - $|\cdot|(x)$ is an extension of the absolute value of k to the residue field $\kappa(\xi)$.
- The completion of $(\kappa(\xi), |\cdot|(x))$ is denoted by $\mathscr{H}(x)$; this is a non-Archimedean extension of k.
- There is a unique point in X^{an} corresponding to a closed point ξ of X, because there is a unique extension of the absolute value to $\kappa(\xi)$ (since $[\kappa(\xi) : k] < \infty$ and k is complete).
- We endow X^{an} with the coarsest topology such that, for any affine open subscheme U of X and any $f \in \mathcal{O}_X(U)$, the subset $U^{an} \subset X^{an}$ is open and the function $U^{an} \longrightarrow \mathbb{R}$, $x \mapsto |f|(x)$ is continuous.

Analytification of an algebraic variety

- If $X = \mathrm{Spec}(A)$ is affine, then X^{an} can equivalently be described as the set of multiplicative k-seminorms on A.
- The sheaf of analytic functions on X^{an} can be thought of as the "completion" of the sheaf $\mathscr{O}_X$ with respect to some seminorms.
- The topology induced by X^{an} on the set of (rational) closed points of X is the metric topology. If the absolute value is non-trivial, these points are dense in X^{an}.
- X^{an} is Hausdorff (resp. compact; resp. connected) iff X is separated (resp. proper; resp. connected).
- The topological dimension of X^{an} is the dimension of X.

Example: the affine line

As a set, $\mathbb{A}_k^{1,\mathrm{an}}$ consists of all multiplicative k-seminorms on $k[\mathrm{t}]$.

- Any $a \in k = \mathbb{A}^1(k)$ defines a point in $\mathbb{A}_k^{1,\mathrm{an}}$, which is the evaluation at a, i.e. $f \mapsto |f(a)|$.
- For any $a \in k$ and any $r \in \mathbb{R}_{\geqslant 0}$, the map

$$\eta_{a,r} : k[t] \to \mathbb{R},\ f = \sum_n a_n (t-a)^n \mapsto \max_n |a_n| r^n$$

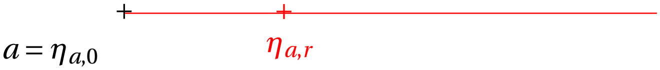
  is a multiplicative k-seminorm, hence a point of $\mathbb{A}_k^{1,\mathrm{an}}$.
- It is an easy exercise to check that

$$\eta_{a,r} = \eta_{b,s} \Longleftrightarrow \begin{cases} r = s \\ |a-b| \leqslant r \end{cases}$$

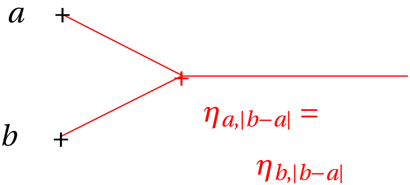
  hence any two points $a, b \in k$ are connected by a path in $\mathbb{A}_k^{1,\mathrm{an}}$.
- If k is algebraically closed and spherically complete, then all the points in $\mathbb{A}_k^{1,\mathrm{an}}$ are of this kind.

Picture: paths

In black (resp. red): points in $\mathbb{A}_k^{1,an}$ over a closed point (resp. the generic point)of $\mathbb{A}_k^1$.

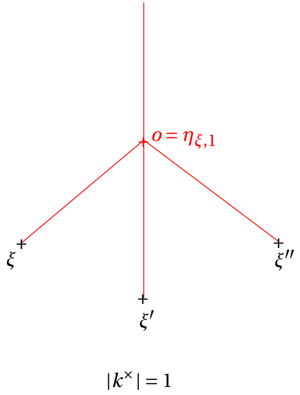


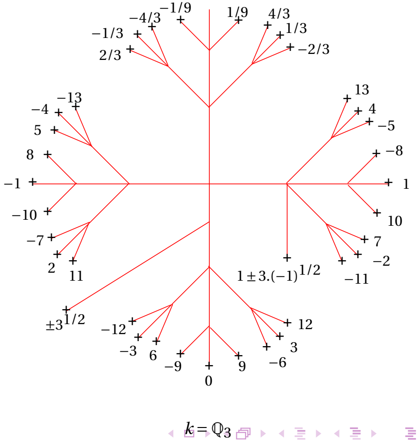
Two points $a, b \in k$ are connected in $\mathbb{A}_k^{1,an}$.



Picture: more paths

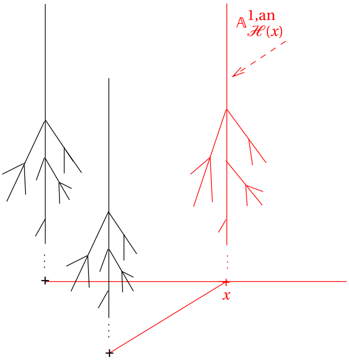
$\mathbb{A}_k^{1,\mathrm{an}}$ looks like a real tree, but equiped with a topology which is much coarser than the usual tree topology.





One last picture

Using a coordinate projection $\mathbb{A}_k^{2,\mathrm{an}} \to \mathbb{A}_k^{1,\mathrm{an}}$, one can try to think of the analytic plane as a bunch of real trees parametrized by a real tree...



The fiber over x is the analytic line over the non-Archimedean field $\mathscr{H}(x)$.

Remark

Even if the valuation of k is trivial, analytic spaces over non-trivially valued fields always spring up in dim $\geqslant 2$.

Hence the trivial valuation is not so trivial!

Homotopy type

BERKOVICH conjectured that any compact k-analytic space is locally contractible and has the homotopy type of a finite polyhedron.

Theorem (BERKOVICH)

(i) *Any smooth analytic space is locally contractible.*

(ii) *If an analytic space X has a poly-stable formal model over k°, then there is a strong deformation retraction of X onto a closed polyhedral subset.*

Recently, E. HRUSHOVSKI and F. LOESER used a model-theoretic analogue of Berkovich geometry to prove:

Theorem (H.-L.)

Let Y be a quasi-projective algebraic variety. The topological space Y^{an} is locally contractible and there is a strong deformation retraction of Y^{an} onto a closed polyhedral subset.

Toric varieties

Analytification of a torus

Let T denote a k-split torus with character group $\mathrm{M} = \mathrm{Hom}(\mathrm{T}, \mathbb{G}_{m,k})$. Its analytification T^{an} is an analytic group, i.e. a group object in the category of k-analytic spaces.

- We have a natural (multiplicative) tropicalization map
$$\tau : \mathrm{T}^{\mathrm{an}} \longrightarrow \mathrm{M}_{\mathbb{R}}^{\vee} = \mathrm{Hom}_{\mathbb{A}}(\mathrm{M}, \mathbb{R}_{>0}), \ \ x \mapsto (\chi \mapsto |\chi|(x)).$$
- The fiber $\mathrm{T}^1 = \left\{x \in \mathrm{T}^{\mathrm{an}} \mid \forall \chi \in \mathrm{M}, |\chi|(x) = 1\right\}$ over 1 is the maximal compact analytic subgroup of T^{an}.
- There is a continuous and $\mathrm{T}(k)$-equivariant section j of τ, defined by
$$\left|\sum_{\chi \in \mathrm{M}} a_\chi \chi\right| (j(u)) = \max_\chi |a_\chi| \cdot \langle u, \chi \rangle.$$

Main point

We thus obtain a canonical realization of the cocharacter space $\mathrm{M}_{\mathbb{R}}^{\vee}$ as a closed subset $\mathfrak{S}(\mathrm{T}) = \mathrm{im}(j)$ of T^{an} (skeleton), together with a retraction $r_{\mathrm{T}} = j \circ \tau : \mathrm{T}^{\mathrm{an}} \to \mathfrak{S}(\mathrm{T})$.

Orbits

Suppose that T^1 acts on some k-analytic space X.

- For each point $x \in X$ with completed residue field $\mathscr{H}(x)$, there exists a canonical rational point $\underline{x}$ in the $\mathscr{H}(x)$-analytic space $X\widehat{\otimes}_k\mathscr{H}(x)$ which is mapped to x by the projection $X\widehat{\otimes}_k\mathscr{H}(x) \to X$. The orbit of x is by definition the image of $T^1_{\mathscr{H}(x)} \cdot \underline{x}$ in X.
- For each $\varepsilon \in [0,1]$, the subset

$$T^1(\varepsilon) = \{x \in T^{an} \mid \forall \chi \in M, |\chi - 1| \leqslant \varepsilon\}$$

  is a compact analytic subgroup of T^1. Moreover, each orbit $T^1(\varepsilon)\cdot x$ contains a distinguished point x^1_ε.
- $(X = T^{an})$ Since $T^1(0) = \{1\}, T^1(1) = T^1, x_0 = x$ and $x^1_1 = r_T(x)$, this leads to a strong deformation retraction

$$[0,1] \times T^{an} \longrightarrow T^{an}, \quad (\varepsilon, x) \mapsto x^1_\varepsilon$$

  onto $\mathfrak{S}(T)$.

Analytification of toric varieties

Let X be a toric variety under the torus T, with open orbit X_0. Let $\mathfrak{S}(X)$ denote the set of T^1-orbits in X^{an} (skeleton).

- The natural map $r_X : X^{an} \to \mathfrak{S}(X)$ has a canonical section $(T^1 \cdot x \mapsto x_1^1)$ which identifies $\mathfrak{S}(X)$ with a closed subset of X^{an}.
- The subset $\mathfrak{S}(X_0) = X_0^{an} \cap \mathfrak{S}(X)$ is an affine space with direction $\mathfrak{S}(T)$.
- The skeleton $\mathfrak{S}(X)$ is the closure of $\mathfrak{S}(X_0)$ in X^{an}. The embedding $\mathfrak{S}(X_0) \hookrightarrow \mathfrak{S}(X)$ is the partial compactification of the affine space $\mathfrak{S}(X_0)$ with respect to the fan of X in $\mathfrak{S}(T)$.
- The stratification of X by T-orbits O corresponds to a stratification of $\mathfrak{S}(X)$ by affine spaces $\mathfrak{S}(O)$ (under quotients of $\mathfrak{S}(T)$).
- There is a canonical strong deformation retraction of X^{an} onto $\mathfrak{S}(X)$.

Analytification *vs* tropicalization

Let X be a toric k-variety under the torus T.

- The partially compactified affine space $\mathfrak{S}(X)$ is the standard tropicalization of X. We realized it as a closed subset of X^{an}, and the retraction $r_X : X^{an} \to \mathfrak{S}(X)$ is the tropicalization map.
- For any closed subscheme Y of X, the subset $r_X(Y^{an}) \subset \mathfrak{S}(X)$ is the standard tropicalization of Y with respect to the toric embedding $Y \hookrightarrow X$.

Let Y be a quasi-projective k-variety. We can consider the category of equivariant embeddings of Y in toric varieties. This leads to an inverse system of maps $r_X : Y^{an} \to r_X(Y)$.

Theorem (S. Payne)

The map $\varprojlim r_X$ *induces a homeomorphism*

$$Y^{an} \xrightarrow{\sim} \varprojlim r_X(Y) \ .$$

Toroidal embeddings

Definition

Let X be a normal variety over k.

An open immersion $X_0 \hookrightarrow X$ is a toroidal embedding without self-intersection if each point of X has a neighborhood U equipped with an étale morphism $\pi : U \to Z$ to a toric variety Z such that $U \cap X_0 = \pi^{-1}(Z_0)$. There is a unique stratification on X lifting locally the toric stratifications.

Example

Assume that X is smooth and let D be a strict normal crossing divisor on X (i.e. locally defined by a product of distinct local coordinates). Then the open immersion $X - D \hookrightarrow X$ is a toroidal embedding.

Analytification

Let k be a field endowed with the trivial absolute value. We consider a toroidal embedding $X_0 \hookrightarrow X$ with X irreducible.

- Since X is irreducible, there is a distinguished point $o \in X^{an}$, corresponding to the trivial absolute value on $k(X)$.

Theorem (Berkovich, T.)

(i) *There exists a unique pair $(\mathfrak{S}(X_0, X), r_X)$, consisting of a closed subset $\mathfrak{S}(X_0, X) \subset X^{an}$ and a retraction $r_X : X^{an} \to \mathfrak{S}(X_0, X)$, which lifts the pair $(\mathfrak{S}(Z), r_Z)$ for any étale chart to a toric variety Z.*

(ii) *The open subset $\mathfrak{S}(X_0, X) \cap X_0^{an}$ is naturally a conical polyhedral complex with integral structure and vertex o.*

Example

If X_0 is the complement of a normal crossing divisor, then $\mathfrak{S}(X_0, X)$ is the cone over the incidence complex of D.

Toroidal deformation

Theorem (Berkovich, T.)

The strong deformation retraction of an analytic toric varietiy Z^{an} onto its skeleton $\mathfrak{S}(Z)$ has a canonical extension to toroidal embeddings.

- For any étale chart $U \to Z$ to a toric variety, the action of the formal torus $\widehat{T_1} \simeq \mathrm{Spf}(k[[t_1, \ldots, t_d]])$ lifts canonically to U.
- This induces an action of
$$\widehat{T_1}^{an} = \bigcup_{\varepsilon \in [0,1)} T^1(\varepsilon)$$
on U^{an}.
- Whereas the action of $\widehat{T_1}$ on U depends on the chart, the orbits of $T^1(\varepsilon)$ are well-defined for any $\varepsilon \in [0,1)$. The strong deformation retraction of X^{an} onto $\mathfrak{S}(X_0, X)$ follows.

An application to singularities

Assume now that X_0 is the complement of a strict normal crossing divisor D on a smooth variety X with incidence complex $\Delta(D)$.

The open subspace $X_0^{an} - r_X^{-1}(o)$ has a deformation retraction onto $\mathfrak{S}(X_0, X) - \{o\} \simeq \Delta(D) \times (0,1)$, hence is homotopy equivalent to $\Delta(D)$.

Theorem (D. Stepanov, T.)

Let X be an irreducible algebraic variety over a perfect field k. For any two proper morphisms $f_i : X_i \to X$ such that X_i is regular, $D_i = f_i^{-1}(Y)_{red}$ is a strict normal crossing divisor and f_i is an isomorphism over $X - Y$, the incidence complexes of D_1 and D_2 have the same homotopy type.

Proof. Both spaces $(X_1)_0^{an} - r_{X_1}^{-1}(o_1)$ and $(X_2)_0^{an} - r_{X_2}^{-1}(o_2)$ are homeomorphic to a punctured tubular neighborhood of Y^{an} in X^{an}.

Homotopy type

Problem

As before, let k be a field endowed with the trivial absolute value.

Given an irreducible k-scheme of finite type X, we would like to understand the homotopy type of X^{an}.

Observation: If X is smooth, then X^{an} is contractible.

Indeed, $X \hookrightarrow X$ is a toroidal embedding and $\mathfrak{S}(X, X) = \{o\}$.

Theorem (J.A. de Jong)

There exist a proper closed subset Z of X and a proper morphism $X' \to X$ endowed with an admissible action of a finite group Γ, such that:

(i) X' *is smooth over a finite extension of* k*;*

(ii) $Z' = f^{-1}(Z)_{red}$ *is a strict normal crossing divisor;*

(iii) *the morphism* $(X' - Z')/\Gamma \to X - Z$ *induced by* f *is radicial.*

Question: is it possible to describe the homotopy type of X^{an} from such a desingularization?

Cubical spaces

De Jong's theorem gives a cartesian diagram of topological spaces

$$\begin{array}{ccc} Z'^{\mathrm{an}}/\Gamma & \overset{j'}{\hookrightarrow} & X'^{\mathrm{an}}/\Gamma \\ {\scriptstyle \pi'^{\mathrm{an}}}\downarrow & & \downarrow{\scriptstyle \pi^{\mathrm{an}}} \\ Z^{\mathrm{an}} & \hookrightarrow & X^{\mathrm{an}} \end{array}$$

where π^{an} is proper and induces a homeomorphism over $X^{\mathrm{an}} - Z^{\mathrm{an}}$.

Fundamental Lemma

If the closed immersion j is a cofibration (homotopy lifting property), then

$$X^{\mathrm{an}} \sim X'^{\mathrm{an}}/\Gamma \sqcup_{\pi,1} \left(Z'^{\mathrm{an}}/\Gamma\right) \times [0,1] \sqcup_{0,\pi'} Z^{\mathrm{an}}.$$

Tubular neighborhoods

Let D be a strict normal crossing divisor on a smooth k-variety X. There exists a function $\tau : X^{an} \to [0,1]$ locally equal to $|f|$ for any local equation f of D.

Definition

The tubular neighborhood of D in X is the k-analytic space

$$T_{X|D} = \tau^{-1}([0,1)).$$

By a careful analysis of the formal torus action on X, one proves the

Theorem (Tubular Theorem, 1)

There is a strong deformation retraction of $T_{X|D}$ *onto* D^{an}. *In particular, the closed immersion* $D^{an} \hookrightarrow X^{an}$ *is a cofibration.*

Another application to singularities

Theorem

Let X be a smooth and irreducible k-variety endowed with an admissible action of a finite group G. Assume that the singular locus of X/G is smooth (e.g. isolated singularities). For any resolution of singularities, the incidence complex of the exceptional divisor is contractible.

Proof. On the one hand, the analytic space X^{an}/G is contractible. On the other hand, it is homotopy equivalent to the suspension of the incidence complex.

Theorem

Let X be any k-variety and let $X_\bullet \to X$ be a cubical resolution of X obtained by iterated resolutions of singularities. Then the geometric realization of $\pi_0(X_\bullet)$ is homotopy equivalent to X^{an}.

Homotopy type of analytic spaces

Let D be a strict normal crossing divisor on a smooth k-variety X.

Theorem (Tubular theorem, 2)

There is a strong deformation retraction of X^{an} *onto* $\mathfrak{S}(X_0, X) \cup D^{an}$.

- By induction on the dimension, this result implies that the analytification of any algebraic variety over k has a strong deformation retraction onto a closed polyhedral subspace.
- Similar arguments apply more generally for any discretely valued non-Archimedean field (work over k°) and give an alternative proof of HRUSHOVSKI-LOESER's theorem.
- Assuming discrete valuation, a suitable version of DE JONG's theorem holds locally for any separated non-Archimedean analytic space (U. HARTL). By standard arguments on cubical spaces, one can deduce that any such space is homotopy equivalent to a locally finite polyhedron.