Thèse présentée en vue de l'obtention de
l'Habilitation à Diriger des Recherches

Spécialité : Mathématiques

Todor Tsankov

Groupes d'automorphismes et leurs actions

soutenue publiquement le 12 décembre 2014

après l'avis des rapporteurs :

M. Emmanuel Breuillard	Université d'Orsay
M. Alain Louveau	CNRS, IMJ–PRG
M. Vladimir Pestov	Université d'Ottawa

devant le jury composé de :

M. Emmanuel Breuillard	Université d'Orsay
M. Damien Gaboriau	CNRS, ENS Lyon
M. Gilbert Levitt	Université de Caen
M. Alain Louveau	CNRS, IMJ–PRG
M. Gilles Pisier	Texas A&M et Univ. Paris 6
Mme Katrin Tent	Universität Münster
M. Stevo Todorcevic	CNRS, IMJ–PRG

Automorphism groups and their actions

Todor Tsankov

Remerciements

Je tiens tout d'abord à remercier mes collaborateurs de ces dernières années : Itaï Ben Yaacov, Manuel Bodirsky, David Evans, Julien Melleray, Lionel Nguyen Van Thé et Michael Pinsker. Ils ont eu un rôle important dans mon évolution mathématique après ma thèse et les recherches présentées dans ce mémoire leur doivent beaucoup.

Je remercie les rapporteurs Emmanuel Breuillard, Alain Louveau et Vladimir Pestov d'avoir accepté cette tâche et d'avoir trouvé le temps d'écrire les rapports. Merci également à Emmanuel Breuillard, Damien Gaboriau, Gilbert Levitt, Alain Louveau, Gilles Pisier, Katrin Tent et Stevo Todorcevic de l'honneur qu'ils me font en acceptant de participer au jury.

Je voudrais remercier les membres de l'équipe d'analyse et de l'équipe de logique de l'IMJ de l'accueil chaleureux qu'ils m'ont réservé et de l'ambiance agréable et stimulante que j'ai trouvée dans le laboratoire. Participer au groupe de travail de théorie descriptive des ensembles a toujours été un plaisir et m'a apporté beaucoup au cours des années. Merci à Françoise Delon et à Martin Hils de toute leur aide et leurs conseils dans les aspects pratiques de la vie en France.

Je suis reconnaissant à Alekos Kechris, mon directeur de thèse, pour m'avoir initié à la recherche ainsi que pour son hospitalité et toutes les discussions que nous avons eues pendant mes visites à Caltech.

Une partie des travaux décrits dans ce mémoire a été accomplie pendant les programmes spéciaux *Thematic Program on Asymptotic Geometric Analysis* à l'Institut Fields à Toronto et *Universality and Homogeneity* à l'Institut Hausdorff à Bonn. Mes séjours à Toronto et à Bonn ont été très agréables et m'ont permis de commencer des collaborations fructueuses. Merci aux organisateurs de m'avoir invité et aux instituts de leur soutien financier.

Je voudrais remercier Alekos Kechris et Julien Melleray pour leur commentaires pertinents sur une version préliminaire de ce mémoire.

Enfin, merci à l'équipe administrative de l'UFR de Mathématiques à Paris 7 et plus spécialement à Pascal Chiettini sans qui la soutenance de cette habilitation n'aurait pas été possible.

Contents

1 Introduction

The goal of this memoir is to survey some recent results and emerging research directions in the theory of automorphism groups of discrete and metric structures and their dynamical systems. The focus is on my own work but I have tried to include enough background and examples in order to make the text accessible and, hopefully, useful for a non-expert reader who would like to get acquainted with the subject.

Understanding the automorphism group of a structure goes hand in hand with understanding the structure itself and if we adopt the perspective of Klein, we can go further and say that a sufficiently homogeneous structure is *defined* by the action of the group of its symmetries. This viewpoint can be formalized in a number of situations that we are interested in and turns out to be quite useful. An important reason why we consider dynamical systems of automorphism groups (apart from their intrinsic interest) is that they provide a framework to describe interactions between the structures and other mathematical objects and offer abstract, powerful tools for studying those interactions.

Most of the structures that we have in mind can be naturally considered as models in first-order logic. Apart from providing a convenient framework that encompasses most examples, this also permits the use of concepts, ideas, and tools from model theory that will be essential throughout this memoir. While initially model theory only dealt with discrete structures, with the development of continuous logic, model-theoretic methods can now also be applied in the metric setting, thus allowing us to study groups of symmetries of structures appearing in analysis. In continuous logic, one often tries to employ ideas and generalize results from classical model theory; however, new phenomena appear that have no analogue in the discrete world and new tools, often adapted from descriptive set theory and functional analysis, are required, thus giving the subject its own distinct flavor.

In order to effectively study large (uncountable) groups, it is often necessary to endow them with some additional structure. As most groups we are interested in appear as groups of isometries, they are naturally equipped with a topology that makes them into topological groups. Group topologies also provide several uniform structures on the groups and their homogeneous spaces that will be important for us.

Symmetry groups in geometry that are studied classically are usually locally compact, for the simple reason that the objects considered are most often of finite dimension. In abstract harmonic analysis as well, the focus is on locally compact groups: even the most basic tools rely on one's ability to take averages with respect to the Haar measure, which is exclusive to them. On the other hand, in functional analysis, model theory, and descriptive set theory, one has to deal with infinite discrete structures and infinite-dimensional spaces whose symmetry groups are almost never locally compact. Since we are mostly interested in structures and groups of the latter kind, it is not surprising that most of the tools that we use come from those three subjects.

A compromise between the very general setting of topological groups and the rather restrictive world of the locally compact ones, a class that includes most groups encountered in analysis, has robust closure properties, and still admits general methods for its study, is the class of *Polish groups* (i.e., completely metrizable, separable topological groups). This class also includes all groups that will be considered in this memoir. The combination of homogeneity, completeness, and the existence of a countable base makes Baire category tools particularly effective and, in a sense, those can be considered the unifying method in the subject. Some examples of how Baire category is used can be found in Section 10.

With the development of the theory of definable equivalence relations in descriptive set theory, it quickly became apparent that a large number of natural equivalence relations arise as (or are bi-reducible with) orbit equivalence relations of Polish group actions. This provided a strong impetus for the development of the theory of Polish dynamical systems and their orbit equivalence relations: after the pioneering work of Becker and Kechris [BK96] and the development of the theory of turbulence by Hjorth [Hjo00], a lot of other work followed; see [Ga009] for a modern account.

From another direction, in model theory, it was realized quite early that automorphism groups of countable homogeneous structures encode all essential information about the structures and are rich objects interesting to study in their own right. The subject has developed deep connections with permutation group theory, combinatorics, and, more recently, theoretical computer science. See the recent survey by Macpherson [Mac11] for general information about classical homogeneous structures and their automorphism groups and the monograph by Bodirsky [Bod12] for the applications in computer science.

As model-theorists seemed to be of a more algebraic mindset, the topology on the automorphism groups, while recognized as useful, was regarded as a secondary structure and they searched to define it solely in terms of the algebra. It turns out that this is possible in a number of cases, via the so-called *small index property*, and the tools that had been developed for studying it led, in the work of Kechris and Rosendal [KR07], to the *automatic continuity property*, which, in a certain sense, is the most general condition that permits the recovery of topological from algebraic information for Polish groups. We discuss automatic continuity in Section 11 below.

A special class of Polish groups that behave particularly well in various settings and play a central role in what follows is the class of *Roelcke precompact* groups (see Subsection 2.3 for the definition). A recurring theme in this memoir is that in many situations, in sharp contrast with what happens for locally compact (non-compact) groups, actions of Roelcke precompact groups tend to be classifiable and, with enough caution, one can let one's intuition be guided by analogy with the simpler situation for compact groups. The classification results are interesting by themselves but also have consequences in combinatorics and probability theory. It is also instructive to compare the theory of the dynamical systems considered here with the theory of orbit equivalence relations, where the actions of those same groups are used to prove non-classifiability results (mostly via Hjorth's theory of turbulence).

Roelcke precompact groups are exactly the automorphism groups of *ω-categorical structures*: separable metric models that are determined, up to isomorphism, by their first-order theory. Those models are canonical, highly symmetric objects whose model-theoretic structure is entirely encoded by the action of the automorphism group. This leads to an interesting correspondence between model-theoretic and dynamical properties, most notably in connection with the theory of representations of dynamical systems on Banach spaces, as developed by Glasner and Megrelishvili. This correspondence, first studied in [BT13], can be exploited in both directions but is not yet completely understood. We discuss this in Section 9, where we also see some examples of using model-theoretic tools and intuition to prove purely topological results.

One of the main topics of this memoir is dynamical systems of automorphism groups, that is, studying their actions on various spaces: more specifically, *topological dynamical systems*, i.e., continuous actions on compact spaces, and *measure-preserving dynamical systems*, measure-preserving actions on standard probability spaces. An important feature of studying dynamical systems for general Polish groups is that new, interesting phenomena appear that have no analogue in the locally compact world, such as extreme amenability and metrizable universal minimal flows as well as the possibility of classification of the dynamical systems of certain groups. Pioneering work in this field was done by Pestov and by Glasner and Weiss.

With the work of Kechris, Pestov, and Todorcevic [KPT05], it became apparent that the topological dynamics of an automorphism group is intimately connected with the Ramsey theory of the corresponding structure and understanding the minimal flows of the group has important combinatorial consequences. Their paper rekindled the interest in structural Ramsey theory and has inspired a lot of new work in both combinatorics and topological dynamics. See Section 3 for a description of this correspondence and the current state of the art.

Measure-preserving dynamical systems of automorphism groups can be interpreted as families of random variables indexed by (*imaginary*) elements of the structure whose joint distribution satisfies certain invariance properties. Such families are studied in probability theory under the name of *exchangeable random variables* and a theory has been developed around *de Finetti's theorem* and its generalizations [Kal05]. The approach to exchangeability theory via dynamical systems of automorphism groups gives a new framework in which different methods, for example, unitary representations, can be applied and suggests interesting generalizations of the setting that is usually considered. In Section 7, we briefly explain this framework and present some preliminary results. Understanding further the measure-preserving actions of Roelcke precompact groups seems a promising direction for future work.

As is often done in dynamics, one can combine topological and measure-theoretic structure in a single system. In this context, *amenability* is an important property of the group as it ensures that every topological system carries an invariant measure. This property has been much studied for discrete and locally compact groups but the theory is less developed for general Polish groups. In Section 5, we present a criterion for amenability of non-archimedean groups inspired by the Reiter condition for discrete groups; unfortunately, verifying this condition in concrete, non-trivial cases seems a rather daunting combinatorial task.

Some topological dynamical systems carry a unique invariant measure (one says that they are *uniquely ergodic*) and in that case, the measure can be used as a tool to understand the topological system even if a priori one is not interested in ergodic theory. It is a fascinating phenomenon, studied in detail by Angel, Kechris, and Lyons [AKL12], that for many automorphism groups, all of their *minimal* flows are uniquely ergodic. In Subsection 7.2, we discuss an alternative proof of some of their results based on a variant of de Finetti's theorem.

Last but not least, *unitary representations* of groups arise naturally in a variety of settings and understanding them can often be useful. It turns out that the unitary representations of Roelcke precompact, non-archimedean groups can be completely classified and the situation is somewhat similar to the one for compact groups. This classification can be used, for example, to show that all such groups have property (T), which, in turn, has dynamical consequences. It is an open question whether such a classification can be carried out for general Roelcke precompact Polish groups. Unitary representations are the topic of Section 6.

Most of this memoir is based on published (or submitted for publication) results; however, in Sections 4, 5, and 7, several new results are presented.

2 Automorphism groups

2.1 Discrete and metric structures

Most groups that we will consider are presented as groups of isometries of separable metric spaces preserving some additional structure and are described most naturally in the framework of first-order (continuous) logic. Even if the group is not initially given as a group of isometries, one can often consider an alternative presentation: for example, the homeomorphism group of a compact space K can be viewed as the automorphism group of the commutative C*-algebra $C(K)$, and the latter is a metric structure. If everything else fails, every Polish group G can be equipped with a left-invariant distance and this action can be used to produce an (approximately homogeneous) metric structure with automorphism group G (see [Mel14, Section 4.2] for details on how to achieve this).

We quickly review classical first-order logic. Let M be a set. A *basic relation* on M is a subset $R \subseteq M^k$, where k is called the *arity* of R. A *discrete structure* is a set M equipped with countably many basic relations. The *language (or signature)* of M is the set of names for the basic relations together with their arities. The language always includes the equality relation. A *definable relation* on M is a relation that can be obtained from the basic ones using Boolean operations, *substitutions* (if R is a definable relation of arity k, then

$$\{(a_1, \ldots, a_n) \in M^n : (a_{i_1}, \ldots, a_{i_k}) \in R\},$$

are definable relations for all choices of $i_1, \ldots, i_k \leq n$), and projections. A definable relation of arity k can also be thought of as a function $M^k \to \{0, 1\}$. Each definable relation has a name, a *first-order formula*, that describes the sequence of operations via which it was obtained. Note that a formula, being a syntactic object, can be evaluated on any structure that has the same signature as M. A *sentence* is a formula without free variables, i.e., a definable 0-ary relation. A *theory* is a collection of sentences with their values. A structure M is a *model of the theory T* (notation $M \models T$) if the sentences of T, when evaluated on M, take the values prescribed by T.

Next we parallel the description above in the metric setting. Let (M, d) be a complete, bounded metric space (if one takes some care, it is also possible to include unbounded spaces but we will not detail this here). A *basic predicate* on M is a uniformly continuous, bounded function $P\colon M^k \to \mathbf{R}$, where k is called the arity of P. A *metric structure* is a space M equipped with countably many basic predicates. The *language* of M is the set of names for the basic predicates together with their arities, bounds, and uniform continuity moduli. The language always includes the distance symbol d. A *definable predicate* on M is a bounded, uniformly continuous function $M^k \to \mathbf{R}$ that can be obtained from the basic predicates by taking continuous combinations, substitutions, *suprema* (if $Q(x_1, \ldots, x_k)$ is a definable predicate, then $\sup_{x_1 \in M} Q(x_1, \ldots, x_k)$ is one too), and uniform limits. Each definable predicate has a name, a *continuous first-order formula*, that describes the sequence of operations via which it was obtained. A *sentence* is a formula without free variables, i.e., a definable 0-ary predicate. The *theory of M* (denoted by $\mathrm{Th}(M)$) is the set of values of all sentences evaluated on M. As we allow arbitrary continuous combinations and taking limits in the construction of formulas, there are uncountably many formulas. However, given a theory T, there is a natural norm on the set of formulas defined by

$$\|\phi\| = \sup_{M \models T, \bar{a} \in M} |\phi^M(\bar{a})|.$$

Equipped with this norm, the set of formulas on n-variables, tensored with the complex numbers and divided by the kernel of the norm, naturally carries the structure of a separable, commutative C*-algebra, denoted by $W_n(T)$. As we are only interested in formulas up to logical equivalence, we can redefine a formula on n variables to be just an element of $W_n(T)$; a *formula on infinitely many variables* is an element of the C*-algebra $W_\infty(T) = \varinjlim W_n(T)$. We refer the reader to [Ben+08] for more details on continuous logic.

Usually one also allows *function symbols* in the language that are interpreted as uniformly continuous functions $M^k \to M$ (and we will have functions in some of our examples). However, omitting them in the definitions entails no loss of generality, as a function $f\colon M^k \to M$ can be replaced by the $k+1$-ary predicate $d(f(\bar{x}), y)$ without losing expressive power of the language.

One sees from the above definitions that discrete structures are a special case of metric ones, where the metric and all basic predicates take only the values 0 and 1. *From here on, we will use the word* structure *without qualifiers to signify a metric structure and we will sometimes refer to discrete structures as* classical. We will also exclusively consider *separable* structures, i.e., structures admitting a countable dense set.

Let M be a structure and $T = \mathrm{Th}(M)$. If $\mathcal{L}$ is a set of definable predicates closed under substitutions, the *$\mathcal{L}$-type* of a tuple $\bar{a} \in M^k$ is the set of values of predicates in $\mathcal{L}$ of arity k evaluated on $\bar{a}$. If $\mathcal{L}$ is omitted, one takes the set of all definable predicates. Alternatively, a *complete k-type* is just an element of the Gelfand space of the algebra $W_k(T)$.

The *automorphism group of M*, denoted by $\mathrm{Aut}(M)$, is the group of all isometries of M that preserve the basic predicates (and therefore all definable predicates). Equipped with the pointwise convergence topology, $G = \mathrm{Aut}(M)$ is a Polish group.

M is called *(approximately) $\mathcal{L}$-homogeneous* if for every two tuples $\bar{a}, \bar{b} \in M^k$ of the same $\mathcal{L}$-type, $G \cdot \bar{a} = G \cdot \bar{b}$ ($\overline{G \cdot \bar{a}} = \overline{G \cdot \bar{b}}$). In this setting, $\mathcal{L}$ will be either the set of definable predicates or the set of basic predicates; in the former case, we call the structure simply *homogeneous* and in the latter, *ultrahomogeneous*. Thus every ultrahomogeneous structure is homogeneous but the converse is not necessarily true.

If X is a metric space and G acts on X by isometries, let

$$X \mathbin{/\!\!/} G = \{\overline{G \cdot x} : x \in X\}$$

(note that $\{(x, y) \in X^2 : x \in \overline{G \cdot y}\}$ is an equivalence relation). We will write $[x]_G$ for $\overline{G \cdot x}$ when we consider it as an element of $X \mathbin{/\!\!/} G$. The set $X \mathbin{/\!\!/} G$ is naturally a metric space if equipped with the distance

$$d([x]_G, [y]_G) = \inf_{g \in G} d(g \cdot x, y).$$

If X is a metric space on which the group G acts, then for each n, we equip X^n with the max distance and the diagonal action of G.

Definition 2.1. Let X be a metric space and let the group G act on X by isometries. The action is called *(approximately) oligomorphic* if for every n, $X^n \mathbin{/\!\!/} G$ is finite (compact).

If M is discrete, then $\mathrm{Aut}(M)$ is a closed group of permutations of a countable set, so embeds as a subgroup of the group of all permutations of M, $S(M)$. When the set M is not important, we will denote this group by S_∞. A characterizing property for closed subgroups of S_∞ is that they admit a basis at the identity consisting of open subgroups (the pointwise stabilizers of finite subsets of M). Groups with this property are also called *non-archimedean*. A discrete structure M is homogeneous iff it is approximately homogeneous and an action $G \curvearrowright M$ on a discrete M is oligomorphic iff it is approximately oligomorphic.

A structure M is called *ω-categorical* if $\mathrm{Th}(M)$ has a unique separable model up to isomorphism. The following theorem is due independently to Ryll-Nardzewski, Engeler, and Svenonius in the classical setting and was generalized to metric structures by Ben Yaacov, Berenstein, Henson, and Usvyatsov [Ben+08]. It is commonly referred to as the Ryll-Nardzewski theorem.

Theorem 2.2. *Let M be a metric structure. Then the following are equivalent:*

(i) *M is ω-categorical;*

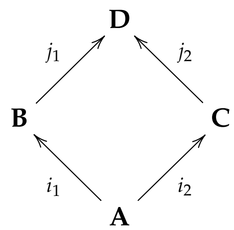
(ii) *The action $\mathrm{Aut}(M) \curvearrowright M$ is approximately oligomorphic.*

Moreover, for every k, the predicates on M^k preserved by $\mathrm{Aut}(M)$ are exactly the k-ary definable predicates and every ω-categorical structure is approximately homogeneous.

This theorem provides a crucial link between model-theoretic notions and the action of $\mathrm{Aut}(M)$ on powers of M.

2.2 Examples

A canonical construction of countable, discrete, ultrahomogeneous structures was described by Fraïssé [Fra54]. An *age* is a countable class of finite structures that is closed under taking substructures and that is *directed* in the the sense that every two structures in the class embed into a third. If M is a countable structure, its age, $\mathrm{Age}(M)$, is the class of its finite substructures; conversely, for every age $\mathcal{A}$, there exists a countable structure M such that $\mathrm{Age}(M) = \mathcal{A}$. Fraïssé isolated a crucial additional property, called the amalgamation property, that ensures that an age $\mathcal{A}$ can be realized as the class of substructures of an *ultrahomogeneous* structure; moreover, this ultrahomogeneous structure is unique up to isomorphism. It is called the *Fraïssé limit* of $\mathcal{A}$. An age $\mathcal{A}$ has the *amalgamation property* if for every $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{A}$ and embeddings $i_1 \colon \mathbf{A} \to \mathbf{B}$ and $i_2 \colon \mathbf{A} \to \mathbf{C}$, there exists $\mathbf{D} \in \mathcal{A}$ and embeddings $j_1 \colon \mathbf{B} \to \mathbf{D}$ and $j_2 \colon \mathbf{C} \to \mathbf{D}$ such that the diagram



commutes.

An analogous (albeit somewhat more technical) construction can also be carried out in the metric case (see Ben Yaacov [Ben12]). The main new features are that the age now forms a Polish metric space and one relaxes the amalgamation property to the *near amalgamation property* where the diagram above is required to commute only approximately.

Below we list several examples of ultrahomogeneous structures that will reappear throughout the text. All of them are easy to describe and the amalgamation property is not difficult to check. This should, however, not mislead the reader: the world of ultrahomogeneous structures is very rich and proving amalgamation can sometimes be a challenging combinatorial problem. A lot of work has been done in the 1980s and 1990s in classifying ultrahomogeneous structures with a given signature: a culminating point of this effort was the work of Cherlin [Che98] who, building on previous results of Lachlan and Woodrow [LW80] and Lachlan [Lac84], classified all ultrahomogeneous directed graphs (of which there are uncountably many).

Discrete examples:

- The Fraïssé limit of all finite sets without structure is a countably infinite set. The corresponding group is S_∞, the group of all permutations of this set.
- The Fraïssé limit of all finite linear orders is the countable dense linear order without endpoints $(\mathbf{Q}, <)$. We denote the corresponding automorphism group by $\mathrm{Aut}(\mathbf{Q})$.
- The Fraïssé limit of all finite Boolean algebras is the countable atomless Boolean algebra which is isomorphic to the algebra of all clopen subsets of the Cantor space $2^{\mathbf{N}}$. The corresponding automorphism group is

Homeo($2^{\mathbf{N}}$), the group of all homeomorphisms of $2^{\mathbf{N}}$.

- The Fraïssé limit of all finite vector spaces over a fixed finite field $\mathbf{F}_q$ is the infinite-dimensional vector space over $\mathbf{F}_q$. The automorphism group is the general linear group $\mathrm{GL}(\infty, \mathbf{F}_q)$.
- The Fraïssé limit of all finite graphs is the *random graph*, the unique countable graph such that for every two finite disjoint sets of vertices U, V, there exists a vertex x which is connected by an edge to all vertices in U and to no vertices in V.
- Let $m \geq 3$ be a natural number. The class of all finite graphs that do not contain a copy of K_m (an m-clique) amalgamates and its Fraïssé limit is *the ultrahomogeneous K_m-free graph.*

Continuous examples:

- The Fraïssé limit of all finite metric spaces is the Urysohn space $\mathbf{U}$; if one considers instead the finite metric spaces of diameter bounded by 1, one obtains the variant $\mathbf{U}_1$ (which is isometric to a sphere of radius $1/2$ in $\mathbf{U}$); the corresponding automorphism groups are denoted by $\mathrm{Iso}(\mathbf{U})$ and $\mathrm{Iso}(\mathbf{U}_1)$.
- The Fraïssé limit of all finite-dimensional Hilbert spaces is the infinite-dimensional, separable Hilbert space $\mathcal{H}$. Its automorphism group is the unitary group $U(\mathcal{H})$.
- The Fraïssé limit of all finite probability algebras is the standard, non-atomic measure algebra (all measurable subsets of $[0, 1]$ up to differences of measure 0 equipped with the Lebesgue measure μ). The automorphism group is the group of all measure-preserving transformations $\mathrm{Aut}(\mu)$.
- The Fraïssé limit of all finite-dimensional Banach spaces is the *Gurarij space* $\mathbf{G}$. As opposed to the previous examples, the Gurarij space is only approximately ultrahomogeneous and not homogeneous.

2.3 Roelcke precompact Polish groups

There are four natural uniformities compatible with the topology on every topological group G: the *left uniformity*, generated by entourages of the form

$$\{(g, gv) : g \in G, v \in V\}, \quad V \text{ is a symmetric neighborhood of } 1_G,$$

the *right uniformity*, generated by the entourages

$$\{(g, vg) : g \in G, v \in V\},$$

the *two-sided uniformity*, generated by intersections of entourages of the left and the right uniformities, and finally, the *Roelcke uniformity* generated by entourages of the form

$$\{(g, v_1 g v_2) : g \in G; v_1, v_2 \in V\}.$$

The Roelcke uniformity was first studied by Roelcke and Dierolf [RD81] and was given its name by Uspenskij.

A topological group is Polish iff it is Hausdorff, second countable, and its two-sided uniformity is complete. In general, the other three uniformities need not be complete. Specifying different properties of these uniformities defines different classes of Polish groups that have been widely studied: for example, the *SIN groups* for which the left and right uniformities coincide (those include the abelian and the compact groups), the larger class of *CLI groups* for which the left (equivalently, the right) uniformity is complete (those include the locally compact groups), and, finally, the *Roelcke precompact* groups for which the Roelcke uniformity is precompact. This latter class plays a central role in this memoir.

An equivalent definition is the following.

Definition 2.3. A topological group G is *Roelcke precompact* if for every neighborhood $V \ni 1_G$, there exists a finite set $F \subseteq G$ such that $VFV = G$.

Roelcke precompact groups were introduced in the book [RD81] by Roelcke and Dierolf and were systematically studied by Uspenskij [Usp98; Usp01; Usp02; Usp08], who also found a number of examples.

The following theorem, proved in [BT13] (and also, independently, by Rosendal [Ros13]), generalizing a similar result for the non-archimedean case [Tsa12], makes it fairly easy to check whether a group of isometries is Roelcke precompact.

Theorem 2.4 ([BT13])**.** *For a Polish group G, the following are equivalent:*

(i) *G is Roelcke precompact;*

(ii) *Whenever G acts continuously by isometries on a complete metric space X and $X /\!\!/ G$ is compact, the action is approximately oligomorphic;*

(iii) *There exists a Polish metric space X and a homeomorphic group embedding $G \hookrightarrow \mathrm{Iso}(X)$ such that the induced action $G \curvearrowright X$ is approximately oligomorphic.*

Theorem 2.4 readily applies in a number of situations to recover some old results: for example, the unitary group of a separable Hilbert space [Usp98], the automorphism group of a standard probability space [Gla12], the isometry group of the bounded Urysohn space [Usp08], and the isometry group of the Gurarij space are all Roelcke precompact. On the other hand, the isometry group of the unbounded Urysohn space $\mathbf{U}$ is not Roelcke precompact: it acts transitively on $\mathbf{U}$ but $\mathbf{U}^2 /\!\!/ \mathrm{Iso}(\mathbf{U}) = \mathbf{R}^+$ is not compact. As for discrete examples, all automorphism groups from Subsection 2.2 are Roelcke precompact, as is the automorphism group of any structure ultrahomogeneous in a finite relational language.

Combining the Ryll-Nardzewski theorem with Theorem 2.4 yields the following corollary.

Corollary 2.5. *Let $G = \mathrm{Aut}(M)$, where M is a metric structure and suppose that $M /\!\!/ G$ is compact. Then the following are equivalent:*

(i) *$\mathrm{Th}(M)$ is ω-categorical;*

(ii) *G is Roelcke precompact.*

It turns out that in a number of situations, Roelcke precompact Polish groups exhibit tame behavior and their actions can be classified: this is the topic of several of the next sections.

3 Topological dynamics and Ramsey theory

Let G be a topological group. A *G-flow* is a compact Hausdorff topological space equipped with a continuous action of G. A flow is *minimal* if it has no proper subflows, or equivalently, if every orbit is dense. A simple application of Zorn's lemma yields that every flow contains a minimal subflow; thus, while there is, in general, no decomposition of flows into minimal subflows, understanding the minimal flows of a certain group G is an important step in studying its topological dynamics.

A *G-ambit* is a G-flow with a distinguished point whose orbit is dense. A continuous map $\pi\colon X \to Y$ is a *morphism of G-flows* if it commutes with the G-action. It is a *morphism of ambits* if, moreover, it sends the distinguished point of X to the distinguished point of Y. Note that there exists at most one morphism between two ambits.

For every group G, there exists a *universal G-ambit*, one that maps onto every other ambit. It can be constructed as the Gelfand space $S(G)$ of the C^*-algebra $\mathrm{RUCB}(G)$ of bounded, right uniformly continuous functions on G (also known as the *Samuel compactification* of G). G embeds naturally as a dense subset of $S(G)$ and it is easy to see that $(S(G), 1_G)$ is a universal ambit. It follows that every minimal subset of $S(G)$ is universal for the minimal flows of G, that is, it maps onto every minimal G-flow. It is a classical result of Ellis that this universal property defines a unique (up to isomorphism) minimal flow, called *the universal minimal flow (UMF) of G* and denoted by $M(G)$. See Uspenskij [Usp00] for a short proof of this fact.

Universal minimal flows are usually complicated and difficult to understand. For example, if G is a discrete group, then its universal ambit is $(\beta G, 1_G)$, where βG is the *Stone–Čech compactification of G*, the space of all ultrafilters on G. Then every minimal subset of βG is isomorphic to $M(G)$; note that if G is infinite, then $M(G)$ is never metrizable simply because βG has no non-trivial convergent sequences. Similarly, for a locally compact, non-compact group G, $M(G)$ is never metrizable.

It is a remarkable fact, discovered by Herer and Christensen [HC75], that there exist Polish groups that admit no non-trivial minimal flows whatsoever, the so called *extremely amenable groups*. Equivalently, a topological group is *extremely amenable* if every time it acts continuously on a compact space, there is a fixed point. The proof of Herer and Christensen proceeds by showing that the group that they construct (the completion of $C(2^{\mathbf{N}})$ with respect to a distance defined using a *pathological* submeasure) admits no non-trivial unitary representations, or is *exotic* according their terminology; as it is amenable (being abelian), every time it acts on a compact space, there is an invariant measure μ and the fact that the natural representation $G \curvearrowright L^2(\mu)$ is trivial implies that μ concentrates on the set of G-fixed points. The terminology used in their paper suggests that they regarded the phenomenon that they had discovered as rather exceptional. It was not until Gromov and Milman [GM83] showed that the unitary group $U(\mathcal{H})$ is extremely amenable by using concentration of measure that it was realized that extreme amenability is, in fact, quite widespread. Soon thereafter, many new examples were found using similar techniques. In fact, all of the continuous examples from Subsection 2.2 except the isometry group $\mathrm{Iso}(\mathbf{G})$ of the Gurarij space are extremely amenable and this can be proved using the concentration of measure technique (see Section 10 for a slightly different approach to these results), and it is an open question whether $\mathrm{Iso}(\mathbf{G})$ is extremely amenable. See the book by Pestov [Pes06] for more background, examples, and references on extreme amenability.

A turning point in the theory of extremely amenable groups was marked by the work of Pestov [Pes98], who used the Ramsey theorem to show that the group $\mathrm{Aut}(\mathbf{Q})$ is extremely amenable. As $\mathrm{Aut}(\mathbf{Q})$ embeds as a dense subgroup of the group of orientation-preserving homeomorphisms of the reals, $\mathrm{Homeo}^+(\mathbf{R})$, this also implies that $\mathrm{Homeo}^+(\mathbf{R})$ is extremely amenable. Since $\mathrm{Homeo}^+(\mathbf{R})$ is isomorphic to the stabilizer of a point in the homeomorphism group of the circle $\mathrm{Homeo}(S^1)$, one obtains that the UMF of $\mathrm{Homeo}(S^1)$ is S^1. Indeed, $S^1 \cong \mathrm{Homeo}(S^1)/\mathrm{Homeo}^+(\mathbf{R})$ is a homogeneous space of $\mathrm{Homeo}(S^1)$ and every time $\mathrm{Homeo}(S^1)$ acts on a compact space X, there is a point $x_0 \in X$ fixed by $\mathrm{Homeo}^+(\mathbf{R})$ and so one obtains a natural continuous map $\mathrm{Homeo}(S^1)/\mathrm{Homeo}^+(\mathbf{R}) \to \mathrm{Homeo}(S^1) \cdot x_0$ which must be onto X if X is minimal. As we will see below, a variation of this technique for calculating universal minimal flows turns out to be quite versatile.

Still using the Ramsey theorem, Glasner and Weiss [GW02] proved that the universal minimal flow of S_∞ is isomorphic to LO, the compact space of all linear orderings on a countable set (on which S_∞ acts naturally by permutations). This provided the first example of a non-transitive, metrizable UMF. Using similar methods, they also calculated [GW03] the UMF of $\mathrm{Homeo}(2^{\mathbf{N}})$.

Inspired by these results, Kechris, Pestov, and Todorcevic [KPT05], working in the framework of ultrahomogeneous structures, described a precise correspondence between the Ramsey property (elaborated by combinatorialists in the 1970s and 1980s) and extreme amenability. They used this correspondence and known results from structural Ramsey theory to find many new examples of extremely amenable, non-archimedean groups: the automorphism groups of the random ordered graph, lexicographically ordered vector spaces over finite fields, the lexicographically ordered atomless Boolean algebra, etc. They also developed tools for calculating the UMF of groups using the extreme amenability of a subgroup (more on this below). Their paper inspired a lot of new work in structural Ramsey theory: see Nguyen Van Thé [Ngu13] for many examples and an account of recent developments.

The equivalence proved by Kechris, Pestov, and Todorcevic also suggested the possibility of proving structural Ramsey theorems using dynamical methods. The idea of using topological dynamics in Ramsey theory is certainly not new: for example, there are well-known dynamical proofs of Van der Waerden's theorem and Hindman's theorem. However, proofs from structural Ramsey theory are significantly more complicated and often not very transparent, so the possibility of a uniform approach to those results using dynamics seems particularly attractive.

3.1 The Ramsey property and extreme amenability

Let M be a classical, ultrahomogeneous structure and let $\mathbf{A}$ and $\mathbf{B}$ be finite structures in the age of M. We will denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the set of all embeddings of $\mathbf{A}$ into $\mathbf{B}$ and by $\binom{M}{\mathbf{A}}$ the set of all embeddings of $\mathbf{A}$ into M.

Definition 3.1. We say that M has the *Ramsey property* if for every natural number k, every pair of finite substructures $\mathbf{A}, \mathbf{B} \subseteq M$, and every coloring $c \colon \binom{M}{\mathbf{A}} \to k$, there exists $f \in \binom{M}{\mathbf{B}}$ such that $c|_{f \circ \binom{\mathbf{B}}{\mathbf{A}}}$ is constant.

Remark 3.2. The property defined above is usually referred to as the *Ramsey property for embeddings*. If finite substructures of M are *rigid* (i.e., have no automorphisms), then the Ramsey property for embeddings is equivalent to the regular Ramsey property where one colors copies of $\mathbf{A}$ in M rather than embeddings, but in general, coloring embeddings is better adapted to our framework.

The classical Ramsey theorem can be stated as the fact that the structure $(\mathbf{Q}, <)$ has the Ramsey property.

The correspondence discovered by Kechris, Pestov, and Todorcevic can now be formulated as follows.

Theorem 3.3 (Kechris–Pestov–Todorcevic)**.** *Let M be a discrete, ultrahomogeneous structure and let $G = \mathrm{Aut}(M)$. Then the following are equivalent:*

(i) *M has the Ramsey property;*

(ii) *G is extremely amenable.*

In connection with Remark 3.2, one should note that G always acts on the compact space of linear orderings on M, so in particular, if G is extremely amenable, then there is always a G-invariant linear order on M, which means that all finite substructures of M are automatically rigid.

While most ω-categorical structures do not have the Ramsey property (as noted above, the Ramsey property for an ω-categorical structure requires a definable ordering), in many cases, it suffices to *expand* the structure by an appropriate ordering and, possibly, some other relations to obtain another structure which does have the Ramsey property and is still ω-categorical. If that happens, we obtain complete information about the Ramsey theory of the original structure and it is easy to calculate the universal minimal flow of its automorphism group as we explain in the next subsection. The question whether such an expansion always exists was raised by Bodirsky, Pinsker and the author in [BPT13].

Question 3.4 ([BPT13]). Let M be an ω-categorical structure. Does there always exist an ω-categorical expansion M' of M with the Ramsey property?

A variant of this question asks whether if M is ultrahomogeneous in a finite relational language, one can find a Ramsey expansion M' with the same property. See Section 8 for some further motivation for those questions.

3.2 Precompact homogeneous spaces and universal minimal flows

If G is a topological group and $H \leq G$ is a closed subgroup, the homogeneous space G/H carries a natural uniform structure (which is the quotient of the right uniform structure on G) whose entourages are of the form

$$\mathcal{U}_V = \{(gH, vgH) : g \in G, v \in V\}, \quad \text{where } V \text{ is a symmetric nbhd of } 1_G.$$

Note that this uniformity is usually not complete; we will denote its completion by $\widehat{G/H}$. As the action $G \curvearrowright G/H$ is by uniform isomorphisms, it extends to a continuous action $G \curvearrowright \widehat{G/H}$.

We say that the homogeneous space G/H is *precompact* (or that H is *co-precompact* in G) if $\widehat{G/H}$ is compact; equivalently, if for every open $V \ni 1_G$, there is a finite set $F \subseteq G$ such that $VFH = G$. Then $(\widehat{G/H}, H)$ is a metrizable G-ambit that has the following universal property: for every G-ambit (X, x_0) such that x_0 is a fixed point of H, there exists a unique morphism $\pi \colon \widehat{G/H} \to X$ such that $\pi(H) = x_0$. Note also that the orbit of H in $\widehat{G/H}$ is a dense G_δ set and a simple Baire category argument shows that, if $\widehat{G/H}$ is minimal, the orbit $G \cdot x_0$ is dense G_δ in any quotient (X, x_0) of $(\widehat{G/H}, H)$.

An easy criterion for co-precompactness of subgroups of automorphism groups is the following. Let M be an ω-categorical structure, $G = \mathrm{Aut}(M)$, and let $H \leq G$. Then G/H is precompact iff the action $H \curvearrowright M$ is approximately oligomorphic.

The notion of a precompact homogeneous space gives us yet another equivalent definition of Roelcke precompactness: a non-archimedean Polish group $G = \mathrm{Aut}(M)$ is Roelcke precompact iff for every open subgroup $V \leq G$, the (discrete) homogeneous space G/V is precompact. In this case, the compactification $\widehat{G/V}$ can be identified with the space of types of the sort determined by V with parameters from the model M. As an example, if $G = \mathrm{Aut}(\mathbf{Q})$ and V is the stabilizer of a single point, $\widehat{G/V}$ can be identified with the unit interval $[0, 1]$ where each rational point except 0 and 1 is split into three. In this case G/V, being discrete, is open in the compactification. This also gives a convenient characterization of ω-stability in terms of the automorphism group: an ω-categorical, discrete structure M with automorphism group G is *ω-stable* if $\widehat{G/V}$ is countable for every open $V \leq G$. See Section 9 for a bit more on stability theory.

If M is an ω-categorical structure and M' is an expansion of M, then $\mathrm{Aut}(M')$ is co-precompact in $\mathrm{Aut}(M)$ iff M' is ω-categorical.

A common strategy for calculating the universal minimal flow of a group G, first elaborated by Pestov [Pes06], is the following. First one finds a closed, co-precompact, extremely amenable subgroup $H \leq G$, for example, by guessing and proving a Ramsey theorem for an appropriate expansion of the original structure. By the extreme amenability of H and the universal property of $\widehat{G/H}$, every minimal subflow of $\widehat{G/H}$ is isomorphic to $M(G)$; in practice, if one has guessed correctly, it often happens that $\widehat{G/H}$ is already minimal. To illustrate this method, consider the case $G = S_\infty$ and $H = \mathrm{Aut}(\mathbf{Q})$: then the homogeneous space G/H is precompact and one can naturally view it as the space of all linear orderings isomorphic to $(\mathbf{Q}, <)$; the completion $\widehat{G/H}$ is isomorphic to the space LO of all linear orderings; by inspection, LO is minimal; finally, as H is extremely amenable, LO is the universal minimal flow of S_∞.

Quite remarkably, *all* known metrizable UMFs are of the form $G \curvearrowright \widehat{G/H}$ for some closed, co-precompact, extremely amenable subgroup $H \leq G$. This suggests the following strategy for exploiting the equivalences from Corollary 2.5 and Theorem 3.3 to answer Question 3.4 in the affirmative.

(i) Prove that the universal minimal flow of a Roelcke precompact, non-archimedean Polish group is always metrizable;

(ii) Show that (possibly under some hypothesis on the group) a metrizable universal minimal flow necessarily has a G_δ orbit (which, by the Baire category theorem, must be unique);

(iii) Prove that the stabilizer of a point in the G_δ orbit of a UMF is extremely amenable.

Item (ii) was also asked as a question by Angel, Kechris, and Lyons in [AKL12].

In [MNT14], jointly with J. Melleray and L. Nguyen Van Thé, we investigated item (iii) and proved the following theorem.

Theorem 3.5 ([MNT14])**.** *Let G be a Polish group and $M(G)$ be its universal minimal flow. Then the following are equivalent:*

(i) *The flow $M(G)$ is metrizable and has a G_δ orbit.*

(ii) *There is a closed, co-precompact, extremely amenable subgroup $H \leq G$ such that $M(G) = \widehat{G/H}$.*

In an independent effort, using very different techniques, Zucker [Zuc14b] proved, in the case of non-archimedean groups, a stronger theorem in which items (ii) and (iii) are dealt with simultaneously.

Theorem 3.6 (Zucker)**.** *Let G be a non-archimedean Polish group with a metrizable universal minimal flow. Then there exists a closed, co-precompact, extremely amenable subgroup $H \leq G$ such that $M(G) \cong \widehat{G/H}$.*

In view of Theorem 3.6 and the discussion above, a positive answer to the following question will imply a positive answer to Question 3.4 (and is equivalent to it in the case of non-archimedean groups).

Question 3.7 ([MNT14]). Is it true that the universal minimal flow of a Roelcke precompact Polish groups is always metrizable?

3.3 General consequences about the minimal flows

If the universal minimal flow of a group G is of the form $\widehat{G/H}$ for some closed, co-precompact $H \leq G$, this gives a lot of information about the other minimal flows of G as the next couple of results show.

The way one proves the uniqueness of the UMF is by showing that it is *coalescent*, i.e., that every endomorphism of the flow is an automorphism. If the universal minimal flow of a group G is of the form $\widehat{G/H}$, then *all* minimal flows of G are coalescent.

Theorem 3.8 ([MNT14])**.** *Let G be a Polish group. Then the following statements hold:*

(i) *Every minimal G-flow of the form $\widehat{G/H}$ is coalescent and has a compact automorphism group;*

(ii) *If $M(G) = \widehat{G/H}$ for some closed, co-precompact $H \leq G$, then the conclusion of (i) is true for every minimal G-flow.*

An important concept in topological dynamics is that of proximality: two points x and y in a G-flow X are *proximal* if there exists a net $\{g_\alpha\}_\alpha$ of elements of G such that $\lim_\alpha g_\alpha \cdot x = \lim_\alpha g_\alpha \cdot y$; a flow is *proximal* if every two points are proximal. Every topological group G admits a *universal minimal proximal flow* of which every other minimal proximal flow is a factor (see Glasner [Gla76] for details).

Theorem 3.9 ([MNT14]). *Let G be a Polish group and H a closed, co-precompact subgroup such that $M(G) = \widehat{G/H}$. Let $N(H)$ denote the normalizer of H in G. Then the universal minimal proximal flow of G is isomorphic to $\widehat{G/N(H)}$.*

4 Classification of the minimal flows

4.1 Smoothness of isomorphism

As we saw in the previous section, once we know that the UMF of a group G is of the form $\widehat{G/H}$, this provides ample information about *all* minimal flows of G. Here we show that the minimal flows of G are classifiable in a sense that we explain below.

Recall that an equivalence relation E on a Polish space X is *smooth* if there is a standard Borel space Y and a Borel function $f\colon X \to Y$ such that

$$x_1 \mathrel{E} x_2 \iff f(x_1) = f(x_2).$$

Smooth equivalence relations are those that can be classified using real numbers (or elements of any other uncountable Polish space) as invariants.

An example of a smooth equivalence relation is measure-theoretic isomorphism of Bernoulli shifts: by a well-known theorem of Ornstein, they are classified by their entropy. However, smoothness of isomorphism is a rather rare phenomenon for dynamical systems of discrete groups and, in general, isomorphism equivalence relations are rather complicated. An example of a recent non-classification result in topological dynamics, due to Gao, Jackson, and Seward [GJS12], is that isomorphism of minimal subshifts of any infinite, countable group is not smooth.

In contrast, if the UMF of a Polish group G is metrizable and has a G_δ orbit (by Zucker's Theorem 3.6, the second condition is automatic if G is non-archimedean), the techniques developed in [MNT14] easily yield that the equivalence relation of isomorphism of minimal flows of G is smooth. First, to make sense of this statement, we need a parametrization of the minimal flows of G by the elements of a Polish space. As $M(G)$ is metrizable and every minimal flow of G is a quotient of $M(G)$ by an invariant closed equivalence relation (*icer*), it is natural to parametrize the minimal flows of G by the space of all icers on $M(G)$ (the space of icers is a G_δ subset of the hyperspace of compact subsets of $M(G)^2$ and therefore Polish). Then one can prove that for two icers R_1 and R_2,

$$M(G)/R_1 \cong M(G)/R_2 \iff \exists h \in \mathrm{Aut}(M(G))\ h \cdot R_1 = R_2.$$

As $\mathrm{Aut}(M(G))$ is a compact group, we obtain the following.

Theorem 4.1 ([MNT14]). *Let G be a Polish group such that $M(G)$ is metrizable with a generic orbit. Then the equivalence relation of isomorphism of minimal flows of G is smooth.*

4.2 The minimal flows of S_∞

The fact that isomorphism of minimal flows of a group G is smooth means that, at least in principle, the minimal flows can be classified. Of course, the simplest case is when G is extremely amenable: then there is only one minimal flow. In this subsection, we study the less trivial example of S_∞ and give a complete description of the category of minimal flows of this group: it turns out that it admits only countably many minimal flows (up to isomorphism) and morphisms between them.

Let G be a group such that its UMF is of the type $\widehat{G/H}$ for some closed, co-precompact subgroup $H \leq G$. If $G \curvearrowright X$ is any minimal G-flow and $\pi\colon \widehat{G/H} \to X$ is a G-map, it is not difficult to see that the orbit of $x_0 = \pi(H)$ in X is also G_δ and the stabilizer G_{x_0} is a closed subgroup of G containing H. In particular, π factors as the composition of the two maps:

$$\widehat{G/H} \to \widehat{G/G_{x_0}} \to X,$$

where the second is one-to-one on the G_δ orbit of $\widehat{G/G_{x_0}}$ (we will call such maps *almost one-to-one*). Thus we see that the task of finding all quotients of $\widehat{G/H}$ splits into two: finding all closed subgroups K between H and G and then identifying the almost one-to-one quotients of each $\widehat{G/K}$.

If H is the automorphism group of an ω-categorical structure M, the closed subgroups between H and S_∞ correspond to the *first-order reducts* of M: structures that can be defined in M using first-order formulas. Reducts have been classified for a number of structures; see Section 8 for more information about them.

Recall that, by the result of Glasner and Weiss [GW02], the universal minimal flow of S_∞ is the space LO of all linear orderings on the natural numbers, which can be represented as $\widehat{S_\infty/H}$ for $H = \mathrm{Aut}(\mathbf{Q})$. The reducts of $(\mathbf{Q}, <)$ have been explicitly described by Cameron [Cam76] (but his results can be deduced from earlier work of Frasnay [Fra65]): the non-trivial ones are the betweenness relation, the circular order, and the separation relation (see below for the definitions), marked in bold in Figure 1.

The most convenient way for us to describe the minimal flows is via equivariant maps $\mathrm{LO} \to 2^{\mathbf{N}^k}$ for various numbers k; then the images of LO by those maps will be necessarily minimal flows and our main theorem states that they will exhaust all minimal flows of S_∞. Glasner and Weiss [GW02] had previously proved that for every k, $2^{\mathbf{N}^k}$ contains only finitely many minimal subsets without describing them; however it is not a priori clear that all minimal flows of S_∞ embed in $2^{\mathbf{N}^k}$ for some k and not even that they are zero-dimensional.

Another easy fact that we will need is that the automorphism group of the flow LO is of order 2; we will denote its non-identity element, *the flip*, by θ.

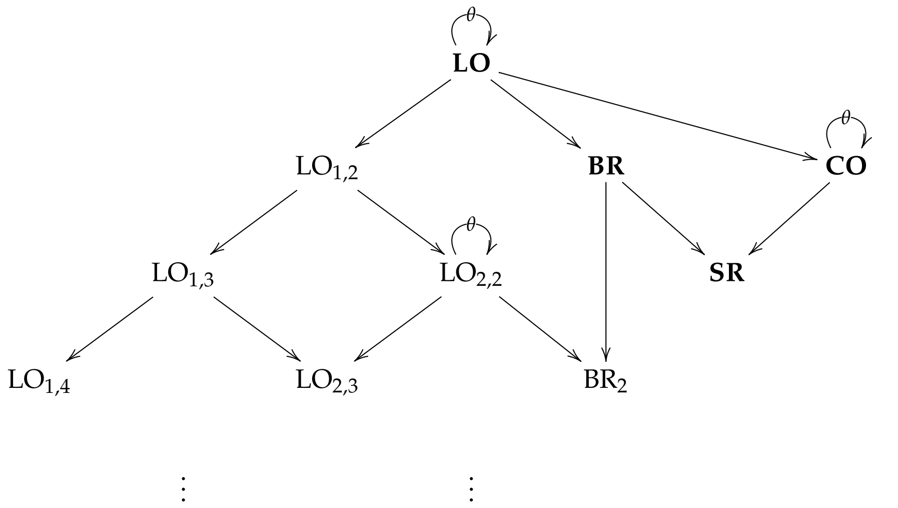


Figure 1: The minimal flows of S_∞ and the morphisms between them

The betweenness relation. For every $x \in \mathrm{LO}$, define the *betweenness relation* $B^x \subseteq \mathbf{N}^3$ by

$$B^x(a,b,c) \iff (a <_x b <_x c) \vee (c <_x b <_x a).$$

Let $\mathrm{BR} = \{B^x : x \in \mathrm{LO}\} \subseteq 2^{\mathbf{N}^3}$. BR can be regarded as the quotient of LO by the flip.

The circular order. If $x \in \mathrm{LO}$, define the *circular order* $K^x \subseteq \mathbf{N}^3$ by

$$K^x(a,b,c) \iff (a <_x b <_x c) \vee (b <_x c <_x a) \vee (c <_x a <_x b).$$

Let $\mathrm{CO} = \{K^x : x \in \mathrm{LO}\} \subseteq 2^{\mathbf{N}^3}$.

The separation relation. For $x \in \mathrm{LO}$, define the *separation relation* $S^x \subseteq \mathbf{N}^4$ by

$$\begin{aligned} S^x(a,b,c,d) &\iff \big(K^x(a,b,c) \wedge K^x(b,c,d) \wedge K^x(c,d,a)\big) \vee \\ &\qquad \big(K^x(d,c,b) \wedge K^x(c,b,a) \wedge K^x(b,a,d)\big) \\ &\iff B^x(a,b,c) \wedge B^x(b,c,d) \wedge B^x(c,d,a) \wedge B^x(d,a,b). \end{aligned}$$

Let $\mathrm{SR} = \{S^x : x \in \mathrm{LO}\}$ and note that $\mathrm{SR} = \mathrm{CO}/\theta$.

Almost one-to-one factors of LO. For $m, n \geq 1$ and $x \in \mathrm{LO}$, define $P^x_{m,n} \subseteq \mathbf{N}^{m+n+1}$ as follows:

$$P^x_{m,n}(a_1, \dots, a_m, b, c_1, \dots, c_n) \iff \\ \big(\bigwedge_i a_i <_x b\big) \wedge \big(\bigwedge_i b <_x c_i\big) \wedge \big(\bigwedge_{i \neq j} a_i \neq a_j\big) \wedge \big(\bigwedge_{i \neq j} c_i \neq c_j\big).$$

Let $\mathrm{LO}_{m,n} = \{P^x_{m,n} : x \in \mathrm{LO}\}$. Denote by $\mathcal{R}^{\mathrm{LO}}_{m,n}$ the equivalence relation on LO given by

$$x \, \mathcal{R}^{\mathrm{LO}}_{m,n} \, y \iff P^x_{m,n} = P^y_{m,n}.$$

It is clear from the definition that $\theta(\mathcal{R}^{\mathrm{LO}}_{m,n}) = \mathcal{R}^{\mathrm{LO}}_{n,m}$, so that $\mathrm{LO}_{m,n}$ and $\mathrm{LO}_{n,m}$ are isomorphic. The automorphism θ of LO factors to an automorphism of $\mathrm{LO}_{m,n}$ iff $m = n$.

The flow $\mathrm{LO}_{m+1,n}$ is a factor of $\mathrm{LO}_{m,n}$ as can be seen from the representation:

$$P^x_{m+1,n}(a_1, \ldots, a_{m+1}, b, c_1, \ldots, c_n) \iff \\ P^x_{m,n}(a_1, \ldots, a_m, b, c_1, \ldots, c_n) \wedge P^x_{m,n}(a_2, \ldots, a_{m+1}, b, c_1, \ldots, c_n).$$

Almost one-to-one factors of BR. For $n \geq 1$ and $x \in \mathrm{LO}$, define $Q^x_n \subseteq \mathbf{N}^{2n+1}$ by

$$\begin{aligned} Q^x_n(a_1, \ldots, a_n, b, c_1, \ldots, c_n) &\iff \bigwedge_{i,j} B^x(a_i, b, c_j) \wedge \bigwedge_{i,j} \neg B^x(a_i, b, a_j) \wedge \\ &\qquad \bigwedge_{i,j} \neg B^x(c_i, b, c_j) \\ &\iff P^x_{n,n}(\bar{a}, b, \bar{c}) \vee P^x_{n,n}(\bar{c}, b, \bar{a}). \end{aligned}$$

Denote $\mathrm{BR}_n = \{Q^x_n : x \in \mathrm{LO}\}$. The first definition shows that BR_n is a factor of BR and the second that it is isomorphic to the quotient $\mathrm{LO}_{n,n}/\theta$. The factor map $\mathrm{BR} \to \mathrm{BR}_n$ is almost one-to-one and it is an isomorphism iff $n = 1$.

The flow BR_{n+1} is a factor of BR_n via the map defined by

$$Q^x_{n+1}(a_1, \ldots, a_{n+1}, b, c_1, \ldots, c_{n+1}) \iff \\ Q^x_{n+1}(a_1, \ldots, a_n, b, c_1, \ldots, c_n) \wedge Q^x_{n+1}(a_2, \ldots, a_{n+1}, b, c_2, \ldots, c_{n+1}).$$

Theorem 4.2. *The diagram on Figure 1 represents the category of minimal flows of S_∞, that is all minimal flows of S_∞ and all morphisms between them, in the sense that every morphism is a composition of arrows on the diagram. (The trivial flow and the morphisms to it are omitted.)*

After this theorem had been proved, I became aware of the work of Frasnay [Fra65] on the classification of *monomorphic relations* and it turns out that the description of the S_∞-equivariant factors of LO can be derived from it. For this reason and because the proof of Theorem 4.2 is rather long and not very enlightening, I do not include it here. Not surprisingly, the main combinatorial tool used in the argument is the Ramsey theorem.

It will be very interesting to describe the minimal flows of other groups for which the UMF is metrizable, for example, $\mathrm{GL}(\infty, \mathbf{F}_q)$ or the automorphism group of the random graph. In problems of this type, it is difficult to conjecture the answer in advance as new objects appear during the process of classification: for instance, the flows $\mathrm{LO}_{m,n}$ and BR_n were discovered during my attempts to prove that non-trivial, almost one-to-one quotients do not exist.

Two more general questions are the following.

Question 4.3. What Polish groups admit only countably many minimal flows?

Of course, here one cannot hope for a complete answer but it will be interesting to have a somewhat general sufficient condition. Good candidates are non-archimedean Polish groups G whose UMFs are *finitely generated*, i.e. there exists a finite clopen partition of $M(G)$ such that its G-translates separate the points of $M(G)$. Some condition of this type is necessary as even very simple compact groups such as $(\mathbf{Z}/2\mathbf{Z})^{\mathbf{N}}$ admit uncountably many minimal flows.

Question 4.4. For which non-archimedean, Polish groups are all of their minimal flows zero-dimensional?

This is true for S_∞ by Theorem 4.2 but only a fortiori. A non-archimedean Roelcke precompact group that admits a connected minimal flow is the automorphism group G of the dense circular order (the stabilizer of a point in the G_δ orbit of CO) as it acts minimally on the circle (with two orbits). In fact, the universal minimal flow of G is $G/\widehat{\mathrm{Aut}(\mathbf{Q})}$ and it has only two other minimal flows: the circle and the one-point flow. A natural condition to impose in order to avoid this type of example is to require that the group be *minimal* in the sense of Subsection 9.2.

5 A criterion for amenability of non-archimedean Polish groups

For a topological group G, denote by $\mathrm{RUCB}(G)$ the algebra of bounded, right uniformly continuous functions on G equipped with the sup norm. The group G acts continuously by isometries on $\mathrm{RUCB}(G)$ by

$$(s \cdot f)(x) = f(s^{-1}x). \tag{5.1}$$

A *mean* on a commutative C*-algebra with unit is a positive linear functional m such that $m(1) = 1$.

Definition 5.1. A topological group G is *amenable* if there exists a G-invariant mean on the algebra $\mathrm{RUCB}(G)$.

Equivalently, G is amenable iff every time it acts continuously on a compact space, there is an invariant measure. For locally compact groups, there is also an equivalent combinatorial definition in terms of almost invariant L^1 functions or Følner sets (see, e.g., [Gre69]). It turns out that there is also a similar combinatorial condition for non-archimedean Polish groups. It is based on the simple fact that if V is an open subgroup of G, then $\ell^\infty(V\backslash G)$ embeds naturally in $\mathrm{RUCB}(G)$ as functions on G constant on right V-cosets and, moreover, if G is non-archimedean, the union of $\ell^\infty(V\backslash G)$ over all open $V \leq G$ is dense in $\mathrm{RUCB}(G)$.

Let G be non-archimedean, $g \in G$ and let $V, K \leq G$ be open subgroups such that $K \leq gVg^{-1}$. Let $\Phi_g\colon \ell^1(G\backslash K) \to \ell^1(V\backslash G)$ denote the linear contraction defined by

$$\Phi_g(\theta)(Vy) = \sum_{\{Kx : Vg^{-1}x = Vy\}} \theta(Kx).$$

The analogue of the Reiter condition for non-archimedean Polish groups is given by the following theorem.

Theorem 5.2. *Let G be a non-archimedean Polish group. Then the following are equivalent:*

(i) *G is amenable;*

(ii) *for every open subgroup $V \leq G$, finite set $Q \subseteq G$ with $1_G \in Q$, and every $\epsilon > 0$, there exists a function $\theta \in \ell^1(K\backslash G)$ such that $\theta \geq 0$, $\|\theta\|_1 = 1$ and for all $g \in Q$,*

$$\|\Phi_g(\theta) - \Phi_{1_G}(\theta)\|_1 < \epsilon,$$

where $K = \bigcap_{g\in Q} gVg^{-1}$.

Proof. (ii) $\Rightarrow$ (i). It suffices for every finite set $f_1, \ldots, f_n \in \mathrm{RUCB}(G)$ with $\|f_i\|_\infty = 1$ for all i, finite $Q \subseteq G$, and $\epsilon > 0$, to find a mean $m \in \mathrm{RUCB}^*(G)$ such that $|m(f_i) - m(g \cdot f_i)| < \epsilon$ for every $g \in Q$ and then apply a compactness argument. Since $f_1, \ldots, f_n \in \mathrm{RUCB}(G)$, there exists an open subgroup $V \leq G$ and functions $\tilde{f}_1, \ldots, \tilde{f}_n \in \ell^\infty(V\backslash G)$ of norm not greater than 1 such that for every $x \in G$ and every i, $|f_i(x) - \tilde{f}_i(Vx)| < \epsilon$. Let now $K \leq G$ and $\theta \in \ell^1(K\backslash G)$ be as in (ii) and for each coset $\tau \in K\backslash G$ choose a representative $x_\tau \in \tau$. Define $m \in \mathrm{RUCB}^*(G)$ by

$$m(f) = \sum_{\tau \in K\backslash G} \theta(\tau) f(x_\tau).$$

We have, for any $g \in Q$ and $i \leq n$,

$$\begin{aligned}
|m(g \cdot f_i) - m(f_i)| &= |\sum_{\tau\in K\backslash G} \theta(\tau) f_i(g^{-1}x_\tau) - \sum_{\tau\in K\backslash G} \theta(\tau) f_i(x_\tau)| \\
&\leq |\sum_{\tau\in K\backslash G} \theta(\tau) \tilde{f}_i(Vg^{-1}x_\tau) - \sum_{\tau\in K\backslash G} \theta(\tau) \tilde{f}_i(Vx_\tau)| + 2\epsilon \\
&= |\sum_{Kx\in K\backslash G} \theta(Kx) \tilde{f}_i(Vg^{-1}x) - \sum_{Kx\in K\backslash G} \theta(Kx) \tilde{f}_i(Vx)| + 2\epsilon \\
&= |\langle \tilde{f}_i, \Phi_g(\theta)\rangle - \langle \tilde{f}_i, \Phi_{1_G}(\theta)\rangle| + 2\epsilon \\
&\leq \|f_i\|_\infty \|\Phi_g(\theta) - \Phi_{1_G}(\theta)\|_1 + 2\epsilon \\
&< 3\epsilon.
\end{aligned}$$

(i) $\Rightarrow$ (ii). Now suppose that m is an invariant mean on $\mathrm{RUCB}(G)$. Fix an open $V \leq G$ and a finite $Q \subseteq G$. Let $K = \bigcap_{g\in Q} gVg^{-1}$. Since $\ell^\infty(K\backslash G)$ embeds in $\mathrm{RUCB}(G)$, we can consider m as an element of $\ell^\infty(K\backslash G)^*$.

Let D be the convex set of all positive functions in $\ell^1(K\backslash G)$ of norm 1 (which we also consider as a subset of $\ell^\infty(K\backslash G)^*$). By a standard Hahn–Banach argument, D is weak* dense in the set of means in $\ell^\infty(K\backslash G)^*$, so in particular, $m \in \overline{D}^{\mathrm{w}^*}$.

Now consider the following subset of $\ell^1(V\backslash G)^Q$:

$$E = \{(\Phi_g(\theta) - \Phi_{1_G}(\theta))_{g\in Q} : \theta \in D\}.$$

Claim. $0 \in \overline{E}^w$.

Proof. Let $\{\theta_\alpha\}_\alpha$ be a net in D that weak* converges to m. Fix $g \in Q$ and let $f \in \ell^\infty(V\backslash G)$. Let the function $f^g \in \ell^\infty(K\backslash G)$ be defined by $f^g(Kx) =$

$f(Vg^{-1}x)$. Note that if one considers both f and f^g as elements of $\mathrm{RUCB}(G)$ and a is the action $G \curvearrowright \mathrm{RUCB}(G)$ given by (5.1), then $f^g = a(g) \cdot f$. Since $\theta_\alpha \to m$, by the invariance of m, we have that

$$\langle f^g, \theta_\alpha \rangle \to \langle f^g, m \rangle = \langle a(g) \cdot f, m \rangle = \langle f, m \rangle. \tag{5.2}$$

We now have

$$\begin{aligned}
\langle f, \Phi_g(\theta_\alpha) - \Phi_{1_G}(\theta_\alpha) \rangle &= \sum_{Vy \in V\backslash G} f(Vy)\big(\Phi_g(\theta_\alpha)(Vy) - \Phi_{1_G}(\theta_\alpha)(Vy)\big) \\
&= \sum_{Vy \in V\backslash G} f(Vy)\big(\sum_{\{Kx : Vg^{-1}x = Vy\}} \theta_\alpha(Kx) - \sum_{\{Kx : Vx = Vy\}} \theta_\alpha(Kx)\big) \\
&= \sum_{Kx \in K\backslash G} \theta_\alpha(Kx) f(Vg^{-1}x) - \sum_{Kx \in K\backslash G} \theta_\alpha(Kx) f(Vx) \\
&= \langle f^g, \theta_\alpha \rangle - \langle f, \theta_\alpha \rangle.
\end{aligned}$$

Taking limits and applying (5.2), one obtains the claim. □

The set E is convex as an affine image of the convex set D. By the Claim and Mazur's theorem, we conclude that 0 is in the norm closure of E, whence the result. □

Remark 5.3. A related condition also appeared in a recent paper of Moore [Moo13]. Theorem 5.2, however, predates his work.

Condition (ii) in Theorem 5.2 seems difficult to verify in practice. Most known non-locally compact, amenable subgroups of S_∞ are either extremely amenable (and one checks this via the Ramsey property) or admit an increasing sequence of compact subgroups whose union is dense. Another practical way to verify whether a group is amenable is to calculate the universal minimal flow: then it is a matter of checking whether it admits an invariant measure. This method was used by Kechris and Sokić [KS12] to show that the automorphism group of the universal ultrahomogeneous partial order, among others, is *not* amenable. Their approach was further refined by Zucker [Zuc14a], who showed that the automorphism groups of the homogeneous directed graph $\mathbf{S}(3)$ and the boron tree homogeneous structure are also not amenable.

6 Unitary representations and property (T)

If G is a topological group, a *unitary representation* of G is an action $G \curvearrowright^\pi \mathcal{H}$ on a complex Hilbert space $\mathcal{H}$ by unitary isomorphisms. We will always assume that representations are continuous, i.e., for every $\xi \in \mathcal{H}$, the map $G \to \mathcal{H}$, $g \mapsto \pi(g) \cdot \xi$ is continuous. An important property of unitary representations is their *complete reducibility*: if $\mathcal{K} \subseteq \mathcal{H}$ is an invariant subspace, then $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$, where $\mathcal{K}^\perp$ is also invariant. A representation π is called *irreducible* if $\mathcal{H}(\pi)$ does not admit non-trivial invariant subspaces.

The main goal of representation theory is to describe the irreducible representations of a given group G and how other important representations are decomposed in terms of irreducibles. A satisfactory theory has been developed for locally compact groups (see, e.g., Folland [Fol95]): the Haar measure on the group ensures the existence of at least one non-trivial representation (the *left regular representation* $G \curvearrowright L^2(G)$), the Gelfand–Raikov theorem affirms that there always exists a sufficient supply of irreducibles, and there is an an elaborate theory for decomposing general representations into direct integrals of irreducibles. Decomposition of representations is particularly well behaved and understood for abelian and compact groups as well as, more generally, for the so called *groups of type I*.

Definition 6.1. A topological group is said to be of *type I* if for every representation π for which the von Neumann algebra generated by π is a factor, it is a factor of type I.

Groups of type I are those whose unitary dual can be classified. Glimm has shown that for a locally compact Polish group, the equivalence relation of isomorphism on the space of irreducible representations is smooth iff the group is of type I. In the the other direction, for discrete groups, Hjorth [Hjo97] showed that when G is not of type I (equivalently, abelian-by-finite [Tho68]), the irreducible representations of G are not classifiable by countable structures. In this section, following [Tsa12], we will see that non-archimedean, Roelcke precompact Polish groups are of type I and will provide an explicit description of their representations.

For non-locally compact groups, many of the basic results in harmonic analysis break down. There are groups that do not admit any non-trivial representations, but sometimes there are subtler problems: for example, the group $L^0(\mathbf{T})$ of measurable maps from the interval to the torus admits plenty of unitary representations but has no irreducible ones (see Solecki [Sol12] for a complete classification).

As for non-archimedean groups, continuous actions on discrete sets separate points, they always admit enough unitary representations. (If $G \curvearrowright M$ is a continuous action on a discrete set, there is a natural associated unitary representation on $\ell^2(M)$.) It turns out that for those groups, there are also plenty of *irreducible* representations and we have the following analogue of the Gelfand–Raikov theorem.

Theorem 6.2 ([Tsa12]). *Let G be a non-archimedean Polish group. Then for every $x, y \in G$, $x \neq y$, there exists a continuous, irreducible, unitary representation π of G such that $\pi(x) \neq \pi(y)$.*

This theorem indicates that for non-archimedean Polish groups, one can still hope for a satisfactory theory.

A natural class of Polish groups that includes the non-archimedean and the locally compact groups and is closed under subgroups and countable products was isolated in recent work of Kwiatkowska and Solecki [KS11], namely, the class of isometry groups of Polish locally compact metric spaces. Perhaps more informatively, a Polish group G belongs to this class iff every neighborhood of 1_G contains a closed subgroup H such that G/H is locally compact and the normalizer of H is open in G (see [KS11] for details). We are therefore led to the following question.

Question 6.3. If G is an isometry group of a locally compact Polish metric space, is it true that irreducible unitary representations of G separate points?

Despite the hurdles outlined above, it is possible to obtain rather explicit classification results for the unitary representations of some special classes of non-locally compact groups. In fact, it seems quite plausible that Roelcke precompact groups behave in a manner similar to compact groups as far as unitary representations are concerned.

Question 6.4. Let G be a Roelcke precompact Polish group. Is it true that G has only countably many irreducible representations and that every representation splits as a direct sum of irreducibles?

The main result of [Tsa12] is that this is indeed the case for Roelcke precompact, non-archimedean, Polish groups and moreover, for those groups, it is possible to describe the irreducible representations quite explicitly. Such descriptions are also known for the unitary group (Kirillov [Kir73] and Olshanski [Ols78]) and $\mathrm{Aut}(\mu)$ (Neretin [Ner96]). There are also many Roelcke precompact Polish groups that do not admit unitary representations whatsoever (see Section 9 for some examples).

In the rest of this section, we proceed to describe the representations in the non-archimedean case.

6.1 A description of the representations

Induction of representations is a standard method to obtain representations of a group G from representations of a subgroup H. One way to view induction is as the left adjoint of the functor that restricts a representation of G to a representation of H. In general, to be able to define an induced representation, one needs a measure on the homogeneous space G/H that is quasi-invariant with respect to the action of G. Such a measure always exists if G is locally compact but not in general. Recently, Ackermann, Freer and Patel [AFP12], following work of Petrov and Vershik [PV10], found a necessary and sufficient condition on H for an invariant probability measure to exist on G/H in the case where $G = S_\infty$; it is an interesting open question to decide what happens in general. In any case, here we will only need to induce from open subgroups, and in that case, one can use the counting measure.

Let H be an open subgroup of a topological group G and let σ be a representation of H on the Hilbert space $\mathcal{H}(\sigma)$. Let $\mathcal{K}$ be the Hilbert space of all functions $f\colon G \to \mathcal{H}(\sigma)$ that are H-equivariant, i.e. such that

$$f(xh^{-1}) = \sigma(h) \cdot f(x) \quad \text{for all } x \in G, h \in H, \tag{6.1}$$

equipped with the norm

$$\|f\|^2 = \sum_{g \in T} \|f(g)\|^2,$$

where T is a subset of G that meets each right H-coset in a single point. Then the action of G on $\mathcal{K}$

$$(g \cdot f)(x) = f(g^{-1}x)$$

defines a representation of G which is denoted by $\mathrm{Ind}_H^G(\sigma)$.

Of course, in general, the induction construction does not preserve irreducibility nor does the induced representation remember the group it was induced from. However, in the special case of inducing from open subgroups of Roelcke precompact groups, this does happen, provided that one takes some care. Say that an open subgroup $H \leq G$ *has no supergroups of finite index* if there is no group H' such that $H < H' \leq G$ and $[H' : H] < \infty$. If M is an ω-categorical, discrete structure and $G = \mathrm{Aut}(M)$, open subgroups of G with no supergroups of finite index are exactly the stabilizers of algebraically closed elements of M^{eq}.

Proposition 6.5 ([Tsa12]). *Let G be Roelcke precompact. Then the following hold:*

(i) *If $H \leq G$ has no supergroups of finite index, $V \trianglelefteq H$ is open, and σ is a representation of H/V, then $\mathrm{Ind}_H^G(\sigma)$ is irreducible iff σ is.*

(ii) *If $H_1, H_2 \leq G$ have no supergroups of finite index, $V_1 \trianglelefteq H_1$, $V_2 \trianglelefteq H_2$ are open, and σ_1, σ_2 are irreducible representations of H_1/V_1, H_2/V_2, respectively, then $\mathrm{Ind}_{H_1}^G(\sigma_1) \cong \mathrm{Ind}_{H_2}^G(\sigma_2)$ iff there exists $g \in G$ such that $H_2 = H_1^g$ and $\sigma_2 \cong \sigma_1^g$, where $H_1^g = gH_1g^{-1}$ and the representation σ_1^g of H_2 is defined by $\sigma_1^g(h) = \sigma_1(g^{-1}hg)$.*

If G is a Roelcke precompact subgroup of S_∞, it turns out that the representations in Proposition 6.5 exhaust all irreducible representations of G.

Theorem 6.6 ([Tsa12]). *Let G be a Roelcke precompact, non-archimedean Polish group. Then every irreducible unitary representation of G is of the form $\mathrm{Ind}_H^G(\sigma)$, where $H \leq G$ is an open subgroup, $V \trianglelefteq H$, is a normal, open subgroup of finite index, and σ is an irreducible representation of the group H/V. Moreover, every unitary representation of G is a sum of irreducibles.*

Special cases of Theorem 6.6 had been known before: Lieberman [Lie72] had classified the representations of S_∞ and Olshanski [Ols85] had described the representations of a certain Roelcke precompact subgroup of $\mathrm{GL}(\infty, \mathbf{F}_q)$.

As every Roelcke precompact, non-archimedean group has only countably many open subgroups, Theorem 6.6 implies that every such group has only countably many irreducible representations.

If M is an ω-categorical structure, Theorem 6.6 allows to describe the representations of $G = \mathrm{Aut}(M)$ in terms of the structure M. In general, this requires understanding the *imaginary elements* of M and we will refer the reader to [Tsa12] for details; instead we will describe several simple examples in which we have *elimination of imaginaries*, so no difficulties appear.

- If $G = \mathrm{Aut}(\mathbf{Q})$, then all irreducible representations of G are of the form $G \curvearrowright \ell^2(\mathbf{Q}^{[n]})$, where $\mathbf{Q}^{[n]}$ denotes the set of n-element subsets of $\mathbf{Q}$.
- If $G = S_\infty$, then the irreducible representations of G are indexed by pairs (n, σ), where n is a natural number and σ is an irreducible representation of the finite symmetric group S_n.
- If G is the automorphism group of the random graph, then the irreducible representations of G (up to isomorphism) are indexed by pairs (A, τ), where A is a finite graph and τ is an irreducible representation of the finite group $\mathrm{Aut}(A)$.

- If $G = \mathrm{GL}(\infty, \mathbf{F}_q)$, the irreducible representations of G are indexed by pairs (n, σ), where n is a natural number and σ is an irreducible representation of $\mathrm{GL}(n, \mathbf{F}_q)$.

6.2 Property (T)

Property (T) was introduced by Kazhdan [Kaz67], where he used it to show that lattices in higher rank, simple Lie groups are finitely generated, and since then has found a large number of applications in group theory, ergodic theory, and operator algebras. We refer the reader to [BHV08] for an account of the basic theory.

Definition 6.7. Suppose G is a topological group, $Q \subseteq G$ and $\epsilon > 0$. If $\pi \colon G \to U(\mathcal{H})$ is a unitary representation of G, we say that a non-zero vector $\xi \in \mathcal{H}$ is *(Q, ϵ)-invariant* for π if $\sup_{x \in Q} \|\pi(x)\xi - \xi\| < \epsilon \|\xi\|$.

We say that (Q, ϵ) is a *Kazhdan pair* for G if every unitary representation π of G that admits a (Q, ϵ)-invariant vector also admits a (non-zero) invariant vector. A compact set $Q \subseteq G$ is a *Kazhdan set* for G if there is $\epsilon > 0$ such that (Q, ϵ) is a Kazhdan pair. G has *Kazhdan's property (T)* if admits a Kazhdan set.

Most of the applications of property (T) have been found in the realm of locally compact groups, most notably discrete groups, where it is, in general, impossible to classify the unitary representations.

There is another closely related property, called property (FH), which states that every affine isometric action of the group on a Hilbert space has a fixed point. In general, property (T) implies property (FH) and the reverse implication is true for locally compact Polish groups. It is easy to see that all Roelcke precompact groups have property (FH) (they have the stronger property that every time they act on a metric space, all orbits are bounded; see Rosendal [Ros09a] for more on this); however proving the stronger property (T) requires some understanding of the representations and combinatorial work.

Property (T) was proved in [Tsa12] for a large class of Roelcke precompact, non-archimedean Polish groups but the question whether it holds for all of them was left open. This question was positively resolved by Evans and the author [ET13], where the following theorem was proved.

Theorem 6.8 ([ET13]). *Let G be a non-archimedean, Roelcke precompact Polish group. Then G has property (T). Moreover, if G has only finitely many open subgroups of finite index, it admits a finite Kazhdan set.*

The main combinatorial result if [ET13] that, combined with the techniques from [Tsa12], permits to deduce Theorem 6.8 is the following.

Theorem 6.9 ([ET13]). *Suppose G is a non-archimedean, Roelcke precompact Polish group and G° is the intersection of the open subgroups of finite index in G. Suppose $G^\circ \neq \{1_G\}$. Then there exist $f, g \in G^\circ$ which generate a (non-abelian) free subgroup F of G° with the property that if $V \leq G$ is open and of infinite index, then F acts freely on the coset space G/V.*

Combining Theorem 6.8 with a result of Glasner and Weiss [GW97], one obtains the following interesting corollary.

Corollary 6.10 ([ET13]). *Let G be a non-archimedean, Roelcke precompact, Polish group and $G \curvearrowright X$ a continuous action on a compact Hausdorff space X. Then the simplex of G-invariant measures on X is a* Bauer simplex*, i.e., the set of its extreme points is closed.*

As we currently lack sufficient information about unitary representations of general Polish Roelcke precompact groups, the following question remains open.

Question 6.11 ([Tsa12]). Does every Roelcke precompact Polish group have property (T)?

7 Invariant measures

7.1 A generalization of de Finetti's Theorem

Every measure-preserving action of a group G on a probability space (X, μ) gives rise to a natural unitary representation on $L^2(X, \mu)$, the *Koopman representation* associated with the action, defined by:

$$(g \cdot f)(x) = f(g^{-1} \cdot x), \quad \text{for } g \in G, f \in L^2(X).$$

The Koopman representation is a classical tool in ergodic theory and many important concepts can be expressed in terms of this representation: for example, ergodicity, weak mixing, mixing, etc. In principle, unitary representations are easier to understand than measure-preserving actions thus provide a useful tool for understanding the original action. In view of this, it is natural to expect that an understanding of the unitary representations of a group will give us some information about its measure-preserving actions.

If $G = \mathrm{Aut}(M)$ is the automorphism group of an ω-categorical, discrete structure M, it turns out that the theory of *exchangeable random variables* [Kal05] gives a natural source of measure-preserving actions of G as follows.

Let $\{\xi_a : a \in M\}$ be a family of random variables indexed by the structure M whose joint distribution satisfies the following condition $(*)$:

> if $\bar{a}$ and $\bar{b}$ are tuples in M of length n that have the same quantifier-free type, then $(\xi_{a_0}, \ldots, \xi_{a_{n-1}})$ and $(\xi_{b_0}, \ldots, \xi_{b_{n-1}})$ have the same distribution.

An easy example of this situation is when the variables $\{\xi_a : a \in M\}$ are independent and the classical de Finetti theorem states that in the case where M is a countable set without structure, this is essentially the only possible situation: every family that satisfies the condition $(*)$ is a mixture of *iid* (independent identically distributed) random variables. Ryll-Nardzewski [Ryl57] generalized this theorem by proving that the same conclusion holds if M is the set of natural numbers equipped with its order (thus weakening the hypothesis).

Without loss of generality, we can assume that the random variables take values in $[0, 1]$, so the question becomes to classify all measures on $[0, 1]^M$

that satisfy the invariance condition (∗). Observe that (∗) depends only on the age of M in the sense that if M and M' have the same age, then any distribution on $[0,1]^M$ satisfying (∗) gives rise to a distribution on $[0,1]^{M'}$ satisfying (∗) and vice versa. If the age satisfies the amalgamation property, it makes sense to assume that M is homogeneous (so, for example, in the theorem of Ryll-Nardzewski consider $(\mathbf{Q}, <)$ in the place of $(\mathbf{N}, <)$) in order to obtain a dynamical system of the group $\mathrm{Aut}(M)$. Then the problem of classifying all joint distributions that satisfy the condition (∗) can be rephrased as follows.

Question 7.1. Assume $G \curvearrowright M$ is an oligomorphic action on a countable set. What are the probability measures on $[0,1]^M$ that are invariant and ergodic with respect to the natural action $G \curvearrowright [0,1]^M$?

Recall that a probability measure μ on a standard Borel space X invariant under an action of a group G is *ergodic* if every measurable set $A \subseteq X$ that satisfies $\mu(A \triangle g \cdot A) = 0$ for all $g \in G$ has measure 0 or 1. The ergodicity assumption in Question 7.1 entails no loss of generality: the ergodic decomposition theorem [Phe01] ensures that every invariant measure can be represented as an integral of ergodic measures.

The following theorem answers this question in a very particular case, where the conclusion is the same as in de Finetti's theorem: the variables must be independent. In order to state it, we will need two auxiliary notions from model theory. We say that an ω-categorical structure M has *no algebraicity* if for every finite set $A \subseteq M$, the pointwise stabilizer G_A of A in G has infinite orbits on $M \setminus A$. M has *weak elimination of imaginaries* if for every open subgroup $V \leq G$, there exists a finite $A \subseteq M$ such that $G_A \leq V$ and $[V : G_A] < \infty$. Of the discrete examples considered in Section 2, the set without structure, the dense linear order, the random graph, and the ultrahomogeneous K_m-free graphs have both of those properties.

Theorem 7.2. *Let M be an ω-categorical structure with no algebraicity and weak elimination of imaginaries and $G = \mathrm{Aut}(M)$. Then the only ergodic, G-invariant measures on $[0,1]^M$ are of the type ν^M, where ν is a Borel measure on $[0,1]$.*

Proof. Let μ be an invariant ergodic measure on $[0,1]^M$. For $a \in \mathbf{M}$, denote by ξ_a the random variable on $[0,1]^M$, which is the projection on the a-th coordinate. It suffices to prove that if A and B are disjoint subsets of M and η_A and η_B are bounded random variables that are measurable with respect to the σ-fields generated respectively by $\{\xi_a : a \in A\}$ and $\{\xi_b : b \in B\}$, then η_A and η_B are uncorrelated, i.e.,

$$\mathbf{E}(\eta_A - \mathbf{E}\,\eta_A)(\eta_B - \mathbf{E}\,\eta_B) = 0.$$

By replacing η_A and η_B by $\eta_A - \mathbf{E}\,\eta_A$ and $\eta_B - \mathbf{E}\,\eta_B$ respectively, we can assume that η_A and η_B have expectation 0, and therefore belong to the Hilbert space

$$L^2_0([0,1]^M) = \{\xi \in L^2([0,1]^M, \mu) : \mathbf{E}\,\xi = 0\}.$$

By ergodicity, $L^2_0([0,1]^M)$ has no invariant vectors, so, by Theorem 6.6 and the assumption of weak elimination of imaginaries, every irreducible subrepresentation π of $L^2_0([0,1]^M)$ has the form $\mathrm{Ind}^G_{G_{(C)}}(\sigma)$, where C is a finite substructure of M, $G_{(C)}$ is the setwise stabilizer of C, and σ is an irreducible representation of $G_{(C)}/G_C$. Let π be such an irreducible subrepresentation and let f_A, f_B be the projections of η_A, η_B on $\mathcal{H}(\pi)$ viewed as functions $G \to \mathcal{H}(\sigma)$ as in (6.1).

Suppose, for contradiction, that $\langle f_A, f_B \rangle \neq 0$. Then there exists $g_0 \in G$ such that $f_A(g_0) \neq 0$ and $f_B(g_0) \neq 0$. As f_A is fixed by $G_{(A)}$, we have that $[G_A : (G_A \cap g_0 G_C g_0^{-1})] < \infty$ (otherwise the norm of f_A would be infinite). By the no algebraicity assumption, $G_A \leq g_0 G_C g_0^{-1}$ and therefore $g_0 \cdot C \subseteq A$. Similarly, $g_0 \cdot C \subseteq B$, which contradicts the assumption that A and B are disjoint. □

Easy examples show that if one wants to obtain independence in the conclusion, the assumptions above are optimal. However, in the cases $G = S_\infty$ and $G = \mathrm{Aut}(\mathbf{Q})$, there are classification results, due to Aldous [Ald81], Hoover [Hoo79], and Kallenberg [Kal89] (see also Austin [Aus08]), where one considers actions on tuples, that is, a classification of the invariant measures on $[0,1]^{\mathbf{N}^k}$ and $[0,1]^{\mathbf{Q}^k}$. In that case, there exist interesting factor maps $[0,1]^{\mathbf{N}^m} \to [0,1]^{\mathbf{N}^k}$ for $m < k$ that produce invariant, ergodic measures on $[0,1]^{\mathbf{N}^k}$ by pushing forward the product measure on $[0,1]^{\mathbf{N}^m}$. A typical example is given by the map $\Phi \colon 2^{\mathbf{N}} \to 2^{\mathbf{N}^2}$ defined by

$$\Phi(x)(i,j) = 1 \iff x(i) \neq x(j).$$

Of course, the actions $S_\infty \curvearrowright \mathbf{N}^k$ and $\mathrm{Aut}(\mathbf{Q}) \curvearrowright \mathbf{Q}^k$ are oligomorphic but they have both algebraicity and imaginaries and the simple-minded approach in the proof of Theorem 7.2 does not work. However it remains plausible that, using more sophisticated techniques, it is possible to classify the invariant measures of the action $G \curvearrowright [0,1]^M$ for an arbitrary oligomorphic action $G \curvearrowright M$.

7.2 Unique ergodicity

In [AKL12], Angel, Kechris, and Lyons studied the phenomenon of unique ergodicity for minimal flows of automorphism groups of classical, ultrahomogeneous structures. Recall that a G-flow X is *uniquely ergodic* if it carries a unique G-invariant measure. If the universal minimal flow of an amenable group G is uniquely ergodic, then every minimal G-flow is uniquely ergodic.[1] Quite remarkably, *all* known metrizable universal minimal flows of amenable Polish groups are uniquely ergodic and it is an open question whether this is always the case.

Question 7.3 ([AKL12]). Let G be an amenable Polish group with a metrizable universal minimal flow $M(G)$. Must $M(G)$ be uniquely ergodic?

The first example of this phenomenon was provided by Glasner and Weiss [GW02] who showed that the flow $S_\infty \curvearrowright \mathrm{LO}$ is uniquely ergodic. Angel, Kechris, and Lyons found many more examples, including the automorphism groups of the random graph and the K_m-free ultrahomogeneous graphs. In this subsection, we provide an alternative proof of their theorem as an application of Theorem 7.2.

[1] I am grateful to Eli Glasner for explaining to me the proof of this fact.

Theorem 7.4 (Angel–Kechris–Lyons)**.** *Let M be the random graph or the K_m-free ultrahomogeneous graph ($m \geq 3$) and let $G = \mathrm{Aut}(M)$. Then the universal minimal flow of G is uniquely ergodic.*

By [KPT05], the universal minimal flow of G is $\mathrm{LO}(M)$, the set of all linear orderings on M. There is a natural measure μ_0 on $\mathrm{LO}(M)$ invariant under the full permutation group $S(M)$ defined as follows. Let $\rho\colon [0,1]^M \to \mathrm{LO}(M)$ be the $S(M)$-equivariant map defined by

$$a <_{\rho(x)} b \iff x(a) < x(b) \tag{7.1}$$

and let μ_0 be the push-forward of λ^M by π, where λ denotes the Lebesgue measure on $[0,1]$. The map ρ is well-defined only on the set

$$\{x \in [0,1]^M : x(a) \neq x(b) \text{ for all } a, b \in M, a \neq b\}$$

but this set has full measure. In fact, one can use in the place of λ any other non-atomic measure ν on $[0,1]$ and will obtain the same measure on $\mathrm{LO}(M)$ because $S(M) \curvearrowright \mathrm{LO}(M)$ is uniquely ergodic [GW02]. To see this, note that, by invariance, one must have, for any $S(M)$-invariant measure μ on $\mathrm{LO}(M)$,

$$\mu(\{x \in \mathrm{LO}(M) : a_1 <_x a_2 <_x \cdots <_x a_n\}) = 1/n!$$

for every n-tuple of distinct elements $a_1, \ldots, a_n \in M$ and those equations determine the measure μ uniquely. Our goal is to show that μ_0 is the only measure invariant under the smaller group G.

Let $F_0 \subseteq F_1 \subseteq \cdots$ be a collection of finite substructures of M with $\bigcup_n F_n = M$. For $a \in M$, denote

$$D(a) = \{b \in M : b \neq a \text{ and } b \text{ and } a \text{ are not connected by an edge}\}.$$

Fix a G-invariant measure μ on $\mathrm{LO}(M)$.

Lemma 7.5. *Let $a \in M$ and let $A \subseteq D(a)$ be a definable, infinite set. Then for μ-a.e. x,*

$$\lim_{n\to\infty} \frac{\#\{b \in F_n \cap A : b <_x a\}}{\# F_n \cap A}$$

exists and is independent of A.

Proof. Denote by G_a the stabilizer of a in G and note that G_a acts oligomorphically on $D(a)$ in a way that satisfies the hypothesis of Theorem 7.2. Let the G_a-map $\pi_a \colon \mathrm{LO}(M) \to 2^{D(a)}$ be defined by

$$\pi_a(x) = \{b \in D(a) : b <_x a\}.$$

Then $(\pi_a)_*\mu$ is a G_a-invariant measure on $2^{D(a)}$ and by Theorem 7.2 and the ergodic decomposition theorem, there exists a measure ν on $[0,1]$ such that

$$(\pi_a)_*\mu = \int \kappa_p \,\mathrm{d}\nu(p),$$

where κ_p denotes the Bernoulli $(p, 1-p)$ measure on $2 = \{0,1\}$. Let $B \subseteq \mathrm{LO}(M)$ be the set of x for which the conclusion of the lemma is satisfied, i.e., the limit exists and is the same for all definable sets A. By the strong law of large numbers, $\kappa_p(\pi_a(B)) = 1$ for all p, whence $\mu(B) = 1$. □

Note that the hypothesis that A is definable is not really important; what matters is that A belongs to a predetermined countable collection of infinite sets.

Proof of Theorem 7.4. For $a \in M$, define

$$\eta_a(x) = \lim_{n\to\infty} \frac{\#\{b \in F_n \cap D(a) : b <_x a\}}{\#F_n \cap D(a)}$$

By Lemma 7.5, η_a is an a.e. well-defined random variable. Let $\pi\colon \mathrm{LO}(M) \to [0,1]^M$ be defined by $\pi(x)(a) = \eta_a(x)$. π is a G-equivariant map and by the ergodicity assumption on μ, $\pi_*\mu$ is a G-invariant ergodic measure on $[0,1]^M$. Applying Theorem 7.2 one more time, we obtain that $\{\eta_a : a \in M\}$ are iid.

First, we check that almost surely,

$$a <_x b \implies \eta_a(x) \le \eta_b(x).$$

Indeed, let x be such that $a <_x b$. By Lemma 7.5, as $D(a) \cap D(b)$ is infinite and definable, almost surely,

$$\eta_a(x) = \lim_{n\to\infty} \frac{\#\{c \in F_n \cap D(a) \cap D(b) : c <_x a\}}{\#F_n \cap D(a) \cap D(b)}.$$

If $c <_x a$, then $c <_x b$, so

$$\eta_a(x) \le \lim_{n\to\infty} \frac{\#\{c \in F_n \cap D(a) \cap D(b) : c <_x b\}}{\#F_n \cap D(a) \cap D(b)} = \eta_b(x).$$

Next, we see that for all $a \neq b$, almost surely $\eta_a \neq \eta_b$. Suppose this is not the case. Then because η_a and η_b are independent, the distribution of η_a must have an atom, i.e., there exists $p \in [0,1]$ such that $\mathbf{P}(\eta_a = p) > 0$. In particular, almost surely, for infinitely many $a \in M$, $\eta_a = p$. Therefore, by invariance, for fixed $a, b, c \in M$ such that there are no edges between a, b, and c, the event

$$a <_x b <_x c \text{ and } \eta_a = \eta_b = \eta_c = p$$

has positive probability. Let

$$q = \mathbf{P}(a <_x b <_x c \mid \eta_a = \eta_c = p \text{ and } a <_x c). \tag{7.2}$$

By the preceding discussion, $q > 0$. If b varies in $D(a) \cap D(c)$, q does not change as, by invariance, it depends only on the type of the triple (a, b, c) in M. (7.2) yields that for every n,

$$\mathbf{E}\Big(\frac{\#\{b \in F_n \cap D(a) \cap D(c) : a <_x b <_x c\}}{\#F_n \cap D(a) \cap D(c)} \;\Big|\; \eta_a = \eta_c = p \text{ and } a <_x c\Big) = q.$$

On the other hand, taking limits,

$$\begin{aligned}
&\mathbf{E}\Big(\lim_{n\to\infty} \frac{\#\{b \in F_n \cap D(a) \cap D(c) : a <_x b <_x c\}}{\#F_n \cap D(a) \cap D(c)} \;\Big|\; \eta_a = \eta_c = p \text{ and } a <_x c\Big) \\
&= \mathbf{E}(\eta_c - \eta_a \mid \eta_a = \eta_c = p \text{ and } a <_x c) \\
&= 0,
\end{aligned}$$

contradiction.

We conclude that almost surely, for all $a, b \in M$,

$$a <_x b \implies \eta_a(x) < \eta_b(x).$$

As $<_x$ is a linear ordering, this implies that

$$a <_x b \iff \eta_a(x) < \eta_b(x). \tag{7.3}$$

Therefore the map $\pi\colon (\mathrm{LO}(M), \mu) \to ([0,1]^M, \nu^M)$, where ν is the distribution of η_a, is measure-preserving and a.e. one-to-one. Let $\rho\colon [0,1]^M \to \mathrm{LO}(M)$ be its inverse. Then ρ is given by (7.3), which is the same as the map defined by (7.1). As $\mu = \rho_*(\nu^M)$, we conclude that $\mu = \mu_0$, thus completing the proof. □

Remark. The proof above does not use in an essential way the hypothesis that the structure M is a graph; however, for the moment, it is not clear what the correct generality is for the statement of Theorem 7.4 and I have preferred to present this most transparent case.

8 Reducts of ω-categorical structures

Let M be a classical, ω-categorical structure. If $R \subseteq M^k$ is a relation on M and $f\colon M \to M$ is a function, say that f *preserves* R if

$$\forall a_1, \ldots, a_k \in M \quad R(a_1, \ldots, a_k) \implies R(f(a_1), \ldots, f(a_k)).$$

It is a consequence of the Ryll-Nardzewski theorem that the relations preserved by the automorphism group of M are exactly the relations that are first-order definable in M.

Recall that a *reduct* of M is a structure with the same underlying set as M and a collection of relations each of which is first-order definable in M. Naturally, the automorphism group of a reduct M' of M is a closed group of permutations of M that contains $\mathrm{Aut}(M)$. Conversely, as a closed group of permutations of M is entirely determined by its orbits on powers of M, it follows from the above remark that every such group corresponds to a reduct of M and two reducts have the same automorphism group iff they are *first-order interdefinable*, i.e., the relations in each reduct are first-order definable in the other. Thus the lattice of closed permutation groups of M between $\mathrm{Aut}(M)$ and $S(M)$ is isomorphic to the lattice of reducts of M with the partial order "is first-order definable in." For many structures of interest, it is possible to completely classify their reducts: for example, we saw in Section 4 a list of all reducts of $(\mathbf{Q}, <)$. Thomas [Tho91] classified the reducts of the random graph and his results prompted him to make the following conjecture.

Conjecture 8.1 (Thomas)**.** *Let M be a classical, ultrahomogeneous structure in a finite, relational language. Then M has only finitely many reducts up to first-order interdefinability.*

This conjecture has motivated a number of papers that have verified it in particular cases but so far no general results have been obtained. The main tool used in the proofs is structural Ramsey theory and it seems that a prerequisite for proving the conjecture (at least by using this type of techniques) is a general Ramsey theorem for homogeneous structures. This was an important motivation for asking Question 3.4 in [BPT13].

In theoretical computer science, and more precisely, in the study of *constraint satisfaction problems* (see Bodirsky [Bod12] for more on that), it is important to consider reducts up to finer notions of equivalence than first-order interdefinability, by restricting the type of formulas used in the definitions. Two of the most important classes of formulas considered are the *existential formulas*, i.e., formulas of the type $\exists \bar{y}\ \phi(\bar{x}, \bar{y})$, where $\phi(\bar{x}, \bar{y})$ is quantifier-free and the subclass of *primitive positive formulas*, where $\phi(\bar{x}, \bar{y})$ must be a conjunction of positive literals. There are analogous preservation theorems for these types of formulas: the relations definable by existential formulas are exactly the ones preserved by the self-embeddings of M and the relations definable by primitive positive formulas are the ones preserved by the polymorphisms of M. Thus the lattice of closed monoids of injective functions $M \to M$ containing the monoid of self-embeddings of M corresponds to the lattice of reducts up to existential interdefinability and the lattice of closed clones containing the polymorphism clone of M corresponds to the lattice of reducts up to primitive positive interdefinability.

The following theorem, proved jointly with M. Bodirsky and M. Pinsker in [BPT13], provides some information about those lattices in a rather general situation. If Γ and Δ are closed monoids of injective functions on M, say that Γ is *minimal above* Δ if $\Delta < \Gamma$ and there is no closed monoid Γ' such that $\Delta < \Gamma' < \Gamma$. If N is a structure, denote by $\mathrm{Emb}(M)$ the monoid of self-embeddings of M.

Theorem 8.2 ([BPT13])**.** *Let M be a structure ultrahomogeneous in a finite relational signature and suppose that it has the Ramsey property. Let M' be a reduct of M in a finite signature. Then there are only finitely many minimal closed monoids above* $\mathrm{Emb}(M')$ *and every closed monoid of injective functions on M properly containing* $\mathrm{Emb}(M')$ *contains a minimal one.*

A similar theorem holds for polymorphism clones.

Theorem 8.2 allows to decide algorithmically the problem of definability in this setting. For this, one needs an additional finiteness assumption that ensures that the original structure is effectively presented. We say that a ultrahomogeneous structure M in a finite relational language is *finitely bounded* if there exists a finite collection $\mathcal{F}$ of finite structures in the language of M such that for every finite structure $\mathbf{A}$,

$$\mathbf{A} \in \mathrm{Age}(M) \iff \text{no element of } \mathcal{F} \text{ embeds into } \mathbf{A}.$$

The elements of $\mathcal{F}$ are called *forbidden configurations*. Typical examples of structures with this property are $(\mathbf{Q}, <)$, the random graph, and the K_m-free graphs.

Theorem 8.3 ([BPT13])**.** *Let M be a finitely bounded, ultrahomogeneous structure with the Ramsey property. Then there exists an algorithm which given formulas $\phi_1, \dots, \phi_n$ and ψ, decides whether ψ can be defined using $\phi_1, \dots, \phi_n$ as basic relations by an existential (primitive positive) formula.*

Unfortunately, our methods do not allow us to replace monoids by groups in Theorem 8.2 and existential formulas by general first-order formulas in Theorem 8.3, so whether the conclusions of those two theorems hold in that situation remains an open problem.

9 Connections with model theory

It is a well-known fact in model theory that was first formulated by Ahlbrandt and Ziegler [AZ86] (but is really already visible in the Ryll-Nardzewski theorem) that the automorphism group of an ω-categorical structure contains all essential information about the structure. More precisely, the category of ω-categorical structures with interpretations (up to homotopy) as morphisms is equivalent to the category of Roelcke precompact Polish groups with group morphisms $\phi\colon G \to H$ such that $\overline{\phi(G)}$ is co-precompact in H. In particular, if M and N are ω-categorical structures, $\mathrm{Aut}(M) \cong \mathrm{Aut}(N)$ iff M and N are bi-interpretable. This correspondence has also recently been extended to the continuous setting by Ben Yaacov and Kaïchouh [BK14].

Remark 9.1. For the equivalence of categories, one does need the continuous setting even for non-archimedean groups: there is no ω-categorical, classical structure that can be associated to a compact infinite group.

This means that all properties that are stable under bi-interpretability (which includes the vast majority of concepts studied in model theory) correspond, in the case of ω-categorical structures, to properties of the automorphism groups. One could thus hope to be able to use the well-developed and sophisticated model-theoretic machinery to gain better understanding about the groups, and, conversely, apply tools from the theory of dynamical systems to obtain results in model theory. This was the starting point of our joint work with Itaï Ben Yaacov [BT13] the results of which are described in this section.

9.1 Functions as formulas

Let M be an ω-categorical structure and let $G = \mathrm{Aut}(M)$. Recall (Theorem 2.4) that then G is Roelcke precompact. By the Ryll-Nardzewski theorem, a formula $\phi(x)$ (in order to avoid complicating the notation, we will only consider formulas on infinitely many variables) is nothing but a bounded, uniformly continuous function $\phi\colon M^{\mathbf{N}} \to \mathbf{C}$ invariant under the diagonal action of G.

Let $a_0 \in M^{\mathbf{N}}$ be a fixed tuple that enumerates a dense subset of M. Every formula $\phi(x, y)$ gives rise to a (both left and right) uniformly continuous, bounded function $\tilde{\phi}\colon G \to \mathbf{C}$ defined by

$$\tilde{\phi}(g) = \phi(a_0, g \cdot a_0). \tag{9.1}$$

Conversely, every function in $\mathrm{UCB}(G)$ is of the form $\tilde{\phi}$ for some ϕ.

The *Roelcke compactification* of G, denoted by $R(G)$, is the completion of the Roelcke uniformity, or equivalently, the Gelfand space of the algebra $\mathrm{UCB}(G)$. $R(G)$ is equipped with a continuous involution $x \mapsto x^*$ that extends the map $G \to G$, $g \mapsto g^{-1}$ and a left and right actions of G that commute. Another way to view $R(G)$ is as the quotient $(\widehat{G}_L \times \widehat{G}_L) /\!\!/ G$, where $\widehat{G}_L$

denotes the completion of (G, d_L) for some left-invariant metric d_L on G and G acts on $\widehat{G}_L \times \widehat{G}_L$ diagonally by isometries. The correspondence between formulas and functions gives us yet another representation of the Roelcke compactification of G: namely, as the space of types of two copies of M (that is, the ways two copies of M can be placed relative to each other in a third one). The reason for this is that the elements of $\widehat{G}_L$ can be naturally identified with the elementary embeddings of M into itself. To illustrate how this works in practice, we briefly describe two simple examples.

- If M is a countable set without structure and $G = S_\infty$, the "relative position" of two copies M_1 and M_2 of M is determined by the information which elements of M_1 coincide with which elements of M_2, that is, a partial bijection between M_1 and M_2. Accordingly, the Roelcke compactification of S_∞ is the set of partial bijections $M \to M$ with the topology inherited from $2^{M \times M}$. (Here we identify both M_1 and M_2 with M via the given isomorphisms.)
- If M is a separable, infinite-dimensional Hilbert space $\mathcal{H}$, the "relative position" of two copies $\mathcal{H}_1$ and $\mathcal{H}_2$ is determined by the values of the inner product $\langle \xi_1, \xi_2 \rangle$, $\xi_1 \in \mathcal{H}_1$, $\xi_2 \in \mathcal{H}_2$. Identifying $\mathcal{H}_1$ and $\mathcal{H}_2$ with $\mathcal{H}$, this defines a sesquilinear form $\langle \cdot, \cdot \rangle_p$ on $\mathcal{H}$ satisfying $|\langle \xi, \eta \rangle_p| \leq 1$ for ξ and η in the unit ball, or, which is the same, a linear contraction $T \colon \mathcal{H} \to \mathcal{H}$ defined by $\langle T\xi, \eta \rangle = \langle \xi, \eta \rangle_p$. We conclude that the Roelcke compactification of $U(\mathcal{H})$ is the space of linear contractions of $\mathcal{H}$ equipped with the weak operator topology. See [BT13] for more examples.

A number of important tameness concepts in model theory, for example, *stability, NIP, simplicity*, etc., are defined *locally*, i.e., in terms of single formulas rather than theories. Typically one considers a formula $\phi(x, y)$ on two groups of variables x and y and declares it to be nice if a certain type of combinatorial configuration does not appear in the directed graph defined by the formula. Most of the concepts were isolated by Shelah while developing his *classification theory* and by now they have found a number of important applications in algebra and combinatorics.

The correspondence (9.1) between functions and formulas allows us to translate directly between those model-theoretic properties and properties of functions on the automorphism group. Often, we arrive at concepts that have been independently considered and studied in dynamical systems and more specifically, in the theory of representations of topological dynamical systems on Banach spaces as developed by Glasner and Megrelishvili (see [GM12] and the references therein for more details). Below we give a couple of examples of how this correspondence can be exploited but it seems likely that more connections and applications will be discovered.

In classical logic, a formula $\phi(x, y)$ is called *stable* if there are no tuples $(a_n)_{n \in \mathbf{N}}$, $(b_n)_{n \in \mathbf{N}}$ in a model M such that

$$M \models \phi(a_i, b_j) \iff i < j,$$

i.e., the formula ϕ cannot order an infinite set. When properly translated to the continuous setting, this becomes a positive statement: ϕ is *stable* iff

$$\lim_{m \to \infty} \lim_{n \to \infty} \phi(a_m, b_n) = \lim_{n \to \infty} \lim_{m \to \infty} \phi(a_m, b_n) \tag{9.2}$$

for all tuples $(a_n)_{n\in\mathbf{N}}$, $(b_n)_{n\in\mathbf{N}}$ for which both limits exist. A theory is *stable* if every formula is.

This exchanging limits condition first appeared in the work of Grothendieck [Gro52] where he showed that it characterizes weak compactness. This gives a very natural correspondence between stable formulas on the model-theoretic side and weakly almost periodic functions on the automorphism group. Recall that a function $f \in \mathrm{RUCB}(G)$ is *weakly almost periodic (WAP)* if the orbit $G \cdot f$ is weakly precompact in the Banach space $\mathrm{RUCB}(G)$. Applying Grothendieck's characterization of weak compactness, one can deduce the following theorem.

Theorem 9.2 ([BT13]). *Let G be the automorphism group of an $\aleph_0$-categorical structure $\mathbf{M}$. Then the following are equivalent:*

(i) $\mathrm{Th}(\mathbf{M})$ *is stable;*

(ii) *Every Roelcke uniformly continuous function on G is weakly almost periodic.*

In the language of Banach space representations, stable formulas correspond to functions coming from representations on *reflexive* spaces.

If the equivalent conditions of Theorem 9.2 are satisfied (for instance, in the two examples above, S_∞ and $U(\mathcal{H})$), one can define an associative, separately continuous multiplication on the Roelcke compactification $R(G)$, so that it becomes a *semitopological semigroup*. This phenomenon is well understood in the theory of dynamical systems: in fact, $W(G)$, the Gelfand space of the algebra $\mathrm{WAP}(G)$, is the maximal compactification of G carrying the structure of a semitopological semigroup; but it is also possible to describe the multiplication on $W(G)$ model-theoretically using the notion of *stable independence*. We refer the reader to [BT13] for more details.

Very recently, in a similar spirit, Ibarlucía [Iba14] proved that NIP formulas correspond to *tame* functions (in the sense of [GM12]) and representations on *Rosenthal* Banach spaces (i.e., those not containing ℓ^1). He also showed that every function on a Roelcke precompact Polish group that comes from a dynamical system representable on an Asplund Banach space is weakly almost periodic.

9.2 Minimality of topological groups

A topological group G is called *minimal* if it admits no coarser Hausdorff group topology, or equivalently, if every bijective continuous homomorphism to another topological group is a homeomorphism. It is called *totally minimal* if all of its quotients are minimal, or equivalently, if every surjective continuous homomorphism to another Hausdorff topological group is open. A Roelcke precompact Polish group is totally minimal iff every continuous homomorphic image of it in another Polish group is closed. (This notion has no connection with the concept of minimality of a dynamical system as discussed in Section 3; unfortunately, the terminology is rather well established.) Minimality has been extensively studied in the theory of topological groups (see the recent survey by Dikranjan and Megrelishvili [DM13] and the references therein for more details). Minimality puts severe restrictions on the ways a group can (continuously) act: the topology on the group induced from the action must be the same as the original group topology.

A typical example of a Roelcke precompact group that is *not* minimal is $\mathrm{Aut}(\mathbf{Q})$ (as it embeds densely in $\mathrm{Homeo}^+(\mathbf{R})$). One of the main contributions of [BT13] is the result that this type of phenomenon cannot occur for automorphism groups of stable structures, as the following theorem shows.

Theorem 9.3 ([BT13])**.** *Let G be a Roelcke precompact group such that $\mathrm{UCB}(G) = \mathrm{WAP}(G)$, or, equivalently, let G be the automorphism group of an ω-categorical, stable structure. Then G is totally minimal.*

Special cases of this theorem had been known before: for example, S_∞ (Gaughan [Gau67]), the unitary group (Stojanov [Sto84]; see also Uspenskij [Usp98]), and $\mathrm{Aut}(\mu)$ (Glasner [Gla12]). However, Theorem 9.3 seems to be the first general result of this kind. Some examples of groups that satisfy the hypothesis of Theorem 9.3 for which minimality had not been known before include automorphism groups of L^p lattices, the automorphism groups of countably dimensional vector spaces over a finite field, and classical, $\aleph_0$-categorical, stable, non-$\aleph_0$-stable examples obtained via the Hrushovski construction ([Wag94, Example 5.3]). We should also note that the phenomenon of minimality is not restricted to automorphism groups of stable structures: some other known minimal groups for which our theorem does not apply are $\mathrm{Iso}(\mathbf{U}_1)$ and the automorphism group of the random graph (Uspenskij [Usp08]), $\mathrm{Homeo}(2^{\mathbf{N}})$ (Gamarnik [Gam91]), etc.

Our proof uses the semigroup structure on the WAP compactification $W(G)$ and is modeled on the proof of Uspenskij [Usp98]. The correspondence between WAP group topologies on G and the central idempotents of $W(G)$, which is at the heart of the argument, can be traced back to Ruppert [Rup90]. The main new contribution in the proof of Theorem 9.3 is the identification of the central idempotents of the semigroup $W(G)$ under the general hypotheses of the theorem and the main technical lemma is based on model-theoretic ideas.

In a different direction, coarser group topologies on non-archimedean groups that are not minimal can be used to produce examples of groups that admit no non-constant weakly almost periodic functions, or equivalently, no non-trivial representations on reflexive Banach spaces. The first example of a group with this property, $\mathrm{Homeo}^+(\mathbf{R})$, was produced by Megrelishvili [Meg01], answering a question of Ruppert. The theorem below, whose proof is based on the Ryll-Nardzewski fixed point theorem, can be used to give a different proof of his result and also provides some new examples.

Theorem 9.4 ([BT13])**.** *Let H be a Roelcke precompact subgroup of S_∞ and let $\pi\colon H \to G$ be a homomorphism to another Polish group with a dense image. Suppose, moreover, that G has no proper open subgroups. Then G admits no non-trivial representations by isometries on a reflexive Banach space.*

This theorem applies to $\mathrm{Homeo}^+(\mathbf{R})$, where $\mathrm{Aut}(\mathbf{Q})$ embeds densely, but also to some other homeomorphism groups of one-dimensional continua. Irwin and Solecki [IS06] constructed a Roelcke precompact, non-archimedean group that embeds densely in the homeomorphism group of the *pseudo-arc* and using a similar technique, Bartošova and Kwiatkowska [BK13] did the same for the *Lelek fan*. Theorem 9.4 then implies that the homeomorphism groups of both the pseudo-arc and the Lelek fan admit no non-trivial representations on a reflexive Banach space.

10 Generic properties of group actions

A *representation* of a countable, discrete group Γ on a structure M is just a homomorphism $\pi\colon \Gamma \to G$, where $G = \mathrm{Aut}(M)$. By varying M, we obtain settings that have rather different flavor. For example, representations on finite-dimensional vector spaces and Hilbert (or, more generally, Banach) spaces are the subject of representation theory, actions by homeomorphisms on compact spaces are studied in topological dynamics, and measure-preserving actions on a standard probability space are the topic of ergodic theory.

In this section, based on on the paper [MT13] joint with J. Melleray, we describe some general techniques based on the Baire category theorem that can be applied in a variety of settings. The space of representations $\mathrm{Hom}(\Gamma, G)$ is naturally equipped with a Polish topology (as a closed subspace of G^Γ) and a continuous G-action

$$(g \cdot \pi)(\gamma) = g\pi(\gamma)g^{-1}$$

by conjugation. In general, if the representations π_1 and π_2 are conjugate, we regard them as isomorphic or equivalent. We will be interested in properties of actions that are conjugacy-invariant, i.e., subsets of $\mathrm{Hom}(\Gamma, G)$ that are unions of orbits of the action. In order to be able to apply Baire category methods, we will also assume that the properties we consider satisfy a mild definability condition: that the sets they define in $\mathrm{Hom}(\Gamma, G)$ have the Baire property; this is certainly true in all cases of interest. A property $\mathcal{P}$ is called *generic* if the set $\{\pi \in \mathrm{Hom}(\Gamma, G) : \pi \text{ has property } \mathcal{P}\}$ is comeager in $\mathrm{Hom}(\Gamma, G)$. In all of the cases that we consider, the action $G \curvearrowright \mathrm{Hom}(\Gamma, G)$ has a dense orbit, so, by the topological 0–1 law (see [Kec95, Theorem 8.46]), for every (conjugacy-invariant) property $\mathcal{P}$, either it or its negation are generic. Real-valued invariants (such as entropy) can also be considered as properties in our sense: we only need to compare the invariant with every rational number and use the fact that a countable union of meager sets is meager.

Some special cases for the group Γ are worth mentioning:

- If $\Gamma = \mathbf{Z}$, the G-space $\mathrm{Hom}(\Gamma, G)$ is isomorphic to G equipped with the conjugation action on itself;
- More generally, if $\Gamma = \mathbf{F}_n$, $\mathrm{Hom}(\Gamma, G) = G^n$, again equipped with the diagonal action by conjugation;
- If $\Gamma = \mathbf{Z}^n$, $\mathrm{Hom}(\Gamma, G)$ is the space of commuting n-tuples of elements of G.

It sometimes happens that the action $G \curvearrowright \mathrm{Hom}(\Gamma, G)$ has a single comeager orbit. In that case, studying generic properties of the actions degenerates into studying a single isomorphism class. This situation arises often when Γ is finite but also for some infinite Γ if the structure M is discrete. When $\Gamma = \mathbf{Z}$, the action $G \curvearrowright \mathrm{Hom}(\Gamma, G)$ having a comeager orbit corresponds to the group G having a comeager conjugacy class, a property that has been extensively studied for automorphism groups of discrete structures. More generally, we say that G has *ample generics* if the action $G \curvearrowright \mathrm{Hom}(\mathbf{F}_n, G)$ has a comeager orbit for every n. This property was introduced by Hodges, Hodkinson, Lascar, and Shelah [Hod+93] in order to study the small index property for automorphism groups (see Section 11 for more details on that).

The authors of [Hod+93] gave a slightly different definition of ample generics; ours is borrowed from [KR07]. See [BT13, Section 4] for a clarification of the relationship between the two definitions. It was proved by Kechris and Rosendal [KR07] that having ample generics implies very strong structural properties for the group G; we discuss those further in Section 11. Groups that are known to have ample generics are the automorphism groups of ω-categorical, ω-stable, discrete structures, the random graph [Hod+93], the measured, countable, atomless Boolean algebra with a measure whose values are the dyadic rationals [KR07], the isometry group of the rational Urysohn space (Solecki [Sol05]), and the homeomorphism group of the Cantor space (Kwiatkowska [Kwi12]). It is an open problem if there exists a Polish group G that is not non-archimedean and that admits ample generics. A recent result of Wesolek [Wes13] states that a non-trivial Polish locally compact group cannot even have a comeager conjugacy class.

We concentrate on three continuous examples for G: $U(\mathcal{H})$, $\mathrm{Aut}(\mu)$, and $\mathrm{Iso}(\mathbf{U})$, but the methods that we develop seem to be general enough to apply to other situations as well. We also restrict ourselves to abelian groups Γ for two reasons: first, those are easier to understand and there are more methods available, and second, some of the questions that we consider (for example, those concerning the centralizers of the actions) are only meaningful or interesting for abelian Γ. An abelian group is called *bounded* if there is an upper bound on the order of its elements, and *unbounded* otherwise. Bounded abelian groups are just direct sums of finite cyclic groups, so below we only state the results for the unbounded case but similar results are also valid for bounded groups; see [MT13] for details.

In the cases that we consider, the action $G \curvearrowright \mathrm{Hom}(\Gamma, G)$ has a dense orbit for every countable Γ, so the topological 0–1 law applies. When $G = U(\mathcal{H})$, orbits are meager for every infinite Γ (Kerr–Li–Pichot [KLP10]), and similarly for $G = \mathrm{Iso}(\mathbf{U})$ (Melleray [Mel14]); when $G = \mathrm{Aut}(\mu)$, this is known for amenable Γ (Foreman–Weiss [FW04]) and open in general. One general method, discovered by Rosendal [Ros09c], for proving meagerness of conjugacy classes that works well for abelian groups (but could possibly be applied to more general situations) is to investigate convergence patterns in $\pi(\Gamma)$, that is, proving that for a fixed $\pi_0 \in \mathrm{Hom}(\Gamma, G)$, the set

$$\{\pi \in \mathrm{Hom}(\Gamma, G) : \pi(\Gamma) \text{ and } \pi_0(\Gamma) \text{ are isomorphic as topological groups}\}$$

is meager.

In view of this, it becomes natural to ask what information one can recover if one forgets the image of Γ and keeps just the Polish group $\overline{\pi(\Gamma)}$. The main focus of [MT13] is studying the closure $\overline{\pi(\Gamma)}$ for the generic $\pi \in \mathrm{Hom}(\Gamma, G)$, that is, generic properties of π that can be read from the group $\overline{\pi(\Gamma)}$. The situation is simplest for the unitary group, where we have the following theorem.

Theorem 10.1 ([MT13]). *Let Γ be an unbounded abelian group. Then the set*

$$\{\pi \in \mathrm{Hom}(\Gamma, U(\mathcal{H})) : \overline{\pi(\Gamma)} \cong L^0(\mathbf{T})\}$$

is comeager in $\mathrm{Hom}(\Gamma, U(\mathcal{H}))$.

Recall that if H is a Polish group, $L^0(H)$ denotes the group of all measurable mappings from a standard probability space to H equipped with pointwise multiplication and the topology of convergence in measure. $\mathbf{T}$ denotes the multiplicative group of complex numbers of absolute value 1. Theorem 10.1, together with the discussion above, leads naturally to the following question.

Question 10.2 ([MT13]). For G either $\mathrm{Aut}(\mu)$ or $\mathrm{Iso}(\mathbf{U})$, does there exist a Polish group H such that for the generic $\pi \in \mathrm{Hom}(\mathbf{Z}, G)$, $\overline{\pi(\mathbf{Z})} \cong H$? In particular, is this true for $H = L^0(\mathbf{T})$?

Christian Rosendal has pointed out that for $G = \mathrm{Iso}(\mathbf{U})$ it is not possible that the generic $\overline{\pi(\mathbf{Z})}$ is isomorphic to $L^0(\mathbf{T})$; however, this remains open for a general H. This question was also asked by Solecki and by Pestov and a related problem was posed by Glasner and Weiss [GW05] (all for $\mathrm{Aut}(\mu)$). Solecki [Sol14] has shown that the closed subgroup generated by a generic element of $\mathrm{Aut}(\mu)$ is a continuous homomorphic image of a closed subspace of $L^0(\mathbf{R})$ and contains an increasing union of finite dimensional tori whose union is dense.

The following is a general theorem that is valid for any Γ and G. Its proof relies on a characterization of extreme amenability due to Pestov.

Theorem 10.3 ([MT13]). *Let Γ be a countable group and G a Polish group. Then the set*

$$\{\pi \in \mathrm{Hom}(\Gamma, G) : \overline{\pi(\Gamma)} \text{ is extremely amenable}\}$$

is G_δ in $\mathrm{Hom}(\Gamma, G)$.

An analogous result for amenability was later obtained by Kaïchouh in [Kaï13].

In particular, Theorem 10.3 shows that if the set of π that generate an extremely amenable group is dense (for example, if there is one such π with a dense orbit), then it is automatically generic, so in some sense, extreme amenability, rather than being a pathological phenomenon, is a prevalent property. This happens, in particular, for abelian Γ and $G = \mathrm{Aut}(\mu)$ or $G = \mathrm{Iso}(\mathbf{U})$.

Theorem 10.4 ([MT13]). *Let Γ be a countable, unbounded, abelian group and G be one of* $\mathrm{Aut}(\mu)$ *or* $\mathrm{Iso}(\mathbf{U})$. *Then the set*

$$\{\pi : \overline{\pi(\Gamma)} \cong L^0(\mathbf{T})\}$$

is dense in $\mathrm{Hom}(\Gamma, G)$ *and therefore, the generic $\overline{\pi(\Gamma)}$ is extremely amenable.*

In the special case of $\Gamma = \mathbf{Z}$ and $G = \mathrm{Iso}(\mathbf{U})$ this theorem answers a question of Glasner and Pestov.

Theorem 10.3 can also be used as a tool to prove extreme amenability for concrete groups. For every Polish group G, the set

$$\{\pi \in \mathrm{Hom}(\mathbf{F}_\infty, G) : \overline{\pi(\mathbf{F}_\infty)} = G\}$$

is dense G_δ; therefore to prove that G is extremely amenable, it suffices to show that

$$\{\pi \in \mathrm{Hom}(\mathbf{F}_\infty, G) : \overline{\pi(\mathbf{F}_\infty)} \text{ is extremely amenable}\}$$

is dense (so, again, it suffices to find a single π with this property that has a dense orbit). We have the following.

Theorem 10.5 ([MT13]). *Let G be either of $U(\mathcal{H})$, $\mathrm{Aut}(\mu)$, $\mathrm{Iso}(\mathbf{U})$; denote by $U(n)$ the unitary group of dimension n. Then the set*

$$\{\pi \in \mathrm{Hom}(\mathbf{F}_\infty, G) : \exists n \ \overline{\pi(\mathbf{F}_\infty)} \cong L^0(U(n))\}$$

is dense in $\mathrm{Hom}(\mathbf{F}_\infty, G)$.

Corollary 10.6 (Gromov–Milman; Giordano–Pestov; Pestov). *The three groups $U(\mathcal{H})$, $\mathrm{Aut}(\mu)$, and $\mathrm{Iso}(\mathbf{U})$ are extremely amenable.*

Thus Theorem 10.5 provides a "uniform" proof for the extreme amenability of those three groups. It is arguable whether this proof can be considered as new: after all, the proof of extreme amenability of $L^0(U(n))$ uses concentration of measure, the same technique that was used in the original proofs of extreme amenability. However, if K is a compact group, the sequence of subgroups $\{K^{2^n} : n \in \mathbf{N}\}$ of $L^0(K)$ is the most basic example of the concentration of measure phenomenon, while in the original proofs, different systems adapted to each situation are used.

Another question one might like to consider is how much the generic properties of $\overline{\pi(\Gamma)}$ depend on the group Γ. Of course, if Γ is abelian, then $\overline{\pi(\Gamma)}$ must also be abelian. However, it turns out that not much more is remembered as the following result indicates.

Theorem 10.7 ([MT13]). *Let G be one of $\mathrm{Aut}(\mu)$ or $\mathrm{Iso}(\mathbf{U})$ and let d be a positive integer. Let $\mathcal{P}$ be a definable property of abelian Polish groups. Then the following are equivalent:*

(i) *for the generic $\pi \in \mathrm{Hom}(\mathbf{Z}, G)$, $\overline{\pi(\mathbf{Z})}$ has property $\mathcal{P}$;*

(ii) *for the generic $\pi \in \mathrm{Hom}(\mathbf{Z}^d, G)$, $\overline{\pi(\mathbf{Z}^d)}$ has property $\mathcal{P}$.*

In particular, the generic $\overline{\pi(\mathbf{Z}^d)}$ is monothetic *(i.e., topologically singly generated).*

In the case $G = \mathrm{Aut}(\mu)$, a result of Ageev allows us to replace $\mathbf{Z}^d$ by any countable abelian Γ containing an infinite cyclic subgroup. It seems plausible that the same is true for $\mathrm{Iso}(\mathbf{U})$; however, our techniques are insufficient to prove this.

The techniques developed for the proof of Theorem 10.7 also permit to study the centralizer of the generic action. If $\pi \colon \Gamma \to G$ is a homomorphism, the *centralizer* $\mathcal{C}(\pi)$ of π is just the stabilizer of π under the conjugation action:

$$\mathcal{C}(\pi) = \{g \in G : g\pi(\gamma) = \pi(\gamma)g \text{ for all } \gamma \in \Gamma\}.$$

$\mathcal{C}(\pi)$ is always a closed subgroup of G and if Γ is abelian, $\overline{\pi(\Gamma)} \subseteq \mathcal{C}(\pi)$. The next theorem shows that, generically, the centralizer is as small as possible.

Theorem 10.8 ([MT13]). *Let G be either of $\mathrm{Aut}(\mu)$ or $\mathrm{Iso}(\mathbf{U})$ and let Γ be a torsion-free abelian group. Then, for the generic $\pi \in \mathrm{Hom}(\Gamma, G)$, $\mathcal{C}(\pi) = \overline{\pi(\Gamma)}$.*

The special case where $\Gamma = \mathbf{Z}$ and $G = \mathrm{Aut}(\mu)$ is a classical theorem of King [Kin86] in ergodic theory (proved by a very different method).

We conclude this section with a few words about the techniques used to prove these results. The proofs of Theorems 10.7 and 10.8 are based on the notion of a category-preserving map (a map $\pi \colon X \to Y$ between Polish spaces X and Y is *category-preserving* if the preimage of a meager set is meager) and a generalization of the classical Kuratowski–Ulam theorem to this setting. In a follow-up paper [Mel12], Melleray further developed these techniques to prove a more general theorem about extensions of measure-preserving actions of abelian groups. See also [Mel14] for further discussion on Baire category methods in the context of Polish groups.

11 Automatic continuity

Many of the results in the previous sections concern the classification of various actions of Polish groups: actions on compact spaces by homeomorphisms, actions on probability spaces by measure-preserving transformations, or representations on Hilbert spaces by unitary isometries. A common feature of those results is that they assume that the actions are *continuous* in the sense that they can be represented as continuous homomorphisms to the symmetry groups of the spaces in question and the proofs exploit in an important way this continuity. It is a classical result in descriptive set theory due to Banach (see, e.g., [Kec95, Theorem 9.10]) that the continuity assumption can be relaxed significantly: every Baire measurable homomorphism from a Polish group to a separable group is continuous. It is a theorem of Shelah [She84] that in order to produce homomorphisms that are not Baire measurable, one needs the axiom of choice, so, in practice, most homomorphisms that arise in a definable fashion are already continuous.

Nevertheless, in the world of locally compact groups, there are several known ways to produce non-continuous homomorphisms. For example, if K is a non-trivial, compact group and $\mathcal{U}$ is a non-principal ultrafilter on $\mathbf{N}$, then the homomorphism $\pi \colon K^{\mathbf{N}} \to K$ defined by

$$\pi((k)_n) = \lim_{n \to \mathcal{U}} k_n$$

is not continuous. Similarly, using the fact that vector spaces of the same dimension over $\mathbf{Q}$ are isomorphic, one sees that the additive groups of $\mathbf{R}$, $\mathbf{R}^2$ and $\mathbf{Q}_2$ are all isomorphic but obviously, there are no continuous isomorphisms. Finally, using non-continuous automorphisms of the field of complex numbers, one sees that $\mathrm{GL}(n, \mathbf{C})$ embeds in S_∞ (see Kallman [Kal00] and Thomas [Tho99]) and the restriction of this embedding to any connected subgroup (for example, $U(n)$) cannot be continuous.

It is an interesting phenomenon that for large Polish groups, similar techniques to construct homomorphisms often cannot be applied.

Definition 11.1. A topological group G has the *automatic continuity property* if every homomorphism from G to a separable topological group is continuous.

The condition of separability on the target group can be somewhat weakened without changing the class of groups under consideration; see [Ros09b] for details.

This property was introduced by Kechris and Rosendal [KR07], who were inspired by a similar property that had been studied for Banach algebras as well as earlier results in model theory about the small index property. A topological group G is said to have the *small index property* if every homomorphism from G to S_∞ (or, equivalently, any action on a countable discrete set) is continuous. One of the reasons model theorists are interested in this property is because, combined with the result of Ahlbrandt and Ziegler [AZ86], it allows to show that automorphism groups of ω-categorical, classical structures up to *algebraic* isomorphism characterize the structures up to bi-interpretability.

Recall that a topological group G is said to have *ample generics* if the diagonal conjugacy action $G \curvearrowright G^n$ has a comeager orbit for every n. This property was introduced by Hodges, Hodkinson, Lascar, and Shelah [Hod+93] in order to prove that automorphism groups of ω-categorical, ω-stable structures and the random graph have the small index property. Kechris and Rosendal showed that ample generics in fact imply the stronger automatic continuity property and thus provided the first examples.

Since then, a number of other groups have been shown to have the automatic continuity property: the automorphism group of $(\mathbf{Q}, <)$ and the homeomorphism group of the Cantor space (Rosendal–Solecki [RS07]), homeomorphism groups of compact 2-manifolds (Rosendal [Ros08]); full groups of ergodic, measure-preserving equivalence relations (Kittrell–Tsankov [KT10]), etc.

Ample generics is the only natural general condition currently known to imply automatic continuity. However, it is an open problem whether every Polish group with ample generics is non-archimedean, and in any case, many of the automorphism groups of metric structures that we are interested in are known to have meager conjugacy classes. Inspired by ideas from continuous logic, Ben Yaacov, Berenstein, and Melleray [BBM13] introduced the weaker notion of *topometric ample generics* that is better adapted to the metric setting.

If $G = \mathrm{Aut}(M)$ is the automorphism group of a metric structure M, then apart from the Polish topology on G (which is just the pointwise convergence topology on M), G also admits another, finer, usually non-separable group topology, namely, the one of uniform convergence on M. This topology is induced by a complete, bi-invariant distance ∂ on G defined by:

$$\partial(g, h) = \sup_{x \in M} d(g \cdot x, h \cdot x).$$

A *Polish topometric group* is a Polish group (G, τ) additionally equipped with a bi-invariant distance ∂ in the fashion described above. G has *topometric ample generics* if the diagonal conjugacy action $G \curvearrowright G^n$ admits an orbit whose ∂-closure is comeager. It was proved in [BBM13] that if G has topometric ample generics and $\phi \colon G \to H$ is a homomorphism to a separable group H that is ∂-continuous, then it is automatically τ-continuous. This result paved the road for two-step proofs of automatic continuity: first show that if ϕ is an arbitrary homomorphism to a separable group, then it is ∂-continuous, and then use topometric ample generics to show that it is τ-continuous. The authors of [BBM13] also developed a method to prove the existence of topometric ample generics and provided several examples: $U(\mathcal{H})$, $\mathrm{Aut}(\mu)$, and $\mathrm{Iso}(\mathbf{U}_1)$.

This two-step strategy for proving automatic continuity has been successfully implemented for two groups. The proof of [KT10] that full groups have automatic continuity easily adapts to show that $(\mathrm{Aut}(\mu), \partial)$ does as well; then Ben Yaacov, Berenstein, and Melleray applied their result to conclude that $\mathrm{Aut}(\mu)$ has automatic continuity for its usual Polish topology. Later, still using topometric ample generics, the following theorem was proved in [Tsa13], answering a question of Rosendal [Ros08].

Theorem 11.2. *The unitary group of a separable, infinite-dimensional Hilbert space has the automatic continuity property.*

More recently, Sabok [Sab13], using a different technique, proved that $\mathrm{Iso}(\mathbf{U}_1)$ also has automatic continuity; his method also adapts to $U(\mathcal{H})$ and $\mathrm{Aut}(\mu)$.

It is interesting to combine automatic continuity results with *minimality* (see Subsection 9.2) in order to obtain that certain groups admit a unique separable group topology. For example, Theorem 11.2 together with the result of Stojanov [Sto84] implies that $U(\mathcal{H})$ has a unique separable group topology, so, if $\phi \colon U(\mathcal{H}) \to G$ is an arbitrary embedding of $U(\mathcal{H})$ in a Polish group G, then its image is closed and ϕ is a homeomorphism. In particular, $U(\mathcal{H})$ does not embed (abstractly) as a subgroup of any CLI Polish group (for example, locally compact or SIN).

Combining the results of [Hod+93], [KR07], and [BT13] one obtains the following corollary.

Corollary 11.3 ([BT13])**.** *Let G be the automorphism group of a classical, ω-categorical, ω-stable structure. Then G admits a unique separable group topology.*

An interesting point about this corollary is that the hypothesis of stability (or the stronger ω-stability) is used in the proofs of automatic continuity and minimality in a completely different fashion.

12 Open questions

Open questions are dispersed throughout the text; in this section, we collect the ones that pertain to actions of large Polish groups, and, in particular, the Roelcke precompact ones, which are the focus of this memoir.

Question 12.1 (Section 3, [MNT14]). Is it true that every Roelcke precompact Polish group has a metrizable universal minimal flow?

A positive answer to Question 12.1 for non-archimedean groups is equivalent to [MNT14; Zuc14b] a positive answer to the following one.

Question 12.2 (Section 3, [BPT13]). Let M be a classical, ω-categorical structure. Does there always exist an ω-categorical expansion M' of M with the Ramsey property?

Question 12.3 (Section 3, [AKL12]). Let G be a Polish group such that $M(G)$ is metrizable. Must $M(G)$ have a G_δ orbit? Equivalently, is every metrizable UMF of the form $\widehat{G/H}$ for a closed, co-precompact, extremely amenable $H \leq G$?

This question was answered positively for non-archimedean groups by Zucker [Zuc14b].

Question 12.4 (Section 3). What are the topological groups that have only countably many minimal flows up to isomorphism? Is this true for all non-archimedean Polish groups with a metrizable universal minimal flow that admits a finite generating partition?

Question 12.5 (Section 6). Is it true that all Roelcke precompact Polish groups are of type I? More specifically, is it true that every representation of a Roelcke precompact Polish group decomposes as a sum of irreducibles and that there are only countably many irreducible representations?

Question 12.6 (Section 6, [Tsa12]). Does every Roelcke precompact Polish group have property (T)?

Question 12.7 (Section 7). Let M be a countable set and $G \curvearrowright M$ be an oligomorphic action. What are the measures on $[0,1]^M$ invariant and ergodic under the action of G?

Question 12.8 (Section 7, [AKL12]). Let G be an amenable Polish group with a metrizable universal minimal flow $M(G)$. Must $M(G)$ be uniquely ergodic?

Liste des travaux présentés dans ce mémoire

[Tsa12] T. Tsankov, *Unitary representations of oligomorphic groups*, Geom. Funct. Anal. **22**.2 (2012), pp. 528–555.

[BT13] I. Ben Yaacov and T. Tsankov, *Weakly almost periodic functions, model-theoretic stability, and minimality of topological groups*, Preprint `arXiv:1312.7757`, 2013.

[BPT13] M. Bodirsky, M. Pinsker, and T. Tsankov, *Decidability of definability*, J. Symbolic Logic **78**.4 (2013), pp. 1036–1054.

[ET13] D. M. Evans and T. Tsankov, *Free actions of free groups on countable structures and property (T)*, Preprint `arXiv:1312.5140`, 2013.

[MT13] J. Melleray and T. Tsankov, *Generic representations of abelian groups and extreme amenability*, Israel J. Math. **198**.1 (2013), pp. 129–167.

[Tsa13] T. Tsankov, *Automatic continuity for the unitary group*, Proc. Amer. Math. Soc. **141**.10 (2013), pp. 3673–3680.

[MNT14] J. Melleray, L. Nguyen Van Thé, and T. Tsankov, *Polish groups with metrizable universal minimal flows*, Preprint `arXiv:1404.6167`, 2014.

Bibliography

[AFP12] N. Ackerman, C. Freer, and R. Patel, *Invariant measures concentrated on countable structures*, Preprint arXiv:1206.4011, 2012.

[AZ86] G. Ahlbrandt and M. Ziegler, *Quasi-finitely axiomatizable totally categorical theories*, Ann. Pure Appl. Logic **30**.1 (1986), Stability in model theory (Trento, 1984), pp. 63–82.

[Ald81] D. J. Aldous, *Representations for partially exchangeable arrays of random variables*, J. Multivariate Anal. **11**.4 (1981), pp. 581–598.

[AKL12] O. Angel, A. S. Kechris, and R. Lyons, *Random orderings and unique ergodicity of automorphism groups*, Preprint arXiv:1208.2389, 2012.

[Aus08] T. Austin, *On exchangeable random variables and the statistics of large graphs and hypergraphs*, Probab. Surv. **5** (2008), pp. 80–145.

[BK13] D. Bartošova and A. Kwiatkowska, *Lelek fan from a projective Fraïssé limit*, Preprint arXiv:1312.7514, 2013.

[BK96] H. Becker and A. S. Kechris, *The descriptive set theory of Polish group actions*, vol. 232, London Mathematical Society Lecture Note Series, Cambridge: Cambridge University Press, 1996, pp. xii+136.

[BHV08] B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan's property (T)*, vol. 11, New Mathematical Monographs, Cambridge: Cambridge University Press, 2008, pp. xiv+472.

[BK14] I. Ben Yaacov and A. Kaïchouh, *Reconstruction of separably categorical metric structures*, Preprint arXiv:1405.4177, 2014.

[BT13] I. Ben Yaacov and T. Tsankov, *Weakly almost periodic functions, model-theoretic stability, and minimality of topological groups*, Preprint arXiv:1312.7757, 2013.

[Ben12] I. Ben Yaacov, *Fraïssé limits of metric structures*, Preprint arXiv:1203.4459, 2012.

[Ben+08] I. Ben Yaacov, A. Berenstein, C. W. Henson, and A. Usvyatsov, *Model theory for metric structures*, in: Model theory with applications to algebra and analysis. Vol. 2, vol. 350, London Math. Soc. Lecture Note Ser. Cambridge: Cambridge Univ. Press, 2008, pp. 315–427.

[BBM13] I. Ben Yaacov, A. Berenstein, and J. Melleray, *Polish topometric groups*, Trans. Amer. Math. Soc. **365**.7 (2013), pp. 3877–3897.

[Bod12] M. Bodirsky, *Complexity classification in infinite-domain constraint satisfaction*, Mémoire d'habilitation, Université Paris Diderot, 2012.

[BPT13] M. Bodirsky, M. Pinsker, and T. Tsankov, *Decidability of definability*, J. Symbolic Logic **78**.4 (2013), pp. 1036–1054.

[Cam76] P. J. Cameron, *Transitivity of permutation groups on unordered sets*, Math. Z. **148**.2 (1976), pp. 127–139.

[Che98] G. L. Cherlin, *The classification of countable homogeneous directed graphs and countable homogeneous n-tournaments*, Mem. Amer. Math. Soc. **131**.621 (1998), pp. xiv+161.

[DM13] D. DIKRANJAN and M. MEGRELISHVILI, *Minimality conditions in topological groups*, in: Recent Progress in General Topology III, ed. by J. VAN MILL and K. HART, Atlantis Press, 2013.

[ET13] D. M. EVANS and T. TSANKOV, *Free actions of free groups on countable structures and property (T)*, Preprint arXiv:1312.5140, 2013.

[Fol95] G. B. FOLLAND, *A course in abstract harmonic analysis*, Studies in Advanced Mathematics, Boca Raton, FL: CRC Press, 1995, pp. x+276.

[FW04] M. FOREMAN and B. WEISS, *An anti-classification theorem for ergodic measure preserving transformations*, J. Eur. Math. Soc. (JEMS) **6**.3 (2004), pp. 277–292.

[Fra65] C. FRASNAY, *Quelques problèmes combinatoires concernant les ordres totaux et les relations monomorphes*, Ann. Inst. Fourier (Grenoble) **15**.fasc. 2 (1965), pp. 415–524.

[Fra54] R. FRAÏSSÉ, *Sur l'extension aux relations de quelques propriétés des ordres*, Ann. Sci. Ecole Norm. Sup. (3) **71** (1954), pp. 363–388.

[Gam91] D. GAMARNIK, *Minimality of the group* $\mathrm{Autohomeom}(C)$, Serdica **17**.4 (1991), pp. 197–201.

[GJS12] S. GAO, S. JACKSON, and B. SEWARD, *Group colorings and Bernoulli subflows*, Preprint arXiv:1201.0513, 2012.

[Gao09] S. GAO, *Invariant descriptive set theory*, vol. 293, Pure and Applied Mathematics (Boca Raton), CRC Press, Boca Raton, FL, 2009, pp. xiv+383.

[Gau67] E. D. GAUGHAN, *Topological group structures of infinite symmetric groups*, Proc. Nat. Acad. Sci. U.S.A. **58** (1967), pp. 907–910.

[GM12] E. GLASNER and M. MEGRELISHVILI, *Representations of dynamical systems on Banach spaces not containing* l_1, Trans. Amer. Math. Soc. **364**.12 (2012), pp. 6395–6424.

[GW97] E. GLASNER and B. WEISS, *Kazhdan's property T and the geometry of the collection of invariant measures*, Geom. Funct. Anal. **7**.5 (1997), pp. 917–935.

[GW02] E. GLASNER and B. WEISS, *Minimal actions of the group* $\mathrm{S}(\mathbb{Z})$ *of permutations of the integers*, Geom. Funct. Anal. **12**.5 (2002), pp. 964–988.

[GW03] E. GLASNER and B. WEISS, *The universal minimal system for the group of homeomorphisms of the Cantor set*, Fund. Math. **176**.3 (2003), pp. 277–289.

[GW05] E. GLASNER and B. WEISS, *Spatial and non-spatial actions of Polish groups*, Ergodic Theory Dynam. Systems **25**.5 (2005), pp. 1521–1538.

[Gla76] S. GLASNER, *Proximal flows*, Lecture Notes in Mathematics, Vol. 517, Berlin: Springer-Verlag, 1976, pp. viii+153.

[Gla12] E. GLASNER, *The group* $\mathrm{Aut}(\mu)$ *is Roelcke precompact*, Canad. Math. Bull. **55**.2 (2012), pp. 297–302.

[Gre69] F. P. GREENLEAF, *Invariant means on topological groups and their applications*, Van Nostrand Mathematical Studies, No. 16, Van Nostrand Reinhold Co., New York-Toronto, Ont.-London, 1969, pp. ix+113.

[GM83] M. GROMOV and V. D. MILMAN, *A topological application of the isoperimetric inequality*, Amer. J. Math. **105**.4 (1983), pp. 843–854.

[Gro52] A. GROTHENDIECK, *Critères de compacité dans les espaces fonctionnels généraux*, Amer. J. Math. **74** (1952), pp. 168–186.

[HC75] W. HERER and J. P. R. CHRISTENSEN, *On the existence of pathological submeasures and the construction of exotic topological groups*, Math. Ann. **213** (1975), pp. 203–210.

[Hjo97] G. HJORTH, *Non-smooth infinite dimensional group representations*, Unpublished note, 1997.

[Hjo00] G. Hjorth, *Classification and orbit equivalence relations*, vol. 75, Mathematical Surveys and Monographs, Providence, RI: American Mathematical Society, 2000, pp. xviii+195.

[Hod+93] W. Hodges, I. Hodkinson, D. Lascar, and S. Shelah, *The small index property for ω-stable ω-categorical structures and for the random graph*, J. London Math. Soc. (2) **48**.2 (1993), pp. 204–218.

[Hoo79] D. N. Hoover, *Relations on probability spaces and arrays of random variables*, Preprint, Institute for Advanced Studies, 1979.

[Iba14] T. Ibarlucía, *The dynamical hierarchy for Roelcke precompact Polish groups*, Preprint arXiv:1405.4613, 2014.

[IS06] T. Irwin and S. Solecki, *Projective Fraïssé limits and the pseudo-arc*, Trans. Amer. Math. Soc. **358**.7 (2006), 3077–3096 (electronic).

[Kaï13] A. Kaïchouh, *Amenability and Ramsey theory in the metric setting*, Preprint, 2013.

[Kal89] O. Kallenberg, *On the representation theorem for exchangeable arrays*, J. Multivariate Anal. **30**.1 (1989), pp. 137–154.

[Kal05] O. Kallenberg, *Probabilistic symmetries and invariance principles*, Probability and its Applications (New York), New York: Springer, 2005, pp. xii+510.

[Kal00] R. R. Kallman, *Every reasonably sized matrix group is a subgroup of S_∞*, Fund. Math. **164**.1 (2000), pp. 35–40.

[Kaz67] D. A. Kazhdan, *On the connection of the dual space of a group with the structure of its closed subgroups*, Funkcional. Anal. i Priložen. **1** (1967), pp. 71–74.

[KPT05] A. S. Kechris, V. G. Pestov, and S. Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geom. Funct. Anal. **15**.1 (2005), pp. 106–189.

[Kec95] A. S. Kechris, *Classical descriptive set theory*, vol. 156, Graduate Texts in Mathematics, New York: Springer-Verlag, 1995, pp. xviii+402.

[KR07] A. S. Kechris and C. Rosendal, *Turbulence, amalgamation and generic automorphisms of homogeneous structures*, Proc. Lond. Math. Soc. **94**.2 (2007), pp. 302–350.

[KS12] A. S. Kechris and M. Sokić, *Dynamical properties of the automorphism groups of the random poset and random distributive lattice*, Fund. Math. **218**.1 (2012), pp. 69–94.

[KLP10] D. Kerr, H. Li, and M. Pichot, *Turbulence, representations, and trace-preserving actions*, Proc. Lond. Math. Soc. (3) **100**.2 (2010), pp. 459–484.

[Kin86] J. King, *The commutant is the weak closure of the powers, for rank-*1 *transformations*, Ergodic Theory Dynam. Systems **6**.3 (1986), pp. 363–384.

[Kir73] A. A. Kirillov, *Representations of the infinite-dimensional unitary group*, Dokl. Akad. Nauk. SSSR **212** (1973), pp. 288–290.

[KT10] J. Kittrell and T. Tsankov, *Topological properties of full groups*, Ergodic Theory Dynam. Systems **30**.2 (2010), pp. 525–545.

[Kwi12] A. Kwiatkowska, *The group of homeomorphisms of the Cantor set has ample generics*, Bull. Lond. Math. Soc. **44**.6 (2012), pp. 1132–1146.

[KS11] A. Kwiatkowska and S. Solecki, *Spatial models of Boolean actions and groups of isometries*, Ergodic Theory Dynam. Systems **31**.2 (2011), pp. 405–421.

[Lac84] A. H. Lachlan, *Countable homogeneous tournaments*, Trans. Amer. Math. Soc. **284**.2 (1984), pp. 431–461.

[LW80] A. H. Lachlan and R. E. Woodrow, *Countable ultrahomogeneous undirected graphs*, Trans. Amer. Math. Soc. **262**.1 (1980), pp. 51–94.

[Lie72] A. Lieberman, *The structure of certain unitary representations of infinite symmetric groups*, Trans. Amer. Math. Soc. **164** (1972), pp. 189–198.

[Mac11] D. Macpherson, *A survey of homogeneous structures*, Discrete Math. **311**.15 (2011), pp. 1599–1634.

[Meg01] M. G. Megrelishvili, *Every semitopological semigroup compactification of the group $H_+[0,1]$ is trivial*, Semigroup Forum **63**.3 (2001), pp. 357–370.

[Mel12] J. Melleray, *Extensions of generic measure-preserving actions*, Preprint `arXiv:1201.4447`, 2012.

[Mel14] J. Melleray, *Groupes polonais et méthodes de Baire*, Mémoire d'habilitation, Université Lyon 1, 2014.

[MNT14] J. Melleray, L. Nguyen Van Thé, and T. Tsankov, *Polish groups with metrizable universal minimal flows*, Preprint `arXiv:1404.6167`, 2014.

[MT13] J. Melleray and T. Tsankov, *Generic representations of abelian groups and extreme amenability*, Israel J. Math. **198**.1 (2013), pp. 129–167.

[Moo13] J. T. Moore, *Amenability and Ramsey theory*, Fund. Math. **220**.3 (2013), pp. 263–280.

[Ner96] Y. A. Neretin, *Categories of symmetries and infinite-dimensional groups*, vol. 16, London Mathematical Society Monographs. New Series, Translated from the Russian by G. G. Gould, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1996, pp. xiv+417.

[Ngu13] L. Nguyen Van Thé, *Théorie de Ramsey structurale et applications en dynamique via la correspondance de Kechris-Pestov-Todorcevic*, Mémoire d'habilitation, Université d'Aix–Marseille, 2013.

[Ols78] G. I. Olshanski, *Unitary representations of the infinite-dimensional classical groups* $\mathrm{U}(p,\infty)$, $\mathrm{SO}_0(p,\infty)$, $\mathrm{Sp}(p,\infty)$, *and of the corresponding motion groups*, Funktsional. Anal. i Prilozhen. **12**.3 (1978), pp. 32–44, 96.

[Ols85] G. I. Olshansky, *Unitary representations of the infinite symmetric group: a semigroup approach*, in: Representations of Lie groups and Lie algebras (Budapest, 1971), Budapest: Akad. Kiadó, 1985, pp. 181–197.

[Pes98] V. Pestov, *On free actions, minimal flows, and a problem by Ellis*, Trans. Amer. Math. Soc. **350**.10 (1998), pp. 4149–4165.

[Pes06] V. Pestov, *Dynamics of infinite-dimensional groups*, vol. 40, University Lecture Series, The Ramsey-Dvoretzky-Milman phenomenon, Revised edition of *Dynamics of infinite-dimensional groups and Ramsey-type phenomena* [Inst. Mat. Pura. Apl. (IMPA), Rio de Janeiro, 2005; MR2164572], Providence, RI: American Mathematical Society, 2006, pp. viii+192.

[PV10] F. Petrov and A. Vershik, *Uncountable graphs and invariant measures on the set of universal countable graphs*, Random Structures Algorithms **37**.3 (2010), pp. 389–406.

[Phe01] R. R. Phelps, *Lectures on Choquet's theorem*, Second, vol. 1757, Lecture Notes in Mathematics, Berlin: Springer-Verlag, 2001, pp. viii+124.

[RD81] W. Roelcke and S. Dierolf, *Uniform structures on topological groups and their quotients*, Advanced Book Program, New York: McGraw-Hill International Book Co., 1981, pp. xi+276.

[Ros08] C. Rosendal, *Automatic continuity in homeomorphism groups of compact 2-manifolds*, Israel J. Math. **166** (2008), pp. 349–367.

[Ros09a] C. Rosendal, *A topological version of the Bergman property*, Forum Math. **21**.2 (2009), pp. 299–332.

[Ros09b] C. Rosendal, *Automatic continuity of group homomorphisms*, Bull. Symbolic Logic **15**.2 (2009), pp. 184–214.

[Ros09c] C. Rosendal, *The generic isometry and measure preserving homeomorphism are conjugate to their powers*, Fund. Math. **205**.1 (2009), pp. 1–27.

[Ros13] C. Rosendal, *Global and local boundedness of Polish groups*, Indiana Univ. Math. J. **62**.5 (2013), pp. 1621–1678.

[RS07] C. Rosendal and S. Solecki, *Automatic continuity of homomorphisms and fixed points on metric compacta*, Israel J. Math. **162** (2007), pp. 349–371.

[Rup90] W. A. F. Ruppert, *On group topologies and idempotents in weak almost periodic compactifications*, Semigroup Forum **40**.2 (1990), pp. 227–237.

[Ryl57] C. Ryll-Nardzewski, *On stationary sequences of random variables and the de Finetti's equivalence*, Colloq. Math. **4** (1957), pp. 149–156.

[Sab13] M. Sabok, *Automatic continuity for isometry groups*, Preprint `arXiv:1312.5141`, 2013.

[She84] S. Shelah, *Can you take Solovay's inaccessible away?*, Israel J. Math. **48**.1 (1984), pp. 1–47.

[Sol05] S. Solecki, *Extending partial isometries*, Israel J. Math. **150** (2005), pp. 315–331.

[Sol12] S. Solecki, *Unitary representations of the groups of measurable and continuous functions with values in the circle*, Preprint, 2012.

[Sol14] S. Solecki, *Closed subgroups generated by generic measure automorphisms*, Ergodic Theory Dynam. Systems **34**.3 (2014), pp. 1011–1017.

[Sto84] L. Stojanov, *Total minimality of the unitary groups*, Math. Z. **187**.2 (1984), pp. 273–283.

[Tho68] E. Thoma, *Eine Charakterisierung diskreter Gruppen vom Typ I*, Invent. Math. **6** (1968), pp. 190–196.

[Tho91] S. Thomas, *Reducts of the random graph*, J. Symbolic Logic **56**.1 (1991), pp. 176–181.

[Tho99] S. Thomas, *Infinite products of finite simple groups. II*, J. Group Theory **2**.4 (1999), pp. 401–434.

[Tsa12] T. Tsankov, *Unitary representations of oligomorphic groups*, Geom. Funct. Anal. **22**.2 (2012), pp. 528–555.

[Tsa13] T. Tsankov, *Automatic continuity for the unitary group*, Proc. Amer. Math. Soc. **141**.10 (2013), pp. 3673–3680.

[Usp98] V. Uspenskij, *The Roelcke compactification of unitary groups*, in: Abelian groups, module theory, and topology (Padua, 1997), vol. 201, Lecture Notes in Pure and Appl. Math. New York: Dekker, 1998, pp. 411–419.

[Usp00] V. Uspenskij, *On universal minimal compact G-spaces*, in: Proceedings of the 2000 Topology and Dynamics Conference (San Antonio, TX), vol. 25, Spring, 2000, pp. 301–308.

[Usp01] V. Uspenskij, *The Roelcke compactification of groups of homeomorphisms*, Topology Appl. **111**.1-2 (2001), pp. 195–205.

[Usp02] V. Uspenskij, *Compactifications of topological groups*, in: Proceedings of the Ninth Prague Topological Symposium (2001), Topol. Atlas, North Bay, ON, 2002, 331–346 (electronic).

[Usp08] V. Uspenskij, *On subgroups of minimal topological groups*, Topology Appl. **155**.14 (2008), pp. 1580–1606.

[Wag94] F. O. Wagner, *Relational structures and dimensions*, in: Automorphisms of first-order structures, Oxford Sci. Publ. New York: Oxford Univ. Press, 1994, pp. 153–180.

[Wes13] P. Wesolek, *Conjugacy class conditions in locally compact second countable groups*, Preprint `arXiv:1310.1561`, 2013.

[Zuc14a] A. Zucker, *Amenability and unique ergodicity of automorphism groups of Fraïssé structures*, Fund. Math. **226**.1 (2014), pp. 41–62.

[Zuc14b] A. Zucker, *Topological dynamics of closed subgroups of S_∞*, Preprint `arXiv:1404.5057`, 2014.