Subequivalence relations and positive-definite functions

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Abstract. We study a positive-definite function associated with a countable, measure-preserving equivalence relation, which can be used to measure quantitatively the proximity of subequivalence relations. Combined with a co-inducing construction introduced by Epstein and earlier work of Ioana, this can be used to construct many mixing actions of countable groups and establish the non-classifiability, in a strong sense, of orbit equivalence of actions of non-amenable groups. We also discuss connections with percolation on Cayley graphs and the theory of costs.

Contents

Consider a *standard probability space* (X, μ), i.e., a space isomorphic to the unit interval with Lebesgue measure. We denote by $\mathrm{Aut}(X, \mu)$ the automorphism group of (X, μ), i.e., the group of all Borel automorphisms of X which preserve μ (where two such automorphisms are identified if they are equal μ-a.e.). A Borel equivalence relation $E \subseteq X^2$ is called *countable* if every E-class $[x]_E$ is countable, and *measure-preserving* if every Borel automorphism T of X for which $T(x) E x$, is measure-preserving. Equivalently, E is countable, measure-preserving iff it is induced by a measure-preserving action of a countable (discrete) group on (X, μ) (see Feldman–Moore [FM]).

To each countable, measure-preserving equivalence relation E one can assign the positive-definite function $\varphi_E(S)$ on $\mathrm{Aut}(X, \mu)$ given by $\varphi_E(S) = \mu(\{x : S(x) E x\})$;

see Section 1. Intuitively, $\varphi_E(S)$ measures the amount by which S is "captured" by E. This positive-definite function completely determines E.

We use this function to measure the proximity of a pair $E \subseteq F$ of countable, measure-preserving equivalence relations. In Section 2, we show, among other things, the next result, where we use the following notation: if a countable group Γ acts on X, we also write γ for the automorphism $x \mapsto \gamma \cdot x$; if $A \subseteq X$ and E is an equivalence relation on X, then $E|A = E \cap A^2$ is the restriction of E to A; if $E \subseteq F$ are equivalence relations, then $[F : E] = m$ means that every F-class contains exactly m classes; if F is a countable, measure-preserving equivalence relation on (X, μ), then $[F]$ is the *full group* of F, i.e., $[F] = \{T \in \operatorname{Aut}(X, \mu) : T(x) F x, \mu\text{-a.e. } (x)\}$.

Theorem 1. *Let Γ be a countable group and consider a measure-preserving action of Γ on (X, μ) with induced equivalence relation $F = E_\Gamma^X$.*

i) If $E \subseteq F$ is a subequivalence relation and $\inf_{\gamma \in \Gamma} \varphi_E(\gamma) = \varphi_E^0 > 0$, then there is an E-invariant Borel set $A \subseteq X$ of positive measure such that $[F|A : E|A] = m \leq \frac{1}{\varphi_E^0}$, so that if $\varphi_E^0 > \frac{1}{2}$, $F|A = E|A$.

ii) If E is any countable, measure-preserving equivalence relation and $\varepsilon > 0$, then $\forall \gamma \in \Gamma(\varphi_E(\gamma) \geq 1 - \varepsilon)$ implies $\forall S \in [F](\varphi_E(S) \geq 1 - 4\varepsilon)$.

Remark. Popa pointed out that some version of part (i) of the preceding theorem was known in the theory of operator algebras, see, for example, the appendix to Popa [PO1]. Actually our initial proof of that theorem was inspired by Popa's technique of conjugating subalgebras in a finite von Neumann algebra (see Section 2 in [PO2]) but, for consistency with the rest of the article, we give another self-contained, ergodic-theoretic proof.

With some additional work, Theorem 1 has the following consequences:

a) In the context of i), if $\varphi_E^0 > 0$, the action of Γ is *free* (i.e., $\gamma \cdot x \neq x$, $\forall \gamma \neq 1$, for almost all x) and E is induced by a free action of a countable group Δ, then Γ, Δ are measure equivalent (ME).

b) Again in the context of i), if $\varphi_E^0 > \frac{1}{2}$ and E is aperiodic (i.e., has no finite classes), then $C_\mu(F) \leq C_\mu(E)$, where $C_\mu(R)$ is the cost of an equivalence relation R (see [G1] or [KM] for the theory of costs).

c) In i) if $\varphi_E^0 > \frac{3}{4}$, then we can find A so that $\mu(A) \geq 4\varphi_E^0 - 3$.

In Section 3, we consider a recent co-inducing construction of Epstein [E]. Given a measure-preserving, ergodic action b_0 of a countable group Γ on (X, μ) with associated equivalence relation $F = E_\Gamma^X$ and a free, measure-preserving action a_0 of a countable group Δ on (X, μ) with associated equivalence relation $E = E_\Delta^X \subseteq F = E_\Gamma^X$, Epstein's construction gives for any measure-preserving action a of Δ on a space (Y, ν), a measure-preserving action b of Γ on a space (Z, ρ), called the *co-induced action of a modulo (a_0, b_0)*, in symbols $b = \operatorname{CInd}(a_0, b_0)_\Delta^\Gamma(a)$. This construction has important applications in the study of orbit equivalence of actions – see Epstein [E].

For further potential applications of this method, it seems that one should have a better understanding of the connection of ergodic properties between a, b as above. We show, for example, that if b_0 is free, mixing and a_0 is ergodic, then: a is mixing $\Longrightarrow$ b is mixing. There are however interesting situations under which b is always mixing for arbitrary a. It turns out that this phenomenon, for given (a_0, b_0), is connected to the positive-definite function discussed earlier. We show the following:

Theorem 2. *If b_0 is mixing, the following are equivalent:*
   (i) *For all actions a of Δ, $b = \mathrm{CInd}(a_0, b_0)_\Delta^\Gamma(a)$ is mixing.*
   (ii) *$\varphi_E(\gamma) \to 0$ as $\gamma \to \infty$.*

The condition (ii) in Theorem 2 somehow asserts that E is "small" relative to F. In the opposite case we have the following fact. If $\varphi_E^0 = \inf_{\gamma \in \Gamma} \varphi_E(\gamma) > 0$, then b is ergodic $\Longrightarrow a$ is ergodic.

It is well known that for any ergodic b_0 as above one can find a free, mixing action a_0 of $\Delta = \mathbb{Z}$ with $E \subseteq F$ (see, e.g., Zimmer [Z], 9.3.2). We show that when b_0 is mixing, one can find such an a_0 so that (ii) of Theorem 2 holds. This gives a method of producing, starting with arbitrary measure-preserving $\mathbb{Z}$ actions, apparently new types of measure-preserving, mixing actions of any infinite group Γ.

Theorem 3. *Let Γ be an infinite countable group, and let b_0 be a free, measure-preserving, mixing action of Γ on (X, μ). Then there is a free, measure-preserving, mixing action a_0 of $\mathbb{Z}$ on (X, μ) such that $E = E_{\mathbb{Z}}^X \subseteq F = E_{\Gamma}^X$ and $\varphi_E(\gamma) \to 0$ as $\gamma \to \infty$.*

When the group Γ is non-amenable, then by work of Gaboriau–Lyons [GL] one can find a free, mixing action b_0 of Γ on (X, μ) and a free ergodic action a_0 of F_2 on (X, μ) with $E = E_{F_2}^X \subseteq F = E_{\Gamma}^X$. We show again that such a_0 can be found so that (ii) of Theorem 2 holds. This is joint work with I. Epstein.

Theorem 4 (with I. Epstein). *Let Γ be a non-amenable countable group. Then there is a free, measure-preserving, mixing action b_0 of Γ on (X, μ) and a free, measure-preserving, ergodic action a_0 of F_2 on (X, μ) such that $E = E_{F_2}^X \subseteq F = E_{\Gamma}^X$ and $\varphi_E(\gamma) \to 0$ as $\gamma \to \infty$.*

Remark. Our proof of Theorem 4 uses the Gaboriau–Lyons [GL] result and additionally the co-inducing construction to produce a pair of actions satisfying Theorem 4. In [GL] the authors actually produce two different pairs of actions as above. After seeing a preliminary version of our article, Lyons pointed out that their first construction can be shown to satisfy Theorem 4, using results of Benjamini–Lyons–Peres–Schramm [BLPS], in particular formula (13.8) in that paper. Subsequently, we realized that the second construction of [GL] also may give a pair of equivalence relations satisfying Theorem 4. More precisely, if one chooses a Cayley graph of

Γ with sufficiently many generators and $p \in (p_c, p_u)$ close enough to p_c, then the subequivalence relation one obtains using the method of [GL] and our Lemma 4.2 will, in fact, satisfy Theorem 4. See [GL], Pak–Smirnova-Nagnibeda [PS] and the proof of Benjamini–Schramm [BS], Theorem 4, for more details. We also note that sometimes the cluster subequivalence relation E for Bernoulli percolation in the non-uniqueness phase does not satisfy $\varphi_E(\gamma) \to 0$ as $\gamma \to \infty$ (see Lyons–Schramm [LS], Remark 1.3).

We use Theorem 4 to study the complexity of the classification problem of free, measure-preserving, ergodic actions of a countable group Γ under *orbit equivalence* (OE). After a series of earlier results that dealt with various important classes of non-amenable groups (see Gaboriau–Popa [GP], Hjorth [H3], Ioana [I], Kida [KI], Monod–Shalom [MS], Popa [PO2]), Epstein [E] finally showed that in general any non-amenable group admits uncountably many non-orbit equivalent free, measure-preserving, ergodic actions. This was proved earlier by Ioana [I] in the case where $F_2 \leq \Gamma$, and his main lemma in that proof could be also used to derive, in this case, the stronger fact that the equivalence relation E_0 (on $2^{\mathbb{N}}$, where $x E_0 y \iff \exists n \forall m \geq n(x(m) = y(m))$ can be Borel reduced to OE on the space of free, measure-preserving, ergodic actions of Γ. Moreover OE on that space cannot be classified by countable structures (see [K], Section 17, (B)). However, it was not known whether this non-classification result extends to *all* non-amenable groups and whether every non-amenable group admits uncountably many non-orbit equivalent free, measure-preserving, *mixing* actions. Putting together Theorems 2, 4 and the work of Epstein [E] leads now to the following positive answer. This is again a joint result with I. Epstein.

Theorem 5 (with I. Epstein). *Let Γ be a non-amenable countable group. Then E_0 can be Borel reduced to OE on the space of free, measure-preserving, mixing actions of Γ and OE in this space cannot be classified by countable structures.*

Thus we have the following strong dichotomy concerning orbit equivalence: If Γ is (infinite) amenable, there is exactly one free, measure-preserving, ergodic action of Γ up to OE, while if Γ is non-amenable, OE of free, measure-preserving, mixing actions of Γ is unclassifiable in a very strong sense.

The proof of Theorem 5 shows that the conclusion in that theorem also holds if OE is replaced by conjugacy (isomorphism) of actions. This fact is also known to be true for abelian Γ (see [K], 5.7, where the proof is presented for $\mathbb{Z}$ but easily generalizes to any abelian Γ).

In Section 4, we review some basic facts concerning invariant bond percolation on Cayley graphs of finitely generated groups (see Lyons–Schramm [LS] or Lyons-Peres [LP]). We also give in Section 5 (C) an alternative proof of Theorem 4.1 (for Cayley graphs) in Lyons–Schramm [LS], using our Theorem 1.

In Section 5, we apply the preceding results to property (T) groups. Recall that a *Kazhdan pair* (Q, ε) for such a group consists of a finite generating set $Q \subseteq \Gamma$ and

a positive ε such that for any unitary representation π of Γ on a Hilbert space $\mathcal{H}$, if there is a vector $\xi \in \mathcal{H}$ with $\|\pi(\gamma)(\xi) - \xi\| < \varepsilon\|\xi\|$, $\forall \gamma \in Q$, then there is a non-0 invariant vector. We state below some sample results.

Theorem 6. *Let Γ be an infinite group with property* (T)*, (Q, ε) a Kazhdan pair and $\boldsymbol{P}$ an invariant, ergodic, insertion-tolerant bond percolation on the Cayley graph $\mathcal{G}_Q$ of Γ (with respect to Q). If the survival probability $\boldsymbol{P}(\{\omega : \omega(e) = 1\})$ of each edge e is $> 1 - \frac{\varepsilon^2}{2}$, then ω has a unique infinite cluster, $\boldsymbol{P}$-a.s. In particular, if $p_u(Q)$ is the critical probability for existence of unique infinite clusters in Bernoulli percolation on this Cayley graph, then $p_u(Q) \leq 1 - \frac{\varepsilon^2}{2}$.*

Theorem 7. *For each $\rho > 0$ and every infinite group Γ with property* (T)*, there is a finite set of generators Q for Γ such that for any invariant, ergodic, insertion-tolerant bond percolation $\boldsymbol{P}$ on $\mathcal{G}_Q$, if the survival probability of each edge is $\geq \rho$, then ω has a unique infinite cluster, $\boldsymbol{P}$-a.s.*

Remark. Lyons–Schramm [LS] had earlier shown that, in the notation of Theorem 6, $p_u(Q) < 1$. Lyons pointed out that one could also easily deduce a version of Theorem 6 from the results of their paper (with perhaps a different constant instead of $1 - \frac{\varepsilon^2}{2}$). Similarly for Theorem 5.9 below. Finally, Lyons mentions that for the special case $\boldsymbol{P} = \boldsymbol{P}_p$, Bernoulli percolation, Theorem 7 was known even for groups Γ for which there exists Q such that $p_u(Q) < 1$ but which do not necessarily satisfy property (T).

Denote below by $C(\Gamma)$ the cost of a countable group Γ. If Γ is an infinite countable group with property (T) and n is the smallest cardinality of a set of generators for Γ, then we have $1 \leq C(\Gamma) < n$ (the strict inequality follows from Gaboriau [G1], since no free, measure-preserving action of Γ is treeable, see Adams–Spatzier [AS]). At this time no example of a property (T) group with $C(\Gamma) > 1$ is known. We obtain here some upper bounds for $C(\Gamma)$ in terms of n, ε, where $n = \text{card}(Q)$ and (Q, ε) is a Kazhdan pair. One such result is the following:

Theorem 8. *Let Γ be an infinite group with property* (T) *and (Q, ε) a Kazhdan pair for Γ. If $\text{card}(Q) = n$, then*

$$C(\Gamma) \leq n\left(1 - \frac{\varepsilon^2}{2}\right) + \frac{n-1}{2n-1}.$$

Another example is the following.

Theorem 9. *Let Γ be an infinite group with property* (T) *and let (Q, ε) be a Kazhdan pair, where Q contains an element of infinite order. Then if $\text{card}(Q) = n$,*

$$C(\Gamma) \leq n - \frac{\varepsilon^2}{2}.$$

In particular, if Γ is torsion-free and 2-generated, then $C(\Gamma) \leq 2 - \frac{(\varepsilon_2)^2}{2}$, where ε_2 is the sup of the ε such that (Q, ε) is a Kazhdan pair with $\mathrm{card}(Q) = 2$.

Remark. Since in this article we work completely in a measure theoretic context, we neglect null sets if there is no danger of confusion. So given a measure space (X, μ), we do not often distinguish between a statement being true for all $x \in X$ or for all $x \in X$, μ-a.e.

1. Equivalence relations and positive-definite functions on Aut(X, μ)

First we recall the standard definitions of positive-definite and negative-definite functions. Given a set Y, a function $\psi \colon Y \times Y \to \mathbb{C}$ is *positive-definite* if for every finite subset $\{y_1, \dots, y_n\} \subseteq Y$ and every $\alpha_i \in \mathbb{C}$, $1 \leq i \leq n$, we have $\sum_{1 \leq i,j \leq n} \bar{\alpha}_i \alpha_j \psi(y_i, y_j) \geq 0$. Such a function is (*conditionally*) *negative-definite* if $\psi(y, z) = \overline{\psi(z, y)}$ and for every finite subset $\{y_1, \dots, y_n\} \subseteq Y$ and every $\alpha_i \in \mathbb{C}, 1 \leq i \leq n$, with $\sum_{i=1}^n \alpha_i = 0$, we have $\sum_{1 \leq i,j \leq n} \bar{\alpha}_i \alpha_j \psi(y_i, y_j) \leq 0$. A function $\varphi \colon G \to \mathbb{C}$ on a group G is positive-definite (resp., negative-definite) if the function $\psi \colon G \times G \to \mathbb{C}$ defined by $\psi(g, h) = \varphi(g^{-1}h)$ is positive-definite (resp., negative-definite).

Let (X, μ) be a standard measure space and Aut(X, μ) the group of measure-preserving automorphisms of (X, μ). Denote by u the uniform topology on Aut(X, μ), induced by the metric

$$\delta_u(S, T) = \mu(\{x : S(x) \neq T(x)\}).$$

Let $E \subseteq X^2$ be a countable, measure-preserving equivalence relation on X. Define on Aut$(X, \mu)^2$:

$$\psi_E(S, T) = \mu(\{x : S^{-1}(x) E T^{-1}(x)\}).$$

(We use S^{-1}, T^{-1} instead of S, T to make ψ_E left-invariant – see below.) So if $E = \Delta$, the equality relation, then $1 - \psi_E(X, T) = \delta_u(S, T)$. We claim that ψ_E is a continuous, positive-definite function on (Aut$(X, \mu), u$). Continuity is straightforward. The proof that ψ_E is positive-definite is similar to that in Aizenman–Newman [AN]. Fix a finite set $\{S_1, \dots, S_n\} \subseteq$ Aut(X, μ) and $\alpha_i \in \mathbb{C}$, $1 \leq i \leq n$, in order to show that

$$\sum_{1 \leq i,j \leq n} \bar{\alpha}_i \alpha_j \psi_E(S_i, S_j) \geq 0.$$

For each $x \in X$, define the equivalence relation $\sim_x$ on $\{1, \ldots, n\}$ by

$$i \sim_x j \iff S_i^{-1}(x) E S_j^{-1}(x).$$

Let $C_1^x, \ldots, C_{m_x}^x$ be the $\sim_x$-classes. Then we have

$$\sum_{1 \le i,j \le n} \bar{\alpha}_i \alpha_j \psi_E(S_i, S_j) = \int \sum_{1 \le i,j \le n} \bar{\alpha}_i \alpha_j \chi_{\{x : S_i^{-1}(x) E S_j^{-1}(x)\}} d\mu$$

$$= \int \sum_{k=1}^{m_x} \Big(\sum_{i,j \in C_k^x} \bar{\alpha}_i \alpha_j \Big) d\mu(x)$$

$$= \int \sum_{k=1}^{m_x} \Big| \sum_{i \in C_k^x} \alpha_i \Big|^2 d\mu(x) \ge 0.$$

Thus $1 - \psi_E(S, T)$ is negative-definite. In particular, if $E = \Delta$, then

$$\delta_u(S, T) = 1 - \psi_\Delta(S, T),$$

so the metric δ_u is negative-definite.

Note also that ψ_E is left-invariant, so

$$\varphi_E(S) = \psi_E(1, S)$$

is a continuous, positive-definite function on $(\text{Aut}(X, \mu), u)$. If $A_E(S) = \{x : S(x) E x\}$, then $\varphi_E(S) = \mu(A_E(S))$, and we view the quantity $\varphi_E(S)$ as measuring the amount by which S is "captured" by E. By the GNS construction (see, e.g., [BHV], p. 355), there is a (unique) triple $(\pi_E, \mathcal{H}_E, \xi_E)$, consisting of a cyclic continuous representation of $(\text{Aut}(X, \mu), u)$ on a Hilbert space $\mathcal{H}_E$ with cyclic unit vector $\xi_E \in \mathcal{H}_E$ such that

$$\varphi_E(S) = \langle \pi_E(S)(\xi_E), \xi_E \rangle.$$

Now if $[E]$ is the full group of E, then

$$S \in [E] \iff \varphi_E(S) = 1,$$

so φ_E completely determines $[E]$ and thus E, i.e., E is encoded in φ_E. Also note that

$$S \in [E] \iff \pi_E(S)(\xi_E) = \xi_E,$$

i.e., $[E]$ is the stabilizer of ξ_E in π_E.

It is not clear how to characterize the continuous, positive-definite functions φ on $(\text{Aut}(X, \mu), u)$, which are of the form φ_E for some E. Clearly any such φ satisfies $0 \le \varphi \le 1$ and $\varphi(1) = 1$. Another necessary condition is that $\ker(\varphi) = \{S \in \text{Aut}(X, \mu) : \varphi(S) = 1\}$ (which is a closed subgroup of $(\text{Aut}(X, \mu), u)$) is separable in the uniform topology. The following observation may also be relevant here. Let

$\Gamma \leq \ker(\varphi)$ be a countable dense subgroup of $\ker(\varphi)$. If φ is of the form φ_E for some E, then Γ is uniformly dense in $\ker(\varphi) = \ker(\varphi_E) = [E]$, so $E = E_\Gamma^X =$ the equivalence relation induced by Γ.

Next consider the negative-definite function

$$\theta_E(S) = 1 - \varphi_E(S)$$

on $\mathrm{Aut}(X, \mu)$. Put also

$$\delta_u(S, [E]) = \inf\{\delta_u(S, T) : T \in [E]\}$$

for the distance (in δ_u) of S to $[E]$. Then we have

Proposition 1.1. $\theta_E(S) = \delta_u(S, [E]) = \inf\{\delta_u(S, T) : T \in [E]\}$ *and moreover this* inf *is attained.*

The proof of Proposition 1.1 uses the following well-known fact:

Lemma 1.2. *Let* $S \in \mathrm{Aut}(X, \mu)$ *and let* E *be a countable, measure-preserving equivalence relation on* X. *Then there is* $T \in [E]$ *such that* $S(x) = T(x)$ *whenever* $S(x)Ex$.

Proof of Proposition 1.1. Given any S, find T as in Lemma 1.2 and note that

$$\delta_u(S, T) = \mu(\{x : \neg S(x)Ex\}) = \theta_E(S).$$

On the other hand, for any $R \in [E]$, $\{x : \neg S(x)Ex\} \subseteq \{x : S(x) \neq R(x)\}$, so $\theta_E(S) \leq \delta_u(S, R)$, thus $\theta_E(S) = \delta_u(S, T) = \delta_u(S, [E])$. □

Proof of Lemma 1.2. Let $A = \{x : S(x)Ex\}$ and $B = S(A)$. It is enough to find a Borel bijection (modulo null sets) $S' : A \cup B \to A \cup B$ with $S'(x) = S(x)$ for $x \in A$ and $S'(x)Ex$, for $x \in A \cup B$. Then we can take $T = S' \cup \mathrm{id}|(X \setminus (A \cup B))$.

Put $Y = A \cup B$ and consider the equivalence relation F on Y induced by $S|A$. Some F-classes C will consist of a cycle $\{x, S(x), \ldots, S^n(x)\}$, where $S^{n+1}(x) = x$. For such C, we have $C \subseteq A$, so we can let $S'(x) = S(x)$, $\forall x \in C$. In every other F-class C, we can define the ordering

$$x <_C y \iff \exists n > 0(S^n(x) = y).$$

The union of the infinite C in which there is a largest or smallest element in $<_C$ has clearly measure 0. So we can assume that $<_C$ is either a finite ordering, with largest and smallest elements b_C, a_C, respectively, in which case $A \cap C = C \setminus \{b_C\}$ or else $<_C$ looks like a copy of the order on $\mathbb{Z}$, in which case $C \subseteq A$. In the first case, we define S' on C by $S'(x) = S(x)$ if $x \neq b_C$ and $S'(b_C) = a_C$. In the second case, we put $S'(x) = S(x)$, $\forall x \in C$. This clearly works. □

Note that if δ_u also denotes the metric induced by δ_u on the homogeneous space $\text{Aut}(X, \mu)/[E]$, i.e.,

$$\delta_u(S[E], T[E]) = \inf\{\delta_u(S', T') : S' \in S[E], T' \in T[E]\}$$
$$= \delta_u(S, T[E]) = \delta_u(T, S[E]),$$

then

$$\delta_u(S[E], T[E]) = \delta_u(S^{-1}T, [E]) = \theta_E(S^{-1}T) = 1 - \psi_E(S, T),$$

so δ_u on $\text{Aut}(X, \mu)/[E]$, with the quotient topology of u, is a continuous, negative-definite function.

And we conclude with some further observations on metrics on $\text{Aut}(X, \mu)$ and certain subgroups of it.

The weak topology w on $\text{Aut}(X, \mu)$ is induced by the metric

$$\delta_w(S, T) = \sum_{n=1}^{\infty} 2^{-n}\mu(S(A_n)\Delta T(A_n)),$$

where $\{A_n\}$ is dense in the measure algebra MALG_μ of (X, μ). Now for each fixed Borel set $A \subseteq X$,

$$\rho_A(S, T) = \mu(S(A)\Delta T(A))$$

is negative-definite, since

$$\rho_A(S, T) = \int |\chi_{S(A)} - \chi_{T(A)}|^2 \, d\mu = \|\chi_{S(A)} - \chi_{T(A)}\|_2^2,$$

and the function $(\xi, \eta) \mapsto \|\xi - \eta\|_2^2$ is negative-definite on $L^2(X, \mu)$. It follows that the left-invariant metric δ_w is negative-definite. In particular, the complete metric $\bar{\delta}_w(S, T) = \delta_w(S, T) + \delta_w(S^{-1}, T^{-1})$ on $\text{Aut}(X, \mu)$ is also negative-definite.

Now consider an aperiodic (i.e., having infinite classes) E and the normalizer $N[E]$ of its full group. Then $N[E]$ has a canonical topology induced by the complete metric

$$\bar{\delta}_{N[E]}(S, T) = \bar{\delta}_w(S, T) + \sum_{n=1}^{\infty} 2^{-n}\delta_u(S\gamma_n S^{-1}, T\gamma_n T^{-1}),$$

where $\{\gamma_n\}$ is a countable subgroup of $\text{Aut}(X, \mu)$ inducing E (see, e.g., Kechris [K]). Since for each n, the function $(S, T) \mapsto \delta_u(S\gamma_n S^{-1}, T\gamma_n T^{-1})$ is negative-definite, so is $\bar{\delta}_{N[E]}(S, T)$ on $N[E]$.

2. Proximity of subequivalence relations

(A) We view the quantity $\varphi_E(S) = \mu(\{x : S(x)Ex\})$ as measuring the amount by which S is captured by E. We will next see that if a countable group Γ acts in a

measure-preserving way on (X, μ) inducing an equivalence relation $F = E_\Gamma^X$, and every element of Γ (viewed as an element of $\text{Aut}(X, \mu)$ via $x \mapsto \gamma \cdot x$, $\gamma \in \Gamma$) is "substantially captured" by E, then E, F are somehow "close" to each other.

Towards this goal we will study a canonical representation associated to a pair $E \subseteq F$ of countable measure-preserving equivalence relations on (X, μ).

Let such E, F be given and decompose $X = \bigsqcup_{N \in \{1,2,\dots,\aleph_0\}} X_N$, where $X_N = \{x : \text{there are exactly } N \text{ } E\text{-classes in } [x]_F\}$, so that X_N is F-invariant. Therefore $[F|X_N : E|X_N] = N$. If F is ergodic, clearly $X = X_N$ for some N. Fix now for each N a sequence of Borel functions $\{C_n^{(N)}\}_{n \in N}$, $C_n^{(N)} : X_N \to X_N$, where we identify N here with $\{0, \dots, N-1\}$ if N is finite, and with $\mathbb{N}$ if $N = \aleph_0$ such that $C_0^{(N)} = \text{id}|X_N$ for each $x \in X_N$, $C_n^{(N)}(x) \neq C_m^{(N)}(x)$, if $m \neq n$, and $\{C_n^{(N)}(x)\}$ is a transversal for the E-classes contained in $[x]_F$. These are called *choice functions*.

Remark 2.1. For further reference, notice that if E is ergodic, so that $X = X_N$ for some N, in which case we can write $C_n^{(N)} = C_n$ if there is no danger of confusion, then we can take the choice functions $C_n^{(N)} = C_n$ to be 1-1, i.e., to be in $\text{Aut}(X, \mu)$. To see this, start with arbitrary $\{C_n^{(N)}\} = \{C_n\}$. Fix $n \in N$ and consider C_n. As it is countable-to-1, let $X = \bigsqcup_{k=1}^\infty Y_k$ be a Borel partition such that $C_n|Y_k$ is 1-1. Let then $Z_k = C_n(Y_k)$, so that $\mu(Z_k) = \mu(Y_k)$. Since E is ergodic, there is $T_k \in [E]$ with $T_k(Z_k) = Y_k$. Let then $D_n(x) = T_k(C_n(x))$, if $x \in Y_k$. We have $D_n(x) E C_n(x)$, $\forall x$, and D_n is 1-1. So $\{D_n\}$ are choice functions and each D_n is 1-1.

Define now the *index cocycle* $\pi_N : F|X_N \to S_N$ ($=$ the symmetric group of N) by the formula

$$\pi_N(x, y)(k) = n \iff [C_k(x)]_E = [C_n(y)]_E$$

(see Feldman–Sutherland–Zimmer [FSZ]). Finally, we can define $\sigma_N : [F|X_N] \times X_N \to S_N$ by

$$\sigma_N(S, x) = \pi_N(x, S(x)).$$

Since $S \in [F|X_N]$ is not a function but an equivalence class of functions identified μ-a.e., σ_N again is to be understood as an equivalence class of functions $(\sigma_N)_S(x) = \sigma_N(S, x)$ identified μ-a.e. We again have the cocycle identity: For each $S, T \in [F|X_N]$,

$$\sigma_N(ST, x) = \sigma_N(S, T(x))\sigma_N(T, x)$$

for almost all $x \in X_N$.

Consider now the Hilbert space

$$\mathcal{H} = \bigoplus_N L^2(X_N \times N),$$

where $X_N \times N$ has the σ-finite measure $(\mu | X_n) \times \mu_N$, with $\mu_N =$ the counting measure on N, and the unitary representation τ of $[F]$ on $\mathcal{H}$ given by

$$\tau(S)(\bigoplus_N f_N) = \bigoplus_N g_N, \quad g_N(x, n) = f_N(S^{-1}(x), \sigma_N(S^{-1}, x)(n)),$$

for $(x, n) \in X_N \times N$, $f_N \in L^2(X_N \times N)$. Clearly each $L^2(X_N \times N)$ is invariant.

Notice that the representation τ is independent of the choice functions $\{C_n^{(N)}\}$, up to unitary equivalence.

Consider the unit vector

$$\xi_0 = \bigoplus_N \chi_{X_N \times \{0\}}$$

in $\mathcal{H}$. Then for $S \in [F]$,

$$\begin{aligned}
\langle \tau(S)(\xi_0), \xi_0 \rangle &= \sum_N \int_{X_N \times N} \xi_0(S^{-1}(x), \sigma_N(S^{-1}, x)(n)) \xi_0(x, n) \, d\mu(x) d\mu_N(n) \\
&= \sum_N \mu(\{x \in X_N : \sigma_N(S^{-1}, x)(0) = 0\}) \\
&= \sum_N \mu(\{x \in X_N : S^{-1}(x) E x\}) \\
&= \sum_N \mu(\{x \in X_N : x E S(x)\}) = \mu(\{x : S(x) E x\}) = \varphi_E(S).
\end{aligned}$$

Thus the representation τ restricted to the closed span of $\{\tau(S)(\xi_0) : S \in [F]\}$ is the GNS representation of $[F]$ associated with φ_E.

If now Γ is a countable group acting in a Borel way on (X, μ) so that $E_\Gamma^X = F$, then, denoting by γ also the map $x \mapsto \gamma \cdot x$, the cocycle σ_N restricts to a cocycle, also denoted by σ_N, from $\Gamma \times X_N$ to S_N: $\sigma_N(\gamma, x) = \pi_N(x, \gamma \cdot x)$. Similarly, the representation τ restricts to a representation, also denoted by τ, of Γ on $\mathcal{H}$.

We now characterize the condition on $E \subseteq F$ under which the representation τ has an invariant non-0 vector. First we note the following:

Proposition 2.2. *A vector ξ is invariant under the Γ-representation iff ξ is invariant under the $[F]$-representation.*

Proof. Suppose ξ is invariant under the Γ-representation, i.e., for $\xi = \bigoplus_N \xi_N$,

$$\xi_N(x, n) = \xi_N(\gamma^{-1} \cdot x, \sigma_N(\gamma^{-1}, x)(n)),$$

for all $x \in X_N$, for all $\gamma \in \Gamma$ (neglecting as usual null sets). Let now $S \in [F]$. Then for each $x \in X_N$ there is $\gamma = \gamma_x \in \Gamma$ with $S^{-1}(x) = \gamma^{-1} \cdot x$. Thus

$$\tau(S)(\xi_N)(x, n) = \xi_N(S^{-1}(x), \sigma_N(S^{-1}, x)(n))$$

and $\sigma_N(S^{-1}, x)(n) = k$, where

$$[C_n(x)]_E = [C_k(S^{-1}(x))]_E = [C_k(\gamma^{-1} \cdot x)]_E,$$

so $\sigma_N(\gamma^{-1}, x)(n) = k = \sigma_N(S^{-1}, x)(n)$, therefore

$$\tau(S)(\xi_N)(x, n) = \xi_N(\gamma^{-1} \cdot x, \sigma_N(\gamma^{-1}, x)(n)) = \xi_N(x, n),$$

i.e., ξ_N and thus ξ is also invariant under $\tau(S)$.   □

We now have:

Proposition 2.3. *The representation τ has an invariant non-0 vector iff there is a Borel set $A \subseteq X$ of positive measure which is E-invariant and $1 \leq m < \infty$ such that $[F|A : E|A] = m$, i.e., on some E-invariant Borel set of positive measure A, there are exactly m E-classes contained in each $F|A$-class. In particular, if E is ergodic, $[F : E] < \infty$.*

Proof. If such an A exists, we can clearly assume that $A \subseteq X_N$ for some N. Then let $B \subseteq X_N \times N$ be defined by

$$(x, n) \in B \iff [C_n^N(x)]_E \subseteq A.$$

Clearly for each $x \in X_N$, $B_x = \{n : (x, n) \in B\}$ has cardinality $\leq m$, so if $\xi = \chi_B$, then $\xi \in L^2(X_N \times N)$ and obviously $\xi \neq 0$. Now we claim that ξ is Γ-invariant, i.e., for $\gamma \in \Gamma$,

$$\xi(x, n) = \xi(\gamma^{-1} \cdot x, \sigma_N(\gamma^{-1}, x)(n)).$$

This is clear, since $[C_n^N(x)]_E = [C_{\sigma_N(\gamma^{-1}, x)(n)}^N(\gamma^{-1} \cdot x)]$ by the definition of σ_N.

Conversely, let $\xi \in \mathcal{H}$ be non-0 and Γ-invariant. Clearly we can assume that $\xi \in L^2(X \times N)$ for some N. Now

$$0 < \int_{X_N} \sum_{n \in N} |\xi(x, n)|^2 \, d\mu < \infty,$$

so $\sum_{n \in N} |\xi(x, n)|^2 < \infty$ for almost all $x \in X_N$. Let then $N_x = \{n \in N : |\xi(x, n)| \text{ is maximal among all } |\xi(x, i)|, i \in N\}$. Let a_x be this maximum. Then N_x is finite, provided that $a_x > 0$. Since ξ is Γ-invariant, we have

$$\xi(x, n) = \xi(\gamma^{-1} \cdot x, \sigma_N(\gamma^{-1}, x)(n)),$$

so $n \in N_x \iff \sigma_N(\gamma^{-1}, x)(n) \in N_{\gamma^{-1} \cdot x}$, $\mathrm{card}(N_x) = \mathrm{card}(N_{\gamma^{-1} \cdot x})$, and $a_x = a_{\gamma^{-1} \cdot x}$, thus $x \mapsto \mathrm{card}(N_x)$, $x \mapsto a_x$, are F-invariant. Also as

$$\int_{X_N} \sum_{n \in N} |\xi(x, n)|^2 \, d\mu > 0,$$

$\{x \in X_N : a_x > 0\}$ has positive measure. So fix $m > 0$ and a set $Y \subseteq X_N$ of positive measure, which is F-invariant, and for $x \in Y$ we have $a_x > 0$ and $m = \mathrm{card}(N_x)$. Let then

$$A = \bigcup \{[C_n^N(x)]_E : x \in Y, n \in N_x\}.$$

Then A is E-invariant, has positive measure and $[F|A : E|A] = m$.   □

Consider now the closed convex hull C of $\{\gamma \cdot \xi_0 : \gamma \in \Gamma\}$, where $S \cdot \xi_0 = \tau(S)(\xi_0)$. Since $\varphi_E(\gamma) = \langle \gamma \cdot \xi_0, \xi_0 \rangle$, we see that if $\inf_{\gamma \in \Gamma} \varphi_E(\gamma) = \varphi_E^0 > 0$, then $\langle \gamma \cdot \xi_0, \xi_0 \rangle \geq \varphi_E^0, \forall \gamma \in \Gamma$, so $\langle \eta, \xi_0 \rangle \geq \varphi_E^0, \forall \eta \in C$. If then ξ is the unique element of least norm in C, we have $\langle \xi, \xi_0 \rangle \geq \varphi_E^0$, and thus $\xi \neq 0$. Clearly ξ is invariant (under Γ and thus $[F]$). Thus we have

Proposition 2.4. *If* $\inf_{\gamma \in \Gamma} \varphi_E(\gamma) = \varphi_E^0 > 0$*, then there is a non-$0$ invariant vector for* τ*.*

(B) We can conclude from 2.3 and 2.4 that if $\varphi_E^0 > 0$, then there is an E-invariant set of positive measure A such that $[F|A : E|A] = m < \infty$. We can in fact obtain an estimate for such m (and prove a somewhat stronger version).

Theorem 2.5. *Let* Γ *be a countable group and consider a measure-preserving action of* Γ *on* (X, μ) *with associated equivalence relation* $F = E_\Gamma^X$*. Let* $E \subseteq F$ *be a subequivalence relation. Let* $S, S' \in [F]$ *and assume that* $\inf_{\gamma \in \Gamma} \varphi_E(S\gamma S') = c > 0$*. Then there is an* E*-invariant Borel set* A *of positive measure such that* $[F|A : E|A] = m \leq \frac{1}{c}$*. In particular, if* $c > \frac{1}{2}, F|A = E|A$*.*

Proof. Since $S\gamma S' = (SS')((S')^{-1}\gamma S')$, by replacing the action $\gamma \cdot x$ of Γ by the conjugate action $\gamma * x = (S')^{-1}(\gamma \cdot S'(x))$, which also induces F, we can assume that $S' = \text{id}$. Thus we have $\inf_{\gamma \in \Gamma} \varphi_E(S\gamma) = c > 0$. So $\langle S\gamma \cdot \xi_0, \xi_0 \rangle \geq c$, or $\langle \gamma \cdot \xi_0, S^{-1} \cdot \xi_0 \rangle \geq c, \forall \gamma \in \Gamma$, thus if C is the closed convex hull of $\{\gamma \cdot \xi_0 : \gamma \in \Gamma\}$, and ξ the element of least norm in C, ξ is invariant for τ and $\langle \xi, S^{-1} \cdot \xi_0 \rangle = \langle S \cdot \xi, \xi_0 \rangle = \langle \xi, \xi_0 \rangle \geq c$.

Now fix $\varepsilon > 0$ and let $\alpha_1, \ldots, \alpha_k \in [0, 1]$, with $\sum_{i=1}^k \alpha_i = 1$, and $\gamma_1, \ldots, \gamma_k \in \Gamma$ be such that if $\xi' = \sum_{i=1}^k \alpha_i(\gamma_i \cdot \xi_0)$, then $\|\xi' - \xi\| \leq \varepsilon$. Then, as ξ is invariant, for any $T \in [F]$ we have

$$\langle T \cdot \xi', \xi_0 \rangle = \langle T \cdot (\xi' - \xi), \xi_0 \rangle + \langle \xi, \xi_0 \rangle \geq c - \varepsilon$$

(note that $\langle T \cdot \xi', \xi_0 \rangle$ is real). Thus for any $T \in [F]$,

$$\sum_{i=1}^k \alpha_i \langle T\gamma_i \cdot \xi_0, \xi_0 \rangle \geq c - \varepsilon$$

or

$$\sum_{i=1}^k \alpha_i \varphi_E(T\gamma_i) \geq c - \varepsilon. \tag{1}$$

We will now use the following lemma.

Lemma 2.6. *Assume* $E \subseteq F$ *are countable, measure-preserving equivalence relations on* (X, μ) *and let* $n \geq 1$*. Then either there is an* E*-invariant Borel set* $A \subseteq X$ *of positive measure such that every* $F|A$*-class contains at most* n $E|A$*-classes or there are* $T_0, \ldots, T_n \in [F]$ *such that* $T_0(x) = x$ *and* $[T_i(x)]_E \neq [T_j(x)]_E$ *if* $i \neq j$*.*

Assuming the lemma, take $n = [\frac{1}{c}]$. If the first case of 2.6 occurs, then the conclusion of the theorem immediately follows, so it is enough to show that no such $T_0, \ldots, T_n$ exist. Otherwise, apply (1) to $T_0, \ldots, T_n$ to get

$$\sum_{j=0}^{n} \sum_{i=1}^{k} \alpha_i \varphi_E(T_j \gamma_i) \geq (n+1)(c - \varepsilon).$$

But notice that

$$\sum_{j=0}^{n} \varphi_E(T_j T) \leq 1, \ \forall T \in [F].$$

This holds since

$$\sum_{j=0}^{n} \varphi_E(T_j T) = \sum_{j=0}^{n} \mu(\{x : T_j T(x) E x\})$$

and the sets $\{x : T_j T(x) E x\}$, $j = 0, \ldots, n$, are pairwise disjoint. Thus

$$(n+1)(c - \varepsilon) \leq \sum_{i=1}^{k} \alpha_i = 1,$$

so, as ε is arbitrary, $n + 1 \leq \frac{1}{c}$, a contradiction.

So it only remains to give the proof of 2.6. □

Proof of Lemma 2.6. Assume first that F is ergodic.

Consider then the ergodic decomposition of E (see, e.g., [KM], Theorem 3.3). This is given by a Borel map $\Sigma \colon X \to \mathcal{E}$, where $\mathcal{E}$ is the standard Borel space of invariant, ergodic probability measures for E such that: (i) Σ is E-invariant; (ii) if $e \in \mathcal{E}$ and $X_e = \Sigma^{-1}(\{e\})$, then $e(X_e) = 1$ and e is the unique E-invariant probability measure on X_e; (iii) if $\Sigma_* \mu = \mu_*$, then $\mu = \int e \, d\mu_*(e)$, i.e., $\mu(B) = \int e(B) \, d\mu_*(e)$ for all Borel sets $B \subseteq X$.

Let $\mathcal{E}_0$ be the atomic part of μ_*, and put $\mathcal{E}_1 = \mathcal{E} \setminus \mathcal{E}_0$. Split $X_1 = \bigcup_{e \in \mathcal{E}_1} X_e$ into E-invariant Borel sets $X_1 = A_0 \sqcup \cdots \sqcup A_n$, where $\mu(A_i) = \mu(A_j), \forall i, j$, and let $\varphi_{i,j} \in [F | X_1]$ be such that $\varphi_{i,j}(A_i) = A_j$, $0 \leq i, j \leq n$. Now let $\psi_0, \ldots, \psi_n \colon \{0, \ldots, n\} \to \{0, \ldots, n\}$ be the bijections defined by

$$\psi_i(m) = (m + i) \mod (n+1).$$

Then define $\varphi_i^{(1)} \in [F | X_1]$ by

$$\varphi_i^{(1)} | A_m = \varphi_{m, \psi_i(m)}$$

(so that $\varphi_i^{(1)}(A_m) = A_{\psi_i(m)}$). Note that $\neg \varphi_i^{(1)}(x) E \varphi_j^{(1)}(x)$ if $i \neq j$. Thus we have found $\varphi_0^{(1)}, \ldots, \varphi_n^{(1)} \in [F | X_1]$ with $\varphi_0^{(1)}(x) = x, \neq \varphi_i^{(1)}(x) E \varphi_j^{(1)}(x)$ if $i \neq j$.

Consider now $e \in \mathcal{E}_0$, so that $\mu(X_e) > 0$. If $[F | X_e : E | X_e] \leq n$, then $A = X_e$ satisfies the first alternative of the lemma. So we can assume that $[F | X_e :$

$E|X_e] \geq n + 1$, $\forall e \in \mathcal{E}_0$. Since $E|X_e$ is ergodic, we can find $\varphi_0^e, \ldots, \varphi_n^e \in [F|X_e]$ with $\varphi_0^e(x) = x$ and $\neq \varphi_i^e(x) E \varphi_j^e(x)$ if $i \neq j$ (see 2.1). Let $\varphi_i^{(0)} = \bigcup_{e \in \mathcal{E}_0} \varphi_i^e$. Thus $\varphi_i^{(0)} \in [F|X_0]$, where $X_0 = \bigcup_{e \in \mathcal{E}_0} X_e = X \setminus X_1$, and $\varphi_0^{(0)}(x) = x$, $\neq \varphi_i^{(0)}(x) E \varphi_j^{(0)}(x)$ if $i \neq j$. Finally let $T_i = \varphi_i^{(0)} \cup \varphi_i^{(1)}$. This clearly works.

If F is not ergodic, consider its ergodic decomposition and apply the preceding argument to each piece of the ergodic decomposition. □

We also have the following result concerning the "proximity" of E to F.

Theorem 2.7. *Let Γ be a countable group and consider a measure-preserving action of Γ on (X, μ) with associated equivalence relation $F = E_\Gamma^X$. Let E be a countable measure-preserving equivalence relation on (X, μ). If $\varepsilon > 0$ is such that $\varphi_E(\gamma) \geq 1 - \varepsilon$, $\forall \gamma \in \Gamma$, then $\varphi_E(S) \geq 1 - 4\varepsilon$, $\forall S \in [F]$.*

Proof. Since $\varphi_E(S) = \varphi_{E \cap F}(S)$ for $S \in [F]$, we can assume that $E \subseteq F$.

In the earlier notation of Section 2 (A) concerning the representation τ of $[F]$, we have that $\langle \gamma \cdot \xi_0, \xi_0 \rangle \geq 1 - \varepsilon$, $\forall \gamma \in \Gamma$ (where we put as before $\tau(S)(\xi) = S \cdot \xi$). If ξ is the element of least norm in the closed convex hull of $\{\gamma \cdot \xi_0 : \gamma \in \Gamma\}$, then ξ is invariant for τ and $\langle \xi, \xi_0 \rangle \geq 1 - \varepsilon$, thus $\|\xi - \xi_0\|^2 \leq 2(1 - \langle \xi, \xi_0 \rangle) \leq 2\varepsilon$. Thus for any $S \in [F]$, $\|\xi - S \cdot \xi_0\|^2 = \|S \cdot \xi - S \cdot \xi_0\|^2 \leq 2\varepsilon$, so $\|S \cdot \xi_0 - \xi_0\| \leq 2\sqrt{2\varepsilon}$, therefore $2(1 - \varphi_E(S)) \leq 8\varepsilon$, and so $\varphi_E(S) \geq 1 - 4\varepsilon$. □

(C) We will next derive some consequences of the preceding results. We refer to Gaboriau [G3] for information about the concept of *measure equivalence* (ME) introduced by Gromov.

Lemma 2.8. *Let Γ, Δ be two countable groups and consider free, measure-preserving actions of Γ, Δ on (X, μ) with associated equivalence relations $E = E_\Delta^X \subseteq F = E_\Gamma^X$. If there is an E-invariant Borel set $A \subseteq X$ such that every $F|A$-class contains at most finitely many $E|A$-classes, then Γ, Δ are ME.*

Proof. By shrinking A we can assume that there is n such that $[F|A : E|A] = n$. Fix Borel $T_1, \ldots, T_n \colon A \to A$ such that the $F|A$-class of x is the union of the $E|A$-classes of $T_i(x)$, $i = 1, \ldots, n$.

Let $\Omega = \{(x, y) : x \in A, y \in X, xFy\} \subseteq F$ and consider on Ω the σ-finite Borel measure $\nu|\Omega$, where ν is the σ-finite Borel measure on F given by $\nu(B) = \int_X \mathrm{card}(B \cap F^x) \, d\mu(x)$ for every Borel $B \subseteq F$. Then Δ acts on Ω by $\delta \cdot (x, y) = (\delta \cdot x, y)$, since A is Δ-invariant, and Γ acts on Ω by $\gamma \cdot (x, y) = (x, \gamma \cdot y)$. These actions clearly preserve $\nu|\Omega$ and commute. So it is enough to show that each of the Δ, Γ actions admits a transversal (fundamental domain) of finite measure. Let $T \colon [A]_F \to A$ be a Borel map such that $T(y) F y$. Then $\Omega_1 = \{(T_i T(y), y) : y \in [A]_F, i = 1, \ldots, n\}$ is a finite measure transversal for the Δ-action and $\Omega_2 = \{(x, x) : x \in A\}$ is a finite measure transversal for the Γ-action. □

Corollary 2.9. *In the context of Lemma 2.8, if there are $S, S' \in [F]$ such that $\inf_{\gamma \in \Gamma} \varphi_E(S \gamma S') > 0$, then Γ, Δ are ME.*

Corollary 2.10. *Let Γ be a countable group and consider a free measure-preserving action of Γ on (X, μ) with associated equivalence relation $F = E_\Gamma^X$. If $E \subseteq F$ is ergodic, treeable and there are $S, S' \in [F]$ such that $\inf_{\gamma \in \Gamma} \varphi_E(S \gamma S') > 0$, then Γ has the Haagerup Approximation Property (HAP).*

Proof. By Hjorth [H4], E is given by a free action of a group Δ. Then Δ has the HAP (see, e.g., Gaboriau [G3]), and Γ, Δ are ME by 2.8, so Γ has the HAP (see again Gaboriau [G3]). $\square$

Below, for an equivalence relation F, we denote by $[[F]]$ the set of measure-preserving bijections $\theta : \operatorname{dom}(\theta) \to \operatorname{rng}(\theta)$ wit h $\operatorname{dom}(\theta), \operatorname{rng}(\theta)$ Borel sets and $\theta(x) F(x)$, for almost all $x \in \operatorname{dom}(\theta)$.

Lemma 2.11. *Let F be a countable, measure-preserving, ergodic equivalence relation on (X, μ) and let $E \subseteq F$. Let $X_\infty = \{x \in X : [x]_E \text{ is infinite}\}$. Assume that there is an E-invariant Borel set A of positive measure such that $F|A = E|A$ (and thus $A \subseteq X_\infty$). Then for every $\varepsilon > 0$, there is $\theta \in [[F]]$ with $\mu(\operatorname{dom}(\theta)) < \varepsilon$ such that if $E \vee \theta$ is the subequivalence relation generated by E and θ, then $(E \vee \theta)|X_\infty = F|X_\infty$ a.e. So if in addition E is aperiodic (i.e., $X_\infty = X$), then $E \vee \theta = F$ a.e.*

Moreover, if F is given by an action of a finitely generated group Γ and $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is a set of generators for Γ, then there are Borel sets $B_1, \ldots, B_n$ with $\mu(B_i) < \varepsilon, \forall i \leq n$, such that if $E' = E \vee \gamma_1|B_1 \vee \cdots \vee \gamma_n|B_n$, then $E'|X_\infty = F|X_\infty$.

Proof. Consider the ergodic decomposition of E, as in the proof of 2.6, whose notation we keep below.

Since $\mu(A) > 0$ and $E|A$ is ergodic, it follows that the measure μ_* has atoms, and $A = X_{e_0}$, for some atom e_0 of μ_*. Let $\mathcal{E}_0$ be the set of atoms of μ_*, $\mathcal{E}_1 = \{e \in \mathcal{E} \setminus \mathcal{E}_0 : e \text{ is non-atomic}\}$ and $\mathcal{E}_2 = \mathcal{E} \setminus (\mathcal{E}_0 \cup \mathcal{E}_1) = \{\sigma \in \mathcal{E} : e \text{ is atomic}\}$. Note that $X_{\mathcal{E}_2} = \bigcup_{e \in \mathcal{E}_2} X_e = X_{\text{fin}} = X \setminus X_\infty$. We can clearly assume that $\mu_*(\mathcal{E} \setminus (\mathcal{E}_2 \cup \{e_0\})) > 0$, otherwise there is nothing to prove.

Fix now $\mu(X_{e_0}) > \varepsilon > 0$. If $e \notin \mathcal{E}_2$, then e is not atomic, so we can find $Y_e \subseteq X_e$ a Borel set with $e(Y_e) = \varepsilon/2$. Then, by ergodicity, Y_e meets every $E|X_e$-class. Let $Y = \bigcup_{e \notin \mathcal{E}_2, e \neq e_0} Y_e$, so that $0 < \mu(Y) < \varepsilon$. Then there is $\theta \in [[F]]$ with $\operatorname{dom}(\theta) = Y$, $\operatorname{rng}(\theta) \subseteq X_{e_0}$. We claim that if $\bar{E} = E \vee \theta$, then $\bar{E}|X_\infty = F|X_\infty$. To see this note that if $y \in X_e$ for some $e \notin \mathcal{E}_2$, $e \neq e_0$, then there is $z \in Y_e$ with $y E z$. Thus $y \bar{E} \theta(z) \in X_{e_0}$. So every $y \in X_\infty$ is $\bar{E}$-equivalent to an element of X_{e_0}. Since $E|X_{e_0} = F|X_{e_0}$, we are done.

For the last assertion, decompose $\operatorname{dom}(\theta) = Y$ into countably many Borel sets of positive measure $\{Y_m\}_{m=1}^\infty$, so that there are words $\{w_m\}_{m=1}^\infty$ in $\{\gamma_1, \ldots, \gamma_n\}$ with

$\theta|Y_m = w_m|Y_m$. Say w_m has length k_m. Then find a Borel set $Z_m \subseteq Y_m$ such that $\mu(Z_m) < \frac{1}{k_m}\mu(Y_m)$ and for every $e \notin \mathcal{E}_2$, $e \neq e_0$, if $e(Y_m \cap X_e) > 0$, then $e(Z_m \cap X_e) > 0$ (again ignoring null sets for μ_*). Let $Z = \bigcup_m Z_m$, so that $e(Z \cap X_e) > 0$ for each $e \neq \mathcal{E}_2$, $e \neq e_0$. Let $\theta' = \theta|Z$. Then, as before, $(E \vee \theta')|X_\infty = F|X_\infty$. Note now that if $w_m = \gamma_{i_1}^{\pm 1}\gamma_{i_2}^{\pm 1}\ldots\gamma_{k_m}^{\pm 1}$, then $\theta|Z_m = \theta'|Z_m$ is a composition of $\gamma_{i_{k_m}}^{\pm 1}|Z_m$, $\gamma_{i_{k_m-1}}^{\pm 1}|\gamma_{i_{k_m}}^{\pm 1}(Z_m),\ldots$. Thus $\mathrm{graph}(\theta|Z_m) \subseteq \gamma_1|B_1^{(m)} \vee \cdots \vee \gamma_n|B_n^{(m)} =$ the equivalence relation generated by $\gamma_i|B_i^{(m)}$, $1 \leq i \leq n$, where $B_i^{(m)}$ are Borel sets with $\mu(B_i^{(m)}) \leq k_m \cdot \frac{1}{k_m} \cdot \mu(Y_m) = \mu(Y_m)$, $\forall i \leq n$. Let $B_i = \bigcup_m B_i^{(m)}$. Then $\mu(B_i) \leq \sum_m \mu(Y_m) < \varepsilon$ and $E \vee \theta' \subseteq E'$, so $E'|X_\infty = F|X_\infty$. $\qquad\square$

Remark 2.12. In the notation of 2.11, it is clear that, under the same hypothesis, if E is also ergodic, then $E = F$.

Below we denote by $C_\mu(E)$ the cost of the countable, measure-preserving equivalence relation E on (X, μ) (see Gaboriau [G1] or Kechris–Miller [KM]).

Corollary 2.13. *Let Γ be a countable group and consider a measure-preserving, ergodic action of Γ on (X, μ) with associated equivalence relation $F = E_X^\Gamma$. Let $E \subseteq F$ be a subequivalence relation. If E is aperiodic and $\inf_{\gamma \in \Gamma} \varphi_E(S\gamma S') > \frac{1}{2}$ for some $S, S' \in [F]$, then $C_\mu(F) \leq C_\mu(E)$.*

Proof. By 2.11 and 2.5. $\qquad\square$

The following gives a quantitative version of (part of) 2.5.

Lemma 2.14. *Let F be a measure-preserving, ergodic equivalence relation on (X, μ) and $E \subseteq F$ a subequivalence relation. Assume that $\forall S \in [F]\ (\varphi_E(S) > 0)$. Then there is an E-invariant Borel set A of positive measure such that $E|A = F|A$. Moreover, for some $S \in [F]$, $\{x : S(x)Ex\} \subseteq A$, so that $\mu(A) \geq \varphi_E(S)$.*

Proof. Consider the ergodic decomposition of E as in the proof of 2.6, whose notation we keep below.

Claim. μ_* *has atoms, i.e., $\mathcal{E}_0 \neq \emptyset$.*

Proof. If μ_* is non-atomic, fix $A_* \subseteq \mathcal{E}$ with $\mu_*(A_*) = \frac{1}{2}$. If $\Sigma^{-1}(A_*) = A$, then $\mu(A) = \frac{1}{2}$ and A is E-invariant. Let $T \in [F]$ be such that $T(A) = \sim A$. Then $\varphi_E(T) = 0$, a contradiction. $\qquad\square$

If $e \in \mathcal{E}_0$, then $\mu(X_e) > 0$. If for all $e \in \mathcal{E}_0$, $E|X_e \neq F|X_e$, then as $E|X_e$ is ergodic, we can find $\varphi_e \in [F|X_e]$ such that $\neq \varphi_e(x)Ex$, $\forall x \in X_e$ (see Remark 2.1). If also $\mu_*(\sim \mathcal{E}_0) > 0$, then we can split $\sim \mathcal{E}_0$ into two pairwise disjoint sets of equal

measure and thus split $\sim X_{\mathcal{E}_0} = \sim \bigcup_{e \in \mathcal{E}_0} X_e$ into two sets of equal measure X_1, X_2 which are E-invariant. Then let $T_1 \in [F]$ be such that $T_1(X_1) = X_2, T_1(X_2) = X_1$. Then $\neg T_1(x) E x$ for $x \notin X_{\mathcal{E}_0}$. If $T \in [F]$ is defined by $T = \left(\bigcup_{e \in \mathcal{E}_0} \varphi_e \right) \cup (T_1 | \sim X_{\mathcal{E}_0})$, then clearly $\neq T(x) E x, \forall x$, a contradiction.

Thus we see that there must be some atom e of μ_* with $E|X_e = F|X_e$.

Enumerate in a sequence (finite or infinite) $\{e_0, e_1, \dots\}$ all elements e of $\mathcal{E}_0$ such that $E|X_e = F|X_e$ in such a way that $\mu(X_{e_i}) \geq \mu(X_{e_{i+1}})$. Put $A = X_{e_0}$. Let $Y = \bigcup_i X_{e_i}, Z = \sim Y$. We have seen that there is $S_1 \in [F|Z]$ with $\neq S_1(z) E z$, $\forall z \in Z$. Let for $n \geq 0, \theta_{n+1} \in [[F]]$ be such that $\mathrm{dom}(\theta_{n+1}) = X_{e_{n+1}}, \mathrm{rng}(\theta_{n+1}) \subseteq X_{e_n}$, and let $\theta^* = \bigcup_{n \geq 0} \theta_n$ so that $\theta^* : \bigcup_{i > 0} X_{e_i} \to \bigcup_i X_{e_i}, \theta^* \in [[F]]$ and $\neq \theta^*(x) E x$. Let $\theta^{**} \in [[F]]$ be such that $\mathrm{dom}(\theta^{**}) = X_{e_0}$ and $\mathrm{rng}(\theta^{**}) = \bigcup_i X_{e_i} \setminus \theta^* \left(\bigcup_{i > 0} X_{e_i} \right)$. Put $S_2 = \theta^* \cup \theta^{**}$, so that $S_2 \in [F|Y]$ and $\neq S_2(y) E y$ if $y \notin X_{e_0}$. Then if $S = S_1 \cup S_2, S \in [F]$ and $\{x : S(x) E x\} \subseteq X_{e_0} = A$. □

Corollary 2.15. *Let Γ be a countable group and consider a measure-preserving, ergodic action of Γ on (X, μ) with associated equivalence relation $F = E_\Gamma^X$. Let $E \subseteq F$ be a subequivalence relation. If $\varphi_E^0 = \inf_{\gamma \in \Gamma} \varphi_E(\gamma) > \frac{3}{4}$, then there is an E-invariant Borel set A with $E|A = F|A$ such that $\mu(A) \geq 4\varphi_E^0 - 3$.*

Proof. By 2.14 and 2.7. □

3. Epstein's co-inducing construction

(A) We will next study some properties of a co-inducing construction of Epstein [E].

We first describe this construction. Fix a standard measure space (X, μ), a countable, measure-preserving equivalence relation F on (X, μ) and a subequivalence relation $E \subseteq F$ such that there is a fixed number $N \in \{1, 2, 3, \dots, \aleph_0\}$ of E-classes in each F-class. This is the case, for example, if F is ergodic. Fix choice functions $\{C_n\}_{n \in N}$, where we identify N here with $\{0, 1, \dots, N-1\}$ if N is finite and with $\mathbb{N}$ if it is infinite, as in Section 2 (A), and let $\pi : F \to S_N$ (= the symmetric group of N) be the index cocycle given by the formula

$$\pi(x, y)(k) = n \iff [C_k(x)]_E = [C_n(y)]_E.$$

Now assume that E as above is induced by a *free* action a_0 of a countable group Δ. Then we can define $\bar{\delta} : F \to \Delta^N$ by

$$\bar{\delta}(x, y)_n \cdot C_{\pi(x,y)^{-1}(n)}(x) = C_n(y).$$

The group S_N of permutations of N acts on Δ^N by shift $(\pi \cdot \bar{\delta})_n = \bar{\delta}_{\pi^{-1}(n)}$, so we can consider the semi-direct product $S_N \ltimes \Delta^N$, whose multiplication is defined by

$$(\pi_1, \bar{\delta}_1)(\pi_2, \bar{\delta}_2) = (\pi_1 \pi_2, \bar{\delta}_1(\pi_1 \cdot \bar{\delta}_2)).$$

It is easy to check that

$$\rho(x, y) = (\pi(x, y), \bar{\delta}(x, y))$$

is a Borel cocycle

$$\rho \colon F \to S_N \ltimes \Delta^N.$$

Now given any measure-preserving action a of Δ on a standard measure space (Y, ν), we can define a measure-preserving action of $S_N \ltimes \Delta^N$ on (Y^N, ν^N) by

$$((\pi, \bar{\delta}) \cdot \bar{y})_n = \bar{\delta}_n \cdot \bar{y}_{\pi^{-1}(n)}.$$

Then we can define a *near-action* of $[F]$ on $(X \times Y^N, \mu \times \nu^N)$, i.e., a continuous homomorphism of $[F]$, with the uniform topology, into the automorphism group $\text{Aut}(X \times Y^N, \mu \times \nu^N)$, with the weak topology, by letting $S \in [F]$ act on $X \times Y^N$ as a skew product via ρ, namely

$$S \cdot (x, \bar{y}) = (S(x), \rho(x, S(x)) \cdot \bar{y}) = (S(x), (n \mapsto \bar{\delta}(x, S(x))_n \cdot \bar{y}_{\pi(x,S(x))^{-1}(n)})).$$

(For information about near-actions, see Glasner–Tsirelson–Weiss [GTW], and concerning the uniform and weak topologies, see Kechris [K].)

In particular, if c_0 is a measure-preserving action of a countable group Λ on (X, μ) with $E_\Lambda^X \subseteq F$, then $\lambda \in \Lambda$ gives rise to an element $x \mapsto \lambda \cdot x$ of $[F]$, so we can define

$$\pi(\lambda, x) = \pi(x, \lambda \cdot x), \quad \bar{\delta}(\lambda, x) = \bar{\delta}(x, \lambda \cdot x), \quad \rho(\lambda, x) = (\pi(\lambda, x), \bar{\delta}(\lambda, x)),$$

and then $\rho \colon \Lambda \times X \to S_N \ltimes \Delta^N$ is a Borel cocycle. The restriction of the near-action of $[F]$ on $(X \times Y^N, \mu \times \nu^N)$ gives then a measure-preserving action of Λ on $(X \times Y^N, \mu \times \nu^N)$, which is the skew product

$$c = c_0 \ltimes_\rho (Y^N, \mu^N)$$

defined by

$$\lambda \cdot (x, \bar{y}) = (\lambda \cdot x, \rho(\lambda, x) \cdot \bar{y}) = (\lambda \cdot x, (n \mapsto \bar{\delta}(\lambda, x)_n \cdot \bar{y}_{\pi(\lambda,x)^{-1}(n)}).$$

Fix now a Borel action b_0 of a countable group Γ on (X, μ) with $F = E_\Gamma^X$. Then applying the above to $\Lambda = \Gamma$, $c_0 = b_0$, we associate to each measure-preserving action a of Δ on (Y, ν) a measure-preserving action b of Γ on $(X \times Y^N, \mu \times \nu^N)$, relative to the fixed pair (a_0, b_0) of the actions of Γ, Δ, respectively, on X (and the choice of $\{C_n\}$ – but it is not hard to check that this action is independent of the choice of $\{C_n\}$, up to isomorphism). We call this the *co-induced action of a, modulo (a_0, b_0)*, in symbols:

$$b = \text{CInd}(a_0, b_0)_\Delta^\Gamma(a).$$

We can view this as an operation from the space $A(\Delta, Y, \nu)$ of measure-preserving actions of Δ on (Y, ν) to the space $A(\Gamma, X \times Y^N, \mu \times \nu^N)$; see, e.g., Kechris [K].

By applying the preceding to $\Lambda = \Delta, c_0 = a_0$, we also have a measure-preserving action a' of Δ on $(X \times Y^N, \mu \times \nu^N)$. Clearly this action gives a subequivalence relation of the equivalence relation given by b. We note that b_0 is a factor of b via $(x, \bar{y}) \mapsto x$, a is a factor of a' via $(x, \bar{y}) \mapsto \bar{y}_0$ (recall here that $C_0(x) = x$, so that $\pi(\gamma, x)(0) = 0$ iff $\gamma \cdot xEx$) and finally a_0 is a factor of a' via $(x, \bar{y}) \mapsto x$. In particular, if b_0 is free, so is b, and a' is always free.

Moreover, for further reference, we note that the map $a \mapsto \mathrm{CInd}(a_0, b_0)^\Gamma_\Delta(a)$ is a continuous map from $A(\Delta, Y, \nu)$ to $A(\Gamma, X \times Y^N, \mu \times \nu^N)$, where each is equipped with the weak topology (see Kechris [K] for its definition).

(B) We will now study some connections between ergodicity properties of an action and its co-induced action. *In the notation of* (A), *if a_0 is ergodic we can choose the choice functions to be* 1-1 *and it will be assumed in this case that the co-inducing construction is done with such choice functions.*

Proposition 3.1. *In the notation of* (A) *above, if b_0 is free, mixing and a_0 is ergodic, then*
$$a \text{ is mixing} \implies b = \mathrm{CInd}(a_0, b_0)^\Gamma_\Delta(a) \text{ is mixing.}$$

Proof. Assume that b_0, a are mixing, b_0 is free, and a_0 is ergodic. Consider the action b of Γ on $(X \times Y^N, \mu \times \nu^N)$. Then Γ acts on $L^2(X \times Y^N, \mu \times \nu^N)$ by $\gamma \cdot f(x, \bar{y}) = f(\gamma^{-1} \cdot (x, \bar{y}))$ and it is enough to show that for $f, g \in L^2(X \times Y^N, \mu \times \nu^N)$, $\int (\gamma^{-1} \cdot f)g \to (\int f)(\int g)$ as $\gamma \to \infty$. Without loss of generality, we can assume that $f(x, \bar{y}) = f_0(x)F_0(\bar{y}_0)\dots F_m(\bar{y}_m)$, $g(x, \bar{y}) = g_0(x)G_0(\bar{y}_0)\dots G_m(\bar{y}_m)$ for bounded $f_0, g_0: X \to \mathbb{C}$, $F_i, G_i: Y \to \mathbb{C}$. Note that $\int f_0(\gamma \cdot x)g_0(x)\, d\mu(x) \to (\int f_0)(\int g_0)$ since b_0 is mixing. Now we have

$$\int (\gamma^{-1} \cdot f)g = \int f_0(\gamma \cdot x)g_0(x)\left[\int \prod_{i=0}^m F_i(\bar{\delta}(\gamma, x)_i \cdot \bar{y}_{\pi(\gamma,x)^{-1}(i)})G_i(\bar{y}_i)\, d\bar{y} \right] dx.$$

Note that if $\{q_n\}$, $q_n: Z \to \mathbb{C}$, and $\{r_n\}$, $r_n: Z \to \mathbb{C}$, are uniformly bounded, where Z is a probability space, $q_n(z) \to a$, $\forall z$, and $\int r_n \to b$, then

$$\int q_n(z)r_n(z) = \int (q_n(z) - a)r_n(z) + a \int r_n(z) \to ab,$$

by Lebesgue Dominated Convergence. So it is enough to show that *for each fixed $x \in X$,*

$$\int \left[\prod_{i=0}^m F_i(\bar{\delta}(\gamma, x)_i \cdot \bar{y}_{\pi(\gamma,x)^{-1}(i)})G_i(\bar{y}_i) \right] d\bar{y} \to \prod_{i=0}^m \left(\int F_i \right)\left(\int G_i \right), \quad (*)$$

as $\gamma \to \infty$.

Fix then $x \in X$ and put

$$S_\gamma = \{(i, \pi(\gamma, x)^{-1}(i)) : i \leq m, \pi(\gamma, x)^{-1}(i) \leq m\}.$$

For each $S \subseteq \{0, \dots, m\}^2$, let

$$\Gamma_S = \{\gamma \in \Gamma : S_\gamma = S\}.$$

Then $\Gamma = \bigsqcup_S \Gamma_S$ is a finite partition of Γ, so it is enough to show that for each fixed S with Γ_S infinite, $(\ast)$ holds as $\gamma \to \infty$, $\gamma \in \Gamma_S$.

For such S and $\gamma \in \Gamma_S$, let

$$I_\gamma = \{i \le m : \pi(\gamma, x)^{-1}(i) \le m\}$$

and let

$$\rho_\gamma(i) = \pi(\gamma, x)^{-1}(i)$$

for $i \in I_\gamma$. Thus $\mathrm{graph}(\rho_\gamma) = S$.

Then

$$\int \Big[\Big(\prod_{i=0}^m F_i(\bar\delta(\gamma, x)_i \cdot \bar y_{\pi(\gamma,x)^{-1}(i)}) G_i(\bar y_i) \Big] d\bar y$$

$$= \int \Big[\prod_{i \in I_\gamma} F_i(\bar\delta(\gamma, x)_i \cdot \bar y_{\rho_\gamma(i)}) G_{\rho_\gamma(i)}(\bar y_{\rho_\gamma(i)}) \Big]$$

$$\cdot \Big[\prod_{i \notin I_\gamma} F_i(\bar\delta(\gamma, x)_i \cdot \bar y_{\pi(\gamma,x)^{-1}(i)}) \prod_{i \notin \rho_\gamma(I_\gamma)} G_i(\bar y_i) \Big] d\bar y.$$

Noticing that if $i \notin I_\gamma, i \le m$, then $\pi(\gamma, x)^{-1}(i) > m$, so that

$$\rho_\gamma(I_\gamma), \{\pi(\gamma, x)^{-1}(i) : i \notin I_\gamma\}, \{0, \dots, m\} \setminus \rho_\gamma(I_\gamma)$$

are pairwise disjoint, and applying independence, we see that the above integral is equal to

$$\Big[\prod_{i \notin I_\gamma} \int F_i \Big] \Big[\prod_{i \notin \rho_\gamma(I_\gamma)} \int G_i \Big] \Big[\int \prod_{i \in I_\gamma} F_i(\bar\delta(\gamma, x)_i \cdot \bar y_{\rho_\gamma(i)}) G_{\rho_\gamma(i)}(\bar y_{\rho_\gamma(i)}) \, d\bar y \Big].$$

Thus for each $i \in I_\gamma$, if $j = \rho_\gamma(i)$, it is enough, by independence again, to show that

$$\lim_{\substack{\gamma \in \Gamma_S \\ \gamma \to \infty}} \int F_i(\bar\delta(\gamma, x)_i \cdot \bar y_j) G_j(\bar y_j) \, d\bar y = \Big(\int F_i \Big) \Big(\int G_j \Big),$$

or equivalently,

$$\int F_i(\bar\delta(\gamma, x)_i \cdot y) G_j(y) \, dy \to \Big(\int F_i \Big) \Big(\int G_j \Big)$$

as $\gamma \to \infty, \gamma \in \Gamma_S$. Using that the action a of Δ on Y is mixing, it is then enough to show that for each i,

$$\gamma \to \infty \implies \bar\delta(\gamma, x)_i \to \infty.$$

Otherwise, there is a finite $K \subseteq \Delta$ such that $\bar{\delta}(\gamma, x)_i \in K$ for infinitely many $\gamma \in \Gamma$. Now $\bar{\delta}(\gamma, x)_i \cdot C_j(x) = C_i(\gamma \cdot x)$, so $C_i(\gamma \cdot x)$ takes only finitely many values for infinitely many $\gamma \in \Gamma$, contradicting the fact that the Γ-action on X is free and C_i is 1-1. $\square$

There is another condition concerning the "smallness" of E in F that actually guarantees that the co-induced action b is mixing for *any* a (mixing or not).

In the context of (A), let for each $\gamma \in \Gamma$, $k, n \in N$,

$$A_E^{k,n}(\gamma) = \{x : \pi(\gamma, x)(k) = n\} = \{x : C_k(x) E C_n(\gamma \cdot x)\}.$$

Thus $A_E^{0,0}(\gamma) = \{x : \gamma \cdot x E x\}$. Put $\varphi_E^{k,n}(\gamma) = \mu(A_E^{k,n}(\gamma))$. Clearly $\varphi_E^{0,0} = \varphi_E$.

Lemma 3.2. *If $\varphi_E(\gamma) \to 0$ as $\gamma \to \infty$, then $\varphi_E^{k,n}(\gamma) \to 0$ for any k, n, as $\gamma \to \infty$.*

Proof. We have $\varphi_E^{k,n}(\gamma) = \mu(\{x : C_k(x) E C_n(\gamma \cdot x)\})$. Put $S = C_k$, $T = C_n$. There is a partition $X = \bigsqcup_{i \in \mathbb{N}} A_i$ and $\gamma_i \in \Gamma$ such that $S = \bigsqcup_i \gamma_i | A_i$. Similarly there is a partition $X = \bigsqcup_{j \in \mathbb{N}} B_j$ and $\delta_j \in \Gamma$ such that $T = \bigsqcup_j \delta_j | B_j$.

Assume that $\varphi_E(\gamma) \to 0$ as $\gamma \to \infty$, and let $\varepsilon > 0$. We will find a finite set $F \subseteq \Gamma$ such that $\varphi_E^{k,n}(\gamma) < \varepsilon$ for $\gamma \notin F$. First find J_0 such that $\sum_{j \geq J_0} \mu(B_j) < \varepsilon/3$. Then fix I_0 such that $\sum_{i \geq I_0} \mu(A_i) < \varepsilon/3|J_0|$. Since $\gamma_i \gamma^{-1} \delta_j \to \infty$ for each i, j, as $\gamma \to \infty$, there is a finite set $F \subseteq \Gamma$ such that $\sum_{i < I_0, j < J_0} \varphi_E(\gamma_i \gamma^{-1} \delta_j^{-1}) < \varepsilon/3$ for $\gamma \notin F$. We show that if $\gamma \notin F$, then $\varphi_E^{k,n}(\gamma) < \varepsilon$.

We have for any γ:

$$
\begin{aligned}
\varphi_E^{k,n}(\gamma) &= \mu(\{x : S(x) E T(\gamma \cdot x)\}) \\
&= \sum_{i,j} \mu(\{x : x \in A_i \wedge x \in \gamma^{-1} \cdot B_j \wedge \gamma_i \cdot x E \delta_j \gamma \cdot x\}) \\
&= \sum_{i,j} \mu(\{x : \gamma^{-1} \delta_j^{-1} \cdot x \in A_i \wedge \gamma^{-1} \delta_j^{-1} \cdot x \in \gamma^{-1} \cdot B_j \wedge \gamma_i \gamma^{-1} \delta_j^{-1} \cdot x E x\}) \\
&= \sum_{i,j} \mu(\{x : \gamma^{-1} \delta_j^{-1} \cdot x \in A_i \wedge \delta_j^{-1} \cdot x \in B_j \wedge \gamma_i \gamma^{-1} \delta_j^{-1} \cdot x E x\}).
\end{aligned}
$$

Let

$$A_{i,j}^{(\gamma)} = \{x : \gamma^{-1} \delta_j^{-1} \cdot x \in A_i \wedge \delta_j^{-1} \cdot x \in B_j\}.$$

Then

$$\mu(A_{i,j}^{(\gamma)}) = \mu(\{x : \gamma^{-1} \cdot x \in A_i \wedge x \in B_j\}) = \mu(\{x : x \in A_i \wedge \gamma \cdot x \in B_j\}),$$

so $\sum_{j \geq J_0} \sum_i \mu(A_{i,j}^{(\gamma)}) = \sum_{j \geq J_0} \mu(B_j) < \varepsilon/3$. Also $\sum_{j < J_0} \sum_{i \geq I_0} \mu(A_{i,j}^{(\gamma)}) \leq \sum_{j < J_0} \sum_{i \geq I_0} \mu(A_i) < \varepsilon/3$. So it follows that

$$\varphi_E^{k,n}(\gamma) \leq \Big[\sum_{i < I_0} \sum_{j < J_0} \varphi_E(\gamma_i \gamma^{-1} \delta_j^{-1}) \Big] + 2\varepsilon/3.$$

Thus, if $\gamma \notin F$ then

$$\varphi_E^{k,n}(\gamma) < \varepsilon. \qquad \square$$

We denote below by i the trivial action of Δ on $(Y, \nu) : \delta \cdot y = y$. We now have:

Theorem 3.3. *In the notation of* (A) *above, and assuming that b_0 is mixing, the following assertions are equivalent:*

(i) $\varphi_E(\gamma) \to 0$ *as* $\gamma \to \infty$;
(ii) $\mathrm{CInd}(a_0, b_0)_\Delta^\Gamma(i)$ *is mixing;*
(iii) $\exists a \in A(\Delta, Y, \nu)$ *(a is not ergodic and* $\mathrm{CInd}(a_0, b_0)_\Delta^\Gamma(a)$ *is mixing);*
(iv) $\forall a \in A(\Delta, Y, \nu)(\mathrm{CInd}(a_0, b_0)_\Delta^\Gamma(a)$ *is mixing).*

Proof. Clearly (ii) $\Longrightarrow$ (iii) and (iv) $\Longrightarrow$ (ii).
  (i) $\Longrightarrow$ (iv): By 3.2 we have that $\varphi_E^{k,n}(\gamma) \to 0$, $\forall k, n$. Let

$$A_E^{j,i}(\gamma) = \{x : \pi(\gamma, x)^{-1}(i) = j\}$$

and

$$A_E^{(m)}(\gamma) = \bigcup_{1 \le i, j \le m} A_E^{j,i}(\gamma).$$

Then going over the proof of 3.1 and keeping its notation, we see that for $x \notin A_E^{(m)}(\gamma)$,

$$\int \left[\prod_{i=0}^m F_i(\bar{\delta}(\gamma, x)_i \cdot \bar{y}_{\pi(\gamma,x)^{-1}(i)}) G_i(\bar{y}_i) \right] d\bar{y} = \prod_{i=1}^m \left(\int F_i \right) \left(\int G_i \right),$$

by independence. Thus, for some bounded $H(x)$,

$$\int (\gamma^{-1} \cdot f)g = \int_{A_E^{(m)}(\gamma)} H(x)\, dx + \int_{\sim A_E^{(m)}(\gamma)} f_0(\gamma \cdot x)g_0(x) \left[\prod_{i=0}^m \left(\int F_i \right) \left(\int G_i \right) \right] dx$$
$$\to \left(\int f \right) \left(\int g \right)$$

as $\gamma \to \infty$, since $\mu(A_E^{(m)}(\gamma)) \to 0$ and b_0 is mixing.
  (iii) $\Longrightarrow$ (i): Fix such an action a and a set $B \subseteq Y$ with $0 < p = \mu(B) < 1$, which is invariant under this action. We will show that $\varphi_E(\gamma) \to 0$. Put $B^{(0)} = \{(x, \bar{y}) : \bar{y}_0 \in B\}$. Since the co-induced action is mixing, we have that

$$(\mu \times \nu^N)(\gamma \cdot B^{(0)} \cap B^{(0)}) \to (\mu \times \nu^N)(B^{(0)}) \cdot (\mu \times \nu^N)(B^{(0)}) = p^2.$$

Now

$$\gamma \cdot B^{(0)} = \{\gamma \cdot (x, \bar{y}) : (x, \bar{y}) \in B^{(0)}\} = \{\gamma \cdot (x, \bar{y}) : \bar{y}_0 \in B\},$$

so

$$\gamma \cdot B^{(0)} \cap B^{(0)} = \{\gamma \cdot (x, \bar{y}) : \bar{y}_0 \in B \wedge (\rho(\gamma, x) \cdot \bar{y})_0 \in B\}$$
$$= \{\gamma \cdot (x, \bar{y}) : \bar{y}_0 \in B \wedge \bar{\delta}(\gamma \cdot x)_0 \cdot \bar{y}_{\pi(\gamma,x)^{-1}(0)} \in B\}$$
$$= \{\gamma \cdot (x, \bar{y}) : \bar{y}_0 \in B \wedge \bar{y}_{\pi(\gamma,x)^{-1}(0)} \in B\}$$
$$= \{\gamma \cdot (x, \bar{y}) : \pi(\gamma, x)(0) = 0 \wedge \bar{y}_0 \in B\}$$
$$\sqcup \{\gamma \cdot (x, \bar{y}) : \pi(\gamma, x)(0) \neq 0 \wedge \bar{y}_0 \in B \wedge \bar{y}_{\pi(\gamma,x)^{-1}(0)} \in B\}.$$

So, by Fubini,

$$(\mu \times \nu^N)(\gamma \cdot B^{(0)} \cap B^{(0)}) = \mu(A_E^{0,0}(\gamma)) \cdot \mu(B) + (1 - \mu(A_E^{0,0}(\gamma)) \cdot \mu(B)^2$$
$$= p\mu(A_E^{0,0}(\gamma)) + p^2(1 - \mu(A_E^{0,0}(\gamma))).$$

Since $(\mu \times \nu^N)(\gamma \cdot B^{(0)} \cap B^{(0)}) \to p^2$ and $0 < p < 1$, it follows that $\mu(A_E^{0,0}(\gamma)) = \varphi_E(\gamma) \to 0$. □

Therefore, if $\varphi_E^{k,n}(\gamma) \not\to 0$ for some k, n, as $\gamma \to \infty$, it follows that, for every $a \in A(\Delta, Y, \nu)$, if $\mathrm{CInd}(a_0, b_0)_\Delta^\Gamma(a)$ is mixing, then a is ergodic. By strengthening the hypothesis, we can obtain the following stronger conclusion.

Proposition 3.4. *With the notation of* (A) *above, if we have* $\inf_{\gamma \in \Gamma} \varphi_E^{k,n}(\gamma) > 0$ *for some k, n, it follows that, for any $a \in A(\Delta, Y, \nu)$, if $b = \mathrm{CInd}(a_0, b_0)_\Delta^\Gamma(a)$ is ergodic, then a is ergodic.*

Proof. Assume that a is not ergodic, in order to show that b is not ergodic. Let $f \in L_0^2(Y) = \{f \in L^2(Y) : \int f = 0\}$ be real such that $\|f\|_2 = 1$ and $\delta \cdot f = f, \forall \delta \in \Delta$. Let $f^{(k)}, f^{(n)} \in L_0^2(X \times Y^N)$ be defined by $f^{(k)}(x, \bar{y}) = f(\bar{y}_k)$, $f^{(n)}(x, \bar{y}) = f(\bar{y}_n)$. Then for the Γ-action on $L_0^2(X \times Y^N)$,

$$\langle \gamma^{-1} \cdot f^{(n)}, f^{(k)} \rangle = \iint f^{(n)}(\gamma \cdot (x, \bar{y})) f^{(k)}(x, \bar{y}) \, dx \, d\bar{y}$$

$$= \iint f^{(n)}(\gamma \cdot x, (n \mapsto \bar{\delta}(\gamma, x)_n \cdot \bar{y}_{\pi(\gamma,x)^{-1}(n)})) f^{(k)}(x, \bar{y}) \, dx \, d\bar{y}$$

$$= \iint f(\bar{\delta}(\gamma, x)_n \cdot \bar{y}_{\pi(\gamma,x)^{-1}(n)}) f(\bar{y}_k) \, dx \, d\bar{y}$$

$$= \int_{A_E^{k,n}(\gamma)} \left[\int f(\bar{\delta}(\gamma, x)_n \cdot \bar{y}_k) f(\bar{y}_k) \, d\bar{y} \right] dx$$

$$\quad + \int_{\sim A_E^{k,n}(\gamma)} \left[\int f(\bar{\delta}(\gamma, x)_n \cdot \bar{y}_{\pi(\gamma,x)^{-1}(n)}) f(\bar{y}_k) d\bar{y} \right] dx$$

$$= \int_{A_E^{k,n}(\gamma)} \left(\int f^2 \right) dx + \int_{\sim A_E^{k,n}(\gamma)} \left(\int f \right)^2 dx$$

$$= \mu(A_E^{k,n}(\gamma)) = \varphi_E^{k,n}(\gamma).$$

Thus $\langle \gamma^{-1} \cdot f^{(n)}, f^{(k)} \rangle \geq c > 0$ for some $c > 0$. If K is the closed convex hull of $\{\gamma \cdot f^{(n)} : \gamma \in \Gamma\}$, then $\langle \xi, f^{(k)} \rangle \geq c, \forall \xi \in K$, so $0 \notin K$. If ξ_1 is the unique element of least norm in K, clearly $0 \neq \xi_1 \in L_0^2(X \times Y^{\mathbb{N}})$, and ξ_1 is Γ-invariant, so the action b is not ergodic. $\qquad\square$

Let b_0 be a free action of a countable group Γ on (X, μ) and let a_0 be a free action of a countable group Δ on (X, μ), so that $E = E_\Delta^X \subseteq F = E_\Gamma^X$. If there exist $S, S' \in [F]$ with $\inf_{\gamma \in \Gamma} \varphi_E(S\gamma S') > 0$, then, by 2.9, Γ, Δ are ME. In particular, if a_0 is ergodic, so that we can take the choice functions $\{C_n\}$ to be in $[F]$, we note that $\varphi_E^{k,n}(\gamma) = \mu(\{x : C_k(x)EC_n(\gamma \cdot x)\}) = \mu(\{x : C_n\gamma C_k^{-1}(x)Ex\}) = \varphi_E(C_n\gamma C_k^{-1})$. Thus we have:

Corollary 3.5. *Let b_0 be a free measure-preserving action of Γ on (X, μ) and let a_0 be a free, ergodic action of Δ on (X, μ) with $E = E_\Delta^X \subseteq F = E_\Gamma^X$. If $\inf_{\gamma \in \Gamma} \varphi_E^{k,n}(\gamma) > 0$ for some k, n, then Γ, Δ are ME. In particular, Γ has property* (T) (*resp. HAP*) *iff Δ has property* (T) (*resp. HAP*).

Remark 3.6. In the context of 3.5, when b_0 is also mixing and Γ does not have the HAP, one can show that Δ does not have the HAP by the following alternative argument, which may be of some independent interest.

We will use the following characterization of groups with HAP; see Kechris [K], 12.7. Below $ERG(\Gamma, X, \mu)$ is the set of ergodic actions of Γ on (X, μ), and $MIX(\Gamma, X, \mu)$ the set of mixing actions. We consider these as subspaces of the space of actions $A(\Gamma, X, \mu)$ with the weak topology.

Theorem 3.7. *Let Γ be an infinite countable group. Then the following conditions are equivalent:*

(i) *Γ does not have the HAP.*
(ii) *$\overline{MIX(\Gamma, X, \mu)} \subseteq ERG(\Gamma, X, \mu)$.*

Consider now the continuous map

$$a \in A(\Delta, Y, \nu) \mapsto b(a) \in A(\Gamma, X \times Y^N, \mu \times \nu^N),$$

where $b(a) = \mathrm{CInd}(a_0, b_0)_\Delta^\Gamma(a)$. Then we have, by 3.1,

$$a \text{ is mixing} \Longrightarrow b(a) \text{ is mixing}$$

and, by 3.4,

$$b(a) \text{ is ergodic} \Longrightarrow a \text{ is ergodic}.$$

Let $C = \{a \in A(\Delta, Y, \nu) : b(a) \in \overline{MIX(\Gamma, X \times Y^N, \mu \times \nu^N)}\}$. Then C is closed and $MIX(\Delta, Y, \nu) \subseteq C$. So $\overline{MIX(\Delta, Y, \nu)} \subseteq C$. If $a \in C$, then $b(a) \in \overline{MIX(\Gamma, X \times Y^N, \mu \times \nu^N)} \subseteq ERG(\Gamma, X \times Y^N, \mu \times \nu^N)$. So $a \in ERG(\Delta, Y, \nu)$, i.e., $\overline{MIX(\Delta, Y, \nu)} \subseteq ERG(\Delta, Y, \nu)$, and thus Δ does not have the HAP.

(C) Let Γ be an infinite countable group and let b_0 be a free mixing action of Γ on (X, μ). It is well known that there is a free, mixing action a_0 of $\mathbb{Z}$ on (X, μ) such that $E = E_{\mathbb{Z}}^X \subseteq F = E_{\Gamma}^X$ (see, e.g., Zimmer [Z], 9.3.2). We will construct below a_0 so that moreover $\varphi_E(\gamma) \to 0$ as $\gamma \to \infty$. Then by 3.3, for every action a of $\mathbb{Z}$ on (Y, ν), the action $\mathrm{CInd}(a_0, b_0)_{\mathbb{Z}}^{\Gamma}(a)$ is mixing, which produces a large supply of seemingly new free, mixing actions of Γ.

Theorem 3.8. *Let Γ be an infinite countable group and let b_0 be a free, measure-preserving, mixing action of Γ on (X, μ). Then there is a free, measure-preserving, mixing action a_0 of $\mathbb{Z}$ on (X, μ) with $E = E_{\mathbb{Z}}^X \subseteq F = E_{\Gamma}^X$ and $\varphi_E(\gamma) \to 0$ as $\gamma \to \infty$.*

We will use the following two lemmas.

Lemma 3.9. *Let Γ be an infinite countable group. Then there exists a positive, symmetric (i.e., invariant under inverses) function $f \in c_0(\Gamma) \setminus \ell^1(\Gamma)$ such that whenever $S \subseteq \Gamma$ and $\sum_{\gamma \in S} f(\gamma) < \infty$, then $\sum_{\gamma \in S} f(\delta_1 \gamma \delta_2) < \infty$ for any $\delta_1, \delta_2 \in \Gamma$ (i.e., the summable ideal associated to f is two-sided invariant).*

Proof. Fix a sequence $\{Q_n\}_{n \geq 0}$ of finite, symmetric subsets of Γ with $Q_0 = \{1\}$, $Q_n \subseteq Q_{n+1}$, $Q_{n+1} \setminus (Q_n)^n \neq \emptyset$ and $\bigcup_n Q_n = \Gamma$. Let $|\gamma| = \min\{n : \gamma \in (Q_n)^n\}$ (this is motivated by an idea in Struble [ST]). Note that $|\gamma| = |\gamma^{-1}|$ and $|\gamma \delta| \leq |\gamma| + |\delta|$, so $|\gamma \delta| \geq |\,|\gamma| - |\delta|\,|$ for any $\gamma, \delta \in \Gamma$. Put now $f(\gamma) = \frac{1}{|\gamma|+1}$. Then clearly $f \in c_0(\Gamma)$ and for every n, there is $\gamma \in \Gamma$ with $|\gamma| = n + 1$, so $f \notin \ell^1(\Gamma)$. Fix now $S \subseteq \Gamma$ with $\sum_{\gamma \in S} f(\gamma) < \infty$. For $\delta \in \Gamma$, let $|\delta| = c$, and notice that

$$\sum_{\gamma \in S} f(\gamma \delta) = \sum_{\gamma \in S} \frac{1}{|\gamma \delta| + 1} \leq \sum_{\{\gamma : |\gamma| \leq c\}} \frac{1}{|\gamma \delta| + 1} + \sum_{\{\gamma \in S : |\gamma| > c\}} \frac{1}{|\gamma| - c + 1}$$

$$\leq \sum \{\gamma : |\gamma| \leq c\} \frac{1}{|\gamma \delta| + 1} + \sum_{\gamma \in S} \frac{c + 1}{|\gamma| + 1} < \infty.$$

Similarly, $\sum_{\gamma \in S} f(\delta \gamma) < \infty$ and we are done. $\square$

Consider now the given free, measure-preserving, mixing action b_0 of Γ on (X, μ) with $F = E_{\Gamma}^X$ the associated equivalence relation.

Lemma 3.10. *Let f be as in Lemma 3.9. Let $R \subseteq F$ be a finite subequivalence relation with uniformly bounded size of its equivalence classes, and let $A, B \subseteq X$ be disjoint with $\mu(A) = \mu(B)$ and such that $A \cup B$ is a section of R (i.e., no two distinct members of $A \cup B$ belong to the same R-class). Suppose also that $\varphi_R(\gamma) \leq f(\gamma)$, $\forall \gamma \in \Gamma$. Then there exists $\theta \in [[F]]$ with $\mathrm{dom}(\theta) = A$, $\mathrm{rng}(\theta) = B$ such that*

$$\varphi_{R \vee \theta}(\gamma) \leq f(\gamma), \ \forall \gamma \in \Gamma. \tag{2}$$

Proof. Let $T \in [F]$ generate R and let N be such that $T^N = 1$. Notice that for any $\theta \in [[F]]$ with $\mathrm{dom}(\theta) \subseteq A, \mathrm{rng}(\theta) \subseteq B$ we have

$$\varphi_{R \vee \theta}(\gamma) \leq \varphi_R(\gamma) + \sum_{i,j=1}^{N} \mu(\{x : T^i \theta T^j(x) = \gamma \cdot x\})$$

$$+ \sum_{i,j=1}^{N} \mu(\{x : T^i \theta^{-1} T^j(x) = \gamma \cdot x\}).$$

Let $\theta \in [[F]]$ be maximal (under inclusion) with $\mathrm{dom}(\theta) \subseteq A, \mathrm{rng}(\theta) \subseteq B$ satisfying (2). We will show that this works, i.e., $\mathrm{dom}(\theta) = A, \mathrm{rng}(\theta) = B$. Otherwise, $A_1 = A \setminus \mathrm{dom}(\theta)$, $B_1 = B \setminus \mathrm{dom}(\theta)$ have positive measure.

For $\rho \in [[F]], \gamma \in \Gamma$ let

$$s_\gamma(\rho) = \sum_{i,j=1}^{N} \mu(\{x : T^i \rho T^j(x) = \gamma \cdot x\}) + \sum_{i,j=1}^{N} \mu(\{x : T^i \rho^{-1} T^j(x) = \gamma \cdot x\}),$$

and put

$$S = \{\gamma : \varphi_R(\gamma) + s_\gamma(\theta) \geq f(\gamma)\}.$$

Let then $K \subseteq \Gamma$ be finite, symmetric such that $\mu\left(\bigcup_i \{x : T^i(x) \notin K \cdot x\}\right) < \frac{\mu(A_1)^2}{4}$, and put $S' = KSK \cup KS^{-1}K$.

Claim. $\sum_{\gamma \in S} f(\gamma) < \infty$.

Proof. First notice that $\sum_{\gamma \in \Gamma} \varphi_R(\gamma) = \sum_{\gamma \in \Gamma} \mu(\{x : \exists i \leq N(T^i(x) = \gamma \cdot x)\}) \leq \sum_{i \leq N} \sum_{\gamma \in \Gamma} \mu(\{x : T^i(x) = \gamma \cdot x\}) \leq N$, as the sets $\{x : T^i(x) = \gamma \cdot x\}, \gamma \in \Gamma$, are pairwise disjoint by the freeness of the action. Similarly $\sum_{\gamma \in \Gamma} s_\gamma(\theta) \leq 2N^2$, since the sets $\{x : T^i \theta T^j(x) = \gamma \cdot x\}, \gamma \in \Gamma$, are pairwise disjoint. Thus $\sum_{\gamma \in S} f(\gamma) \leq \sum_{\gamma \in S} \varphi_R(\gamma) + \sum_{\gamma \in S} s_\gamma(\theta) < \infty$. $\square$

So by Lemma 3.9, $\sum_{\gamma \in S'} f(\gamma) < \infty$ and hence $\Gamma \setminus S'$ is infinite. Since the action is mixing, there is $\gamma_0 \in \Gamma \setminus S'$ such that $\mu(\gamma_0 \cdot A_1 \cap B_1) \geq (3/4)\mu(A_1)^2$. Then

$$(K\gamma_0 K \cup K\gamma_0^{-1} K) \cap S = \emptyset.$$

Put

$$D = (A_1 \cap \gamma_0^{-1}(B_1)) \setminus \left(\bigcup_i \{x : T^i(x) \notin K \cdot x\} \cup \gamma_0^{-1} \cdot \bigcup_i \{x : T^i(x) \notin K \cdot x\}\right).$$

Then $\mu(D) \geq (3/4)\mu(A_1)^2 - (2/4)\mu(A_1)^2 > 0$. Also for any i, j,

$$T^i(\gamma_0|D)T^j(x) \in K\gamma_0 K \cdot x$$

if the left-hand side is defined. Indeed for such x, $T^j(x) \in D$, so $x = T^{N-j}T^j(x) \in K \cdot T^j(x)$ and $T^i\gamma_0 T^j(x) \in K \cdot \gamma_0 T^j(x)$. Thus $T^j(x) \in K \cdot x$ and $T^i\gamma_0 T^j(x) \in K\gamma_0 K \cdot x$. Similarly $T^i(\gamma_0^{-1}|\gamma_0(D))T^j(x) \in K\gamma_0^{-1}K \cdot x$, whenever the left-hand side is defined. In particular,

$$T^i(\gamma_0|D)T^j(x), \quad T^i(\gamma_0^{-1}|\gamma_0(D))T^j(x)$$

are not in $S \cdot x$, when they are defined. Take now $D' \subseteq D$ with

$$0 < \mu(D') \leq \min_{\gamma \in K\gamma_0 K \cup K\gamma_0^{-1}K}(f(\gamma) - \varphi_R(\gamma) - s_\gamma(\theta))/(2N^2).$$

This makes sense because $\gamma \in K\gamma_0 K \cup K\gamma_0^{-1}K$ implies that $\gamma \notin S$, so $f(\gamma) - \varphi_R(\gamma) - s_\gamma(\theta) > 0$. Let $\theta_0 = \gamma_0|D'$ and $\theta' = \theta \cup \theta_0$. We claim that this contradicts the maximality of θ. We need to verify that θ' satisfies (2). Now

$$\varphi_{R\vee\theta'}(\gamma) \leq \varphi_{R\vee\theta}(\gamma) + \sum_{i,j=1}^{N} \mu(\{x : T^i\theta_0 T^j(x) = \gamma \cdot x\})$$
$$+ \sum_{i,j=1}^{N} \mu(\{x : T^i(\theta_0)^{-1}T^j(x) = \gamma \cdot x\}).$$

But if $T^i\theta_0 T^j(x) = T^i(\gamma_0|D')T^j(x) = \gamma \cdot x$, then $\gamma \in K\gamma_0 K$. Therefore, if $\{x : T^i\theta_0 T^j(x) = \gamma \cdot x\}$ is not empty, then $\gamma \in K\gamma_0 K$, and similarly if $\{x : T^i(\theta_0)^{-1}T^j(x) = \gamma \cdot x\}$ is not empty, then $\gamma \in K\gamma_0^{-1}K$. Thus for any $\gamma \notin K\gamma_0 K \cup K\gamma_0^{-1}K, \varphi_{R\vee\theta'}(\gamma) \leq \varphi_{R\vee\theta}(\gamma) \leq f(\gamma)$ and for $\gamma \in K\gamma_0 K \cup K\gamma_0^{-1}K$,

$$\varphi_{R\vee\theta'}(\gamma) \leq \varphi_R(\gamma) + s_\gamma(\theta) + 2\sum_{i,j=1}^{N}\mu(D')$$
$$\leq \varphi_R(\gamma) + s_\gamma(\theta) + 2N^2\frac{f(\gamma) - \varphi_R(\gamma) - s_\gamma(\theta)}{2N^2}$$
$$\leq f(\gamma).$$

Hence the proof of the lemma is complete. $\qquad\square$

Proof of Theorem 3.8. We follow the argument of [Z], 9.3.2 (or [KM], 7.13), which shows how to construct an ergodic, hyperfinite subequivalence relation $E \subseteq F$. That proof proceeds by constructing a sequence of finite equivalence relations $E_1 \subseteq E_2 \subseteq \ldots$ each with classes of bounded size such that $E_1 =$ equality and $E_{n+1} = E_n \vee \theta_n$, where θ_n is *any* $\theta \in [[F]]$, with $\text{dom}(\theta) = A_n$, $\text{rng}(\theta) = B_n$, where A_n, B_n are some appropriately chosen Borel sets with $\mu(A_n) = \mu(B_n)$ and $A_n \cup B_n$ a section of E_n. Choose now f as in Lemma 3.9 and by Lemma 3.10 choose inductively θ_n so that $\varphi_{E_{n+1}}(\gamma) = \varphi_{E_n\vee\theta_n}(\gamma) \leq f(\gamma), \forall\gamma$ (we can of course assume that $f(1) = 1$, so that $\varphi_{E_1}(\gamma) \leq f(\gamma), \forall\gamma$). Then $\varphi_E(\gamma) = \lim_{n\to\infty}\varphi_{E_n}(\gamma) \leq f(\gamma)$, and thus $\varphi_E(\gamma) \to 0$ as $\gamma \to \infty$.

Finally note that, since by Dye's Theorem all ergodic, hyperfinite equivalence relations are isomorphic, we can find a free mixing action a_0 of $\mathbb{Z}$ that induces E. $\square$

It is an old problem of Schmidt [S] to find out whether there exist infinite countable groups Γ for which every ergodic action is mixing. One possible approach towards showing the non-existence of such groups is the following. Fix a free, mixing action b_0 of a countable infinite group Γ on (X, μ). Then there is a free, mixing action a_0 of $\mathbb{Z}$ on (X, μ) such that $E = E_{\mathbb{Z}}^X \subseteq F = E_{\Gamma}^X$. There is a weakly mixing action a of $\mathbb{Z}$ on (Y, ν) which is not mixing. One might hope that by constructing judiciously a_0, $\mathrm{CInd}(a_0, b_0)_{\mathbb{Z}}^{\Gamma}(a)$ might also be weakly mixing but not mixing.

Consider now the case of non-amenable groups Γ. Gaboriau and Lyons [GL] have shown that if Γ is a non-amenable group, there is a free, measure-preserving, mixing action b_0 of Γ on (X, μ) and a free, measure-preserving, ergodic action of $F_2 = \langle a, b \rangle$ on (X, μ) such that $E = E_{F_2}^X \subseteq F = E_{\Gamma}^X$. We will show below that one can find such a pair of actions so that moreover $\varphi_E(\gamma) \to 0$ as $\gamma \to \infty$.

Theorem 3.11 (with I. Epstein). *Let Γ be a non-amenable countable group. Then there is a free, measure-preserving, mixing action b_0 of Γ on (X, μ) and a free, measure-preserving, ergodic action a_0 of F_2 on (X, μ) with $E = E_{F_2}^X \subseteq F = E_{\Gamma}^X$ and $\varphi_E(\gamma) \to 0$ as $\gamma \to \infty$.*

We will postpone the proof of this theorem until Section 4 (D), as it will require some ideas from percolation on Cayley graphs.

(D) We will finally show how the combination of 3.11 and the work of Epstein [E], who showed that any non-amenable countable group Γ has uncountably many non-orbit equivalent free, measure-preserving, ergodic actions, provides the following strengthening to a non-classification result and also sharpens it by restricting to mixing actions. We refer the reader to [H2] for background information concerning Borel reducibility and the theory of turbulence and classification by countable structures.

Theorem 3.12 (with I. Epstein). *Let Γ be a non-amenable countable group. Then E_0 can be Borel reduced to OE on the space of free, measure-preserving, mixing actions of Γ and OE on this space cannot be classified by countable structures.*

Proof. We fix a free, measure-preserving, mixing action b_0 of Γ on (X, μ) and a free, measure-preserving, ergodic action a_0 of F_2 on (X, μ) with $E = E_{F_2}^X \subseteq F = E_{\Gamma}^X$ such that $\varphi_E(\gamma) \to 0$. Then for any action $a \in A(F_2, Y, \nu)$ we have the co-induced action $b = \mathrm{CInd}(a_0, b_0)_{F_2}^{\Gamma}(a) \in A(\Gamma, Z, \rho)$, where $Z = X \times Y^N$, $\rho = \mu \times \nu^N$ and the action $a' \in A(F_2, Z, \rho)$, which gives a subequivalence relation of that given by b. The action b is free and by 3.3 it is mixing. Also a' is free.

From Epstein [E], and the fact that in our case b is ergodic, we also have the following additional properties:

(*) For any Borel homomorphism $g: Y \to \bar{Y}$ of a to a *free* action $\bar{a} \in A(F_2, \bar{Y}, \bar{\nu})$, we have

$$\rho(\{z \in Z : \exists \gamma \neq 1(g \circ f(\gamma \cdot z) = g \circ f(z))\}) = 0,$$

where $f: Z \to Y$ is defined by $f(x, \bar{y}) = \bar{y}_0$.

(**) For every a'-invariant Borel set $A \subseteq Z$ of positive measure, if $\rho_A = \frac{\rho|A}{\rho(A)}$ is the normalized restriction of ρ to A, then $f_* \rho_A = \nu$, thus a is a factor of $(a'|A, \rho_A)$.

Consider now the standard action of $\mathrm{SL}_2(\mathbb{Z})$ on $(\mathbb{T}^2, \lambda)$ with the usual product measure λ, and fixing a copy of F_2 of finite index in $\mathrm{SL}_2(\mathbb{Z})$ let $\bar{a}_0$ be the restriction of this action to F_2. We will use the following basic lemma originally proved in Ioana [I], when $F_2 \leq \Gamma$, but realized to hold as well in the more general case stated below in Epstein [E].

Lemma 3.13. *Let Γ be a countable group and let $\{b_i\}_{i \in I}$ be an uncountable family of OE free, measure-preserving, ergodic actions of Γ on (Z, ρ) such that for each $i \in I$ there is a free, measure-preserving action a_i' of F_2 on (Z, ρ) with the following two properties:*

(i) *$E_{a_i'} \subseteq E_{b_i}$ (where E_c is the equivalence relation induced by an action c).*

(ii) *There is a Borel homomorphism $f_i: Z \to \mathbb{T}^2$ of a_i' to $\bar{a}_0$ such that*

$$\rho(\{z : \exists \gamma \neq 1(f_i(\gamma \cdot z) = f_i(z))\}) = 0.$$

Then there is an uncountable $J \subseteq I$ such that given any $i, j \in J$, there are a_i'-, a_j'-invariant Borel sets A_i, A_j of positive measure, so that the actions $a_i'|A_i, a_j'|A_j$ are isomorphic (with respect to the normalized measures ρ_{A_i}, ρ_{A_j}).

Denote by $\mathrm{Irr}(F_2, \mathcal{H})$ the Polish space of irreducible unitary representation of F_2 on a separable, infinite-dimensional Hilbert space $\mathcal{H}$ (see, e.g., [K], Appendix H). By a result of Hjorth [H1] (see also [K], Appendix H) there is a conjugacy invariant dense G_δ set $G(\Gamma, \mathcal{H}) \subseteq \mathrm{Irr}(F_2, \mathcal{H})$ such that the conjugacy action of the unitary group $U(\mathcal{H})$ on G is turbulent. As a consequence, if $\cong$ denotes isomorphism between representations, then $\cong |G(\Gamma, \mathcal{H})$ is not classifiable by countable structures.

Finally for each unitary representation π of F_2 on $\mathcal{H}$ denote by a_π the corresponding Gaussian action of F_2 (on a space (Ω, τ); see [K], Appendix E). It has the following two properties:

(1) $\pi \cong \rho \implies a_\pi \cong a_\rho$;

(2) if $\kappa_0^{a_\pi}$ is the Koopman representation on $L_0^2(\Omega, \tau)$ associated to a_π, then $\pi \leq \kappa_0^{a_\pi}$.

Given now any $\pi \in G(\Gamma, \mathcal{H})$, consider the (diagonal) product action $a(\pi) = \bar{a}_0 \times a_\pi$ on $(\mathbb{T}^2 \times \Omega, \lambda \times \tau) = (Y, \nu)$. Let then $b(\pi) \in A(\Gamma, Z, \rho)$ be the co-induced action of $a(\pi)$ and $a'(\pi)$ the associated F_2-action. Thus $\pi \mapsto b(\pi)$ is a Borel function from $G(\Gamma, \mathcal{H})$ into the space of free, measure-preserving, mixing actions of Γ on (Z, ρ). Put for $\pi, \rho \in G(\Gamma, \mathcal{H})$:

$$\pi R \rho \iff b(\pi) \mathrm{OE} b(\rho).$$

Then R is an equivalence relation on $G(\Gamma, H)$ and $\pi \cong \rho \implies \pi R \rho$.

Claim. *R has countable index over $\cong$.*

Granting this, the proof is completed as follows. First, to see that E_0 can be Borel reduced to OE on the space of free, measure-preserving, mixing actions of Γ it is of course enough to show that it can be Borel reduced to R. The equivalence relation R is analytic with meager classes (as each $\cong$-class in $G(\Gamma, \mathcal{H})$ is meager in $G(\Gamma, \mathcal{H})$, and every R-class contains only countably many $\cong$-classes), so R is meager and contains the equivalence relation $\cong$ induced by the conjugacy action of $U(\mathcal{H})$ on $G(\Gamma, \mathcal{H})$ which has dense orbits (being turbulent). Then E_0 is Borel reducible to R by the argument in Becker–Kechris [BK], 3.4.5.

To prove non-classification by countable structures, it is of course enough to show that R has the same property. If this fails, towards a contradiction, there is Borel $F \colon G(\Gamma, \mathcal{H}) \to X_L$, where X_L is the standard Borel space of countable models of a countable language L, such that

$$\pi R \rho \iff F(\pi) \cong F(\rho),$$

so that, in particular,

$$\pi \cong \rho \implies F(\pi) \cong F(\rho).$$

But then, by turbulence, there is a comeager set $A \subseteq G(\Gamma, \mathcal{H})$ and $M_0 \in X_L$ with

$$F(\pi) \cong M_0, \ \forall \pi \in A.$$

Because every R-class is meager, there are R-inequivalent $\pi, \rho \in A$, so that $F(\pi) \not\cong F(\rho)$, a contradiction. $\qquad\square$

Proof of the Claim. Assume, towards a contradiction, that there is an uncountable family $\{\pi_i\}_{i \in I} \subseteq G(\Gamma, \mathcal{H})$ of pairwise non-isomorphic representations such that if we put $b_i = b(\pi_i)$, then $\{b_i\}_{i \in I}$ are OE. Let $a'_i = a'(\pi_i)$, so that $E_{a'_i} \subseteq E_{b_i}$. Moreover if $f_i \colon Z \to \mathbb{T}^2$ is given by $f_i = g \circ f$, where $g \colon Y \to \mathbb{T}^2$ is the projection, then f_i is a Borel homomorphism of a'_i to $\bar{a}_0$ such that

$$\rho(\{z : \exists \gamma \neq 1 (f_i(\gamma \cdot z) = f_i(z))\}) = 0$$

(by property (*) of the co-induced action mentioned earlier). So, by Lemma 3.13, there is an uncountable $J \subseteq I$ such that given any $i, j \in J$, there are a'_i-, a'_j-invariant Borel sets A_i, A_j of positive measure, so that the actions $a'_i | A_i, a'_j | A_j$ are isomorphic. Note that we also have, by property (**) of the co-induced action, that $a(\pi_i) = \bar{a}_0 \times a_{\pi_i}$ is a factor of $a'_i | A_i$.

Fix any $i_0 \in J$. Then for any $j \in J$, fixing A_{i_0}, A_j as above, $\pi_j \leq \kappa_0^{a_{\pi_j}} \leq \kappa_0^{a_0 \times a_{\pi_j}} \leq \kappa_0^{a'_j | A_j} \cong \kappa^{a'_{i_0} | A_{i_0}} \leq \kappa^{a'_{i_0}}$. This produces an uncountable family $\{\pi_j\}_{j \in J}$ of pairwise non-isomorphic irreducible subrepresentations of $\kappa^{a'_{i_0}}$, which is impossible. $\qquad\square$

4. Percolation on Cayley graphs of groups

(A) Let Γ be a finitely generated group, $Q = \{\gamma_1, \ldots, \gamma_n\}$ a set of generators for Γ not containing 1, and let $\mathcal{G}_Q = \langle \Gamma, E_Q \rangle$ be the left Cayley graph of Γ, with respect to Q, whose set of edges E_Q consists of all $\{\gamma, \gamma_i \gamma\}$, $1 \leq i \leq n$, $\gamma \in \Gamma$. Note that the edges of our Cayley graphs are unordered pairs of vertices and that we do not require the set Q to be symmetric. In fact, for most of our applications, the set $Q \cap Q^{-1}$ will only contain elements of order 2.

The group Γ acts on Γ and thus on the Cayley graph by right translations: $\delta \cdot \{\gamma, \gamma_i \gamma\} = \{\gamma \delta^{-1}, \gamma_i \gamma \delta^{-1}\}$. So Γ acts also on the space $\Omega_Q = 2^{E_Q}$ by shift:

$$(\gamma \cdot \omega)(\{\delta, \varepsilon\}) = \omega(\gamma^{-1} \cdot \{\delta, \varepsilon\}) = \omega(\{\delta\gamma, \varepsilon\gamma\}).$$

Every $\omega \in \Omega_Q$ can be viewed as the subgraph of $\mathcal{G}_Q$ with vertex set Γ and an edge set ω (here we identify, as usual, elements of 2^{E_Q} with subsets of E_Q). The connected components of ω are called the *clusters* of ω.

An *invariant bond percolation* on this Cayley graph is a Γ-invariant probability Borel measure P on Ω_Q. The percolation P is *ergodic* if the action of Γ is ergodic with respect to P.

Consider now a free measure-preserving action of Γ on (X, μ) and fix Borel subsets $A_1, \ldots, A_n$ of X satisfying the condition:

$$\text{if } \gamma_i, \gamma_j \in Q \text{ and } \gamma_j = \gamma_i^{-1}, \text{ then } \gamma_i \cdot A_i = A_j.$$

Define $\Phi \colon X \to \Omega_Q$ by

$$\Phi(x)(\{\delta, \gamma_i \delta\}) = 1 \iff \delta \cdot x \in A_i.$$

The condition above ensures that Φ is well defined. It is also easy to check that Φ is Γ-equivariant and thus if $P = \Phi_* \mu$, then P is an invariant bond percolation on the Cayley graph $\mathcal{G}_Q$.

Note that if $E = \gamma_1 | A_1 \vee \cdots \vee \gamma_n | A_n$ is the subequivalence relation of $F = E_\Gamma^X$ generated by $\gamma_1 | A_1, \ldots, \gamma_n | A_n$ and if $x \in X$ is such that $\Phi(x) = \omega$, then the map $\gamma \mapsto \gamma \cdot x$ is a 1-1 correspondence of Γ with $\Gamma \cdot x = [x]_F = \{y : yFx\}$, which sends the clusters of ω to the E-classes contained in $[x]_F$. So the structure of the E-classes in $[x]_F$ is equivalent to the structure of clusters of $\Phi(x) = \omega$. The E-class of x corresponds to the cluster of 1 in ω. We let

$$X_\infty = \{x \in X : [x]_E \text{ is infinite}\}.$$

For $\gamma, \delta \in \Gamma$, put

$$C_{\gamma, \delta} = \{\omega \in \Omega_Q : \gamma, \delta \text{ are in the same cluster of } \omega\}$$

and (as in Lyons–Schramm [LS]) let

$$\tau(\gamma, \delta) = P(C_{\gamma, \delta}).$$

Note that by invariance

$$\tau(\gamma, \delta) = \tau(\gamma\varepsilon, \delta\varepsilon), \ \forall \varepsilon \in \Gamma,$$

so $\tau(\gamma, \delta) = \tau(1, \delta\gamma^{-1})$. If

$$C_\gamma = C_{1,\gamma}$$

and

$$A_\gamma = \{x : \Phi(x) \in C_\gamma\},$$

then $x \in A_\gamma$ iff $\gamma \cdot x \in [x]_E$ iff $x \in A_E(\gamma)$ and so

$$\varphi_E(\gamma) = \mu(A_E(\gamma)) = P(C_\gamma) = \tau(1, \gamma),$$

where we identify here γ with $x \mapsto \gamma \cdot x$.

(B) Conversely, let P be an invariant bond percolation on $\mathcal{G}_Q$. The action of Γ on Ω_Q might not be free P-a.e. So fix a free, measure-preserving action of Γ on some space (Y, ν) and consider the product action of Γ on $(X, \mu) = (\Omega_Q \times Y, P \times \nu)$, which is clearly free. (If the action of Γ on Ω_Q is already free, we can simply take $(X, \mu) = (\Omega_Q, P)$.) Let

$$C_i = \{\omega \in \Omega_Q : \omega(1, \gamma_i) = 1\},$$
$$A_i = C_i \times Y \subseteq X.$$

Consider $\gamma_1|A_1, \ldots, \gamma_n|A_n$ and $\Phi \colon X \to \Omega_Q$ defined as before. Then if $x = (\omega, y)$, we have

$$\begin{aligned}
\Phi(x)(\{\delta, \gamma_i\delta\}) = 1 &\iff \delta \cdot x \in A_i \\
&\iff (\delta \cdot \omega, \delta \cdot y) \in A_i \\
&\iff \delta \cdot \omega \in C_i \\
&\iff \delta \cdot \omega(\{1, \gamma_i\}) = 1 \\
&\iff \omega(\{\delta, \gamma_i\delta\}) = 1,
\end{aligned}$$

i.e., $\Phi(x) = \Phi(\omega, y) = \omega$. We can also define F, E, and X_∞ the same way as before.

(C) For further reference, we discuss some additional concepts and results concerning percolation.

In the context of (B), and assuming that the Γ-action on (X, μ) is ergodic and $\mu(X_\infty) > 0$, we say that P has *indistinguishable infinite clusters* if $E|X_\infty$ is ergodic (this is not the standard definition but is justified by Gaboriau–Lyons [GL], Prop. 5).

For any invariant bond percolation P on $\mathcal{G}_Q$ and edge $e \in E_Q$, define $\pi_e \colon \Omega_Q \to \Omega_Q$ by $\pi_e(\omega) = \omega \cup \{e\}$. Then we say that P is *insertion-tolerant* if $P(A) > 0 \implies P(\pi_e(A)) > 0$ for all $e \in E_Q$ and $A \subseteq \Omega_Q$. An example of insertion-tolerant, ergodic percolation is Bernoulli percolation.

It is well known, see Newman–Schulman [NS] or Lyons–Schramm [LS], 3.8, that if P is ergodic and insertion-tolerant, then exactly one of the following happens: ω has no infinite clusters, P-a.s.; ω has infinitely many infinite clusters, P-a.s.; ω has exactly one infinite cluster P-a.s. Thus in the context of (B), if P is insertion-tolerant, then either the E-classes are finite, μ-a.e., or there are infinitely many infinite E-classes in each F-class, μ-a.e., or there is exactly one infinite E-class in each F-class, μ-a.e.

Moreover, again in the context of (B), Lyons–Schramm [LS] and Gaboriau-Lyons [GL], Prop. 6, show that if P is ergodic and insertion-tolerant, and has infinite clusters, P-a.s., then P has indistinguishable infinite clusters.

Finally, Lyons–Schramm [LS], Theorem 4.1, show that for any ergodic, insertion-tolerant, invariant bond percolation P,

$$\inf_{\gamma \in \Gamma} \tau(1, \gamma) > 0 \Longrightarrow \omega \text{ has a unique infinite cluster, } P\text{-a.s.}$$

This implies that in the context of (B), if P is ergodic and insertion-tolerant, then if $\inf_{\gamma \in \Gamma} \varphi_E(\gamma) > 0$, there is a unique infinite E-class in each F-class, μ-a.e., and thus $E|X_\infty = F|X_\infty$, μ-a.e.

We note here that one can give an alternative proof of Theorem 4.1 in [LS] by using the results in the present article and the indistinguishability of infinite clusters for insertion-tolerant percolations. Indeed assume that $\inf_{\gamma \in \Gamma} \tau(1, \gamma) > 0$. In the notation of (A), (B) above, since $\tau(1, \gamma) = \varphi_E(\gamma)$, this means that $\varphi_E^0 = \inf_{\gamma \in \Gamma} \varphi_E(\gamma) > 0$, so by 2.5, there is an E-invariant Borel set $A \subseteq X$ of positive measure such that $[F|A : E|A] < \infty$. By taking in (B) the action of Γ on (Y, ν) to be weakly mixing, we have that $F = E_\Gamma^X$ is ergodic and thus A meets every F-class. It follows that $A \subseteq X_\infty$ and thus, since P is insertion-tolerant, so that $E|X_\infty$ is ergodic, we have that $A = X_\infty$. So ω has finitely many infinite clusters, P-a.s. and therefore exactly one.

Remark 4.1. Note that for any free, measure-preserving action of an infinite group Γ on (X, μ) and any subequivalence relation $E \subseteq F = E_\Gamma^X$, if E has finite classes, then $\varphi_E(\gamma) \to 0$, as $\gamma \to \infty$. Indeed, $\varphi_E(\gamma) = \int f(\gamma, x) \, d\mu(x)$, where $f(\gamma, x) = 1$ if $\gamma \cdot x \in [x]_E$ and $f(\gamma, x) = 0$ if $\gamma \cdot x \notin [x]_E$. Since $f(\gamma, x) \to 0$ for each x, this conclusion follows by the Lebesgue Dominated Convergence Theorem.

(D) The rest of this section is devoted to proving Theorem 3.11 whose statement we recall below.

Theorem 3.11 (with I. Epstein). *Let Γ be a non-amenable countable group. Then there is a free, measure-preserving, mixing action b_0 of Γ on (X, μ) and a free, measure-preserving, ergodic action a_0 of F_2 on (X, μ) with $E = E_{F_2}^X \subseteq F = E_\Gamma^X$ and $\varphi_E(\gamma) \to 0$ as $\gamma \to \infty$.*

Proof. We start with the following lemma.

Lemma 4.2. *Consider a free, measure-preserving, ergodic action of a countable group Γ on (X, μ), a Borel set $A \subseteq X$ of positive measure and E_A a Borel equivalence relation on A satisfying $E_A \subseteq E_\Gamma^X$ and*

$$\lim_{\gamma \to \infty} \mu(\{x \in A : \gamma \cdot x \in A \text{ and } \gamma \cdot xE_Ax\}) = 0.$$

Then there exists a Borel equivalence relation $E \subseteq E_\Gamma^X$ on X with $E|A = E_A$ such that $\varphi_E(\gamma) \to 0$ as $\gamma \to \infty$. Moreover, if E_A is treeable (resp., ergodic), then E is treeable (resp. ergodic).

Proof. Let $X \setminus A = \bigsqcup_{\gamma \in \Gamma} D_\gamma$, with $\gamma(D_\gamma) \subseteq A$, $\forall \gamma \in \Gamma$. This exists since A is complete section for E_Γ^X. Let $E = E_A \vee \{\gamma|D_\gamma : \gamma \in \Gamma\}$ be the equivalence relation generated by E_A and $\{\gamma|D_\gamma : \gamma \in \Gamma\}$. Clearly E has all the required properties except perhaps that $\varphi_E(\gamma) \to 0$ as $\gamma \to \infty$, which we now proceed to verify.

Define $f : X \to \Gamma$ by

$$f(x) = \gamma \iff x \in D_\gamma \text{ or } (x \in A \text{ and } \gamma = 1).$$

Fix $\varepsilon > 0$. Let $K \subseteq \Gamma$ be finite, symmetric such that $\mu(\{x : f(x) \notin K\}) < \varepsilon$. Let $F \subseteq \Gamma$ be finite such that $\mu(\{x \in A : \gamma \cdot x \in A \text{ and } \gamma \cdot xE_Ax\}) < \varepsilon/|K|^2$ for $\gamma \notin F$ (where $|K| = \text{card}(K)$). We will show that $\varphi_E(\gamma) < 3\varepsilon$ if $\gamma \notin KFK$. Indeed, fix such γ. Notice that for each x such that $\gamma \cdot xEx$, γ can be uniquely written as $f(\gamma \cdot x)^{-1}\gamma'_x f(x)$ for some γ'_x. We have

$$\mu(\{x : \gamma \cdot xEx\}) < 2\varepsilon + \mu(\{x : \gamma \cdot xEx \text{ and } f(x) \in K \text{ and } f(\gamma \cdot x)^{-1} \in K\}).$$

Now it only remains to notice that if x is in the second set above, $\gamma'_x \notin F$ and $\gamma'_x \in K\gamma K$. So, by the choice of F, the second summand is bounded by ε and we are done. $\square$

By Gaboriau–Lyons [GL], we fix a free, measure-preserving, mixing action $\bar{b}_0$ of Γ on (Y, ρ) and a free, measure-preserving, ergodic action $\bar{a}_0$ of F_2 on (Y, ρ) with $E_{\bar{a}_0} \subseteq E_{\bar{b}_0}$ and such that moreover if $F_2 = \langle g_1, g_2 \rangle$ then $\langle g_1 \rangle$ also acts ergodically on (Y, ρ) and thus, by Dye's Theorem, we can in fact assume that the action of $\langle g_1 \rangle$ is mixing. Let $N = [E_{\bar{b}_0} : E_{\bar{a}_0}]$. We can of course assume that $N > 1$.

Let $\mathcal{G} = \langle F_2, E \rangle$ be the Cayley graph of F_2 corresponding to the set of generators $Q = \{g_1, g_2\}$. We consider on $\mathcal{G}$ Bernoulli p-percolation P_p with $1/3 < p < 1$, so that ω has infinitely many infinite clusters P_p-a.s., see [LP]. Consider the usual shift action a of F_2 on (Ω, ν), where $\Omega = 2^E$, $\nu = P_p$, and the co-induced action $b_0 = \text{CInd}(\bar{a}_0, \bar{b}_0)_{F_2}^\Gamma(a)$ on $(X, \mu) = (Y \times \Omega^N, \rho \times \nu^N)$. This action is mixing by 3.1. Consider also the associated action a' of F_2 on (X, μ), so that $E_{a'} \subseteq E_{b_0} = E_\Gamma^X = F$ and a is a factor of a' via $\varphi(y, \bar{\omega}) = \bar{\omega}_0$, so, in particular, a' is free. The same argument as in the proof of 3.1 also shows that the action of $\langle g_1 \rangle$ on X is mixing, hence a' is ergodic as well.

Recall that the action a' of F_2 on (X, μ) is given (in the notation of Section 3) by

$$\delta \cdot (y, \bar\omega) = (\delta \cdot y, (n \mapsto \bar\delta(\delta, y)_n \cdot \bar\omega_{\pi(\delta, y)^{-1}(n)})),$$

Here we have that $\pi(\delta, y)(k) = n \iff [C_k(y)]_{E_{\bar a_0}} = [C_n(\delta \cdot y)]_{E_{\bar a_0}}$ and $\bar\delta(\delta, y)_n \cdot C_{\pi(\delta, y)^{-1}(n)}(y) = C_n(\delta \cdot y)$. Since $C_0(y) = y$, we have $\pi(\delta, y)(0) = 0$. Moreover $\bar\delta(\delta, y)_0 \cdot y = \delta \cdot y$, so $\bar\delta(\delta, y)_0 = \delta$. It follows that

$$\delta \cdot (y, \bar\omega) = (\delta \cdot y, \delta \cdot \bar\omega_0, (n > 0 \mapsto \bar\delta(\delta, y)_n \cdot \bar\omega_{\pi(\delta, y)^{-1}(n)})).$$

Therefore (up to an obvious isomorphism) $X = \Omega \times (Y \times \Omega^{N \setminus \{0\}})$, and the action a' is the product action of a and a free, measure-preserving action of F_2 on $(Y \times \Omega^{N \setminus \{0\}}, \rho \times \nu^{N \setminus \{0\}})$. The projection of $X = Y \times \Omega^N$ to the first factor Ω in the above product is of course $\varphi(y, \bar\omega) = \bar\omega_0$. Thus we are in the situation of Section 4 (B).

We can then define the associated subequivalence relation $E' \subseteq E_{a'}$ by

$$x_1 E' x_2 \iff \exists \delta \in F_2(\delta \cdot x_1 = x_2 \text{ and } 1, \delta \text{ are in the same } \varphi(x_1) \text{ cluster}),$$

i.e., $E' = g_1|A_{g_1} \vee g_2|A_{g_2}$, where $A_{g_i} = \{x : \varphi(x)(\{1, g_i\}) = 1\}$. Let

$$A = X_\infty = \{x : [x]_{E'} \text{ is infinite}\}$$
$$= \{x : \text{ the cluster of } 1 \text{ in } \varphi(x) \text{ is infinite}\}.$$

Since P_p has indistinguishable infinite clusters, $E_A = E'|A$ is ergodic. It is also clear that E_A is treeable and non-hyperfinite, as the canonical treeing on it has infinitely many ends, a.e. (see [LP], 7.29).

We next show that E_A satisfies the hypothesis of 4.2. Indeed we have by Fubini

$$\mu(\{(y, \bar\omega) : \gamma \cdot (y, \bar\omega) E_A(y, \bar\omega)\}) = \int \nu^N(\{\bar\omega : \gamma \cdot (y, \bar\omega) E_A(y, \bar\omega)\}) \, d\rho(y),$$

and it is enough to show that the function under the integral converges to 0 as $\gamma \to \infty$ for any $y \in Y$. Fix $y \in Y$ and consider an arbitrary sequence $\gamma_n \to \infty$. Notice that if $\gamma \cdot (y, \bar\omega) E_A(y, \bar\omega)$, then $\gamma \cdot (y, \bar\omega) E'(y, \bar\omega)$, so there is $\delta \in F_2$ with $\gamma \cdot (y, \bar\omega) = \delta \cdot (y, \bar\omega)$ and thus $\gamma \cdot \bar y = \delta \cdot \bar y$. Since we can clearly assume that $\{\bar\omega : \gamma_n \cdot (y, \bar\omega) E_A(y, \bar\omega)\} \neq \emptyset$ for each n, there is (unique) $\delta_n \in F_2$ with $\gamma_n \cdot y = \delta_n \cdot y$ and $\gamma_n \cdot (y, \bar\omega) E_A(y, \bar\omega) = \Rightarrow \gamma_n \cdot (y, \bar\omega) = \delta_n \cdot (y, \bar\omega)$. Clearly $\delta_n \to \infty$. So

$$\nu^N(\{\bar\omega : \gamma_n \cdot (y, \bar\omega) E_A(y, \bar\omega)\}) = \nu^N(\{\bar\omega : \delta_n \cdot (y, \bar\omega) E_A(y, \bar\omega)\}) \leq \tau(1, \delta_n) \to 0.$$

Thus by 4.2 there is a treeable, ergodic, non-hyperfinite equivalence relation $E_1 \subseteq E_\Gamma^X$ with $E_1|A = E_A$ such that $\varphi_{E_1}(\gamma) \to 0$ as $\gamma \to \infty$. In particular, the cost of E_1 is > 1. Then by [GL], Proposition 13, one can find a free, measure-preserving, ergodic action a_0 of F_2 on (X, μ) such that, letting $E = E_{F_2}^X$ be its associated equivalence relation, we have $E \subseteq E_1 \subseteq F = E_\Gamma^X$. Clearly $\varphi_E(\gamma) \to 0$ as $\gamma \to \infty$, and the proof of 3.11 is complete. $\qquad\square$

5. Property (T) groups

(A) Let now Γ be an infinite group with Kazhdan's property (T). A *Kazhdan pair* for Γ is a pair (Q, ε), where Q is a finite generating set for Γ and $\varepsilon > 0$ is such that for any unitary representation π of Γ on a Hilbert space $\mathcal{H}$, if there is a vector $\xi \in H$ with $\|\pi(\gamma)(\xi) - \xi\| < \varepsilon\|\xi\|$, $\forall \gamma \in Q$, then there is a non-0 Γ-invariant vector. The group Γ having property (T) is equivalent to the assertion that there is a Kazhdan pair (Q, ε) and also equivalent to the assertion that for every finite generating set Q there is $\varepsilon > 0$ with (Q, ε) a Kazhdan pair. Let

$$\varepsilon_Q = \max\{\varepsilon : (Q, \varepsilon) \text{ is a Kazhdan pair}\} > 0$$

be the *maximal Kazhdan constant* associated to Q (this is sometimes denoted by $\mathcal{K}(Q, \Gamma)$). It is easy to see that $\varepsilon_Q \leq \sqrt{2}$ and Shalom (private communication) has shown that $\sup_Q \varepsilon_Q = \sqrt{2}$ (where the sup is over all finite generating sets). We state below a more precise quantitative version.

Proposition 5.1 (Shalom). *Let Γ be a countable group satisfying property* (T)*. Let (Q, ε) be a Kazhdan pair for Γ, with Q symmetric containing* 1*. Let* $\mathrm{card}(Q) = k$. *Then for every $n \geq 1$,*

$$\left(Q^n, \sqrt{2\left[1 - \left(\frac{k - \varepsilon^2/2}{k}\right)^n\right]} \right)$$

is also a Kazhdan pair. In particular, $\sup_Q \varepsilon_Q = \sqrt{2}$, *where the* sup *is over all the finite generating sets.*

Proof. Observe that $(\bar{Q}, \bar{\varepsilon})$ is a Kazhdan pair iff for any unitary representation $\pi : \Gamma \to U(\mathcal{H})$ which has no non-0 invariant vectors, we have

$$\max_{\gamma \in Q} \|\pi(\gamma)(\xi) - \xi\| \geq \bar{\varepsilon}$$

for every unit vector $\xi \in \mathcal{H}$, or equivalently

$$\min_{\gamma \in Q} \mathrm{Re}\langle \pi(\gamma)(\xi), \xi \rangle \leq 1 - \frac{\bar{\varepsilon}^2}{2} \tag{$*$}$$

for any unit vector $\xi \in \mathcal{H}$.

So fix $Q' = Q^n$, $\varepsilon' = \sqrt{2[1 - (\frac{k - \varepsilon^2/2}{k})^n]}$ in order to show that $(*)$ holds for (Q', ε') and any π without invariant non-0 vectors. For this define the averaging operator

$$T = \frac{1}{k} \sum_{\gamma \in Q} \pi(\gamma).$$

Since T is self-adjoint, we have

$$\|T\| = \sup_{\|\xi\|=1} |\langle T(\xi), \xi\rangle| = \sup_{\|\xi\|=1} \frac{1}{k}\Big|1 + \sum_{\gamma \in Q\setminus\{1\}} \mathrm{Re}\langle \pi(\gamma)(\xi), \xi\rangle\Big|$$

$$\leq \sup_{\|\xi\|=1} \frac{1}{k}\Big((k-1) + \Big(1 - \frac{\varepsilon^2}{2}\Big)\Big) = \frac{k - \frac{\varepsilon^2}{2}}{k}$$

(by $(*)$ for (Q, ε)).

Then for every $n \geq 1$ and every unit vector $\xi \in \mathcal{H}$, we have

$$\frac{1}{k^n} \sum_{\gamma_1,\ldots,\gamma_n \in Q} \mathrm{Re}\langle \pi(\gamma_1 \cdots \gamma_n)(\xi), \xi\rangle = \langle T^n(\xi), \xi\rangle$$

$$\leq \|T^n\| \leq \|T\|^n \leq \Big(\frac{k - \frac{\varepsilon^2}{2}}{k}\Big)^n.$$

This gives

$$\min_{\gamma \in Q^n} \mathrm{Re}\langle \pi(\gamma)(\xi), \xi\rangle \leq \Big(\frac{k - \frac{\varepsilon^2}{2}}{k}\Big)^n$$

for every unit vector $\xi \in \mathcal{H}$. $\qquad\square$

We can also see that restricting the number of generators to a fixed size n gives an upper bound strictly less than $\sqrt{2}$. More precisely we have for each infinite group Γ with property (T):

$$\varepsilon_n(\Gamma) = \sup\{\varepsilon_Q : Q \text{ generates } \Gamma, \mathrm{card}(Q) \leq n\} \leq \sqrt{2} \cdot \sqrt{\frac{2n-1}{2n+1}}.$$

To see this fix Q with $\mathrm{card}(Q) \leq n$ and consider the left regular representation λ of Γ, let $\bar{Q} = Q \cup \{1\} \cup Q^{-1}$, so that $\mathrm{card}(\bar{Q}) = m \leq 2n + 1$, and let $\xi \in \ell^2(\Gamma)$ be the unit vector $\xi = \frac{1}{\sqrt{m}}\chi_{\bar{Q}}$. Then, as $\gamma\bar{Q} \cap \bar{Q}$ contains $\{1, \gamma\}$ for each $\gamma \in Q$, we have $\langle \gamma \cdot \xi, \xi\rangle = \frac{1}{m}\mathrm{card}(\gamma\bar{Q} \cap \bar{Q}) \geq \frac{2}{m} \geq \frac{2}{2n+1}$, so $\|\gamma \cdot \xi - \xi\|^2 = 2(1 - \langle \gamma \cdot \xi, \xi\rangle) \leq 2(1 - \frac{2}{2n+1}) = 2(\frac{2n-1}{2n+1})$. Since λ has no non-0 invariant vectors, $\varepsilon_Q \leq \sqrt{2} \cdot \sqrt{\frac{2n-1}{2n+1}}$. We do not know if the above upper bound for $\varepsilon_n(\Gamma)$ is best possible.

We see from 5.1 that if Q is a symmetric generating set with $\mathrm{card}(Q) = k$ and (Q, ε) is a Kazhdan pair for Γ, then

$$\varepsilon_{k^n}(\Gamma) \geq \sqrt{2} \cdot \sqrt{1 - \Big(\frac{k - \varepsilon^2/2}{k}\Big)^n}, \quad \forall n \geq 1.$$

In a preliminary version of this article, we raised the following question: Fix $n \geq 2$. Is $\inf\{\varepsilon_n(\Gamma) : \Gamma \text{ is an } n\text{-generated infinite group with property (T)}\} > 0$? Shalom

pointed out that this is false, with the counterexamples being products of two groups with one factor a finite cyclic group. The question remains open whether there are such counterexamples if one considers only property (T) groups which are perfect (i.e., groups equal to their commutator subgroups).

The following fact is well known.

Proposition 5.2 (see [BHV], 1.1.9). *Let (Q, ε) be a Kazhdan pair for Γ. Then for any $1 \geq \delta > 0$, any unitary representation $(\pi, \mathcal{H})$ of Γ and $\xi \in \mathcal{H}$, if $\forall \gamma \in Q$ $(\|\pi(\gamma)(\xi) - \xi\| < \delta\varepsilon\|\xi\|)$, there is a Γ-invariant vector η with $\|\xi - \eta\| \leq \delta\|\xi\|$.*

The next result is a consequence of Deutsch–Robertson [DR], but we will give the short proof for the reader's convenience.

Proposition 5.3. *Let Γ be a group with property* (T), *(Q, ε) a Kazhdan pair for Γ, $\delta > 0$, and φ a positive-definite function on Γ with $\varphi(1) = 1$. Then $\forall \gamma \in Q$ $(\mathrm{Re}\, \varphi(\gamma) \geq 1 - \frac{\delta^2\varepsilon^2}{2})$ implies that $\forall \gamma \in \Gamma$ $(\mathrm{Re}\, \varphi(\gamma) \geq 1 - 2\delta^2)$.*

Proof. Let $(\pi, \mathcal{H}, \xi) = (\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be the GNS representation of Γ associated to φ, so that $\langle \pi(\gamma)(\xi), \xi \rangle = \varphi(\gamma)$. In particular ξ is a unit vector as $\varphi(1) = 1$. Also $\|\pi(\gamma)(\xi) - \xi\|^2 = 2(1 - \mathrm{Re}\, \varphi(\gamma))$. So if $\mathrm{Re}\, \varphi(\gamma) \geq 1 - \frac{\delta^2\varepsilon^2}{2}$ for $\gamma \in Q$, then $\|\pi(\gamma)(\xi) - \xi\|^2 \leq \delta^2\varepsilon^2$, so there is a Γ-invariant vector η with $\|\xi - \eta\| \leq \delta$. Then $\|\pi(\gamma)(\xi) - \eta\| \leq \delta$, so $\|\pi(\gamma)(\xi) - \xi\| \leq 2\delta$, thus $2(1 - \mathrm{Re}\, \varphi(\gamma)) \leq 4\delta^2$, $\forall \gamma \in \Gamma$, or $\mathrm{Re}\, \varphi(\gamma) \geq 1 - 2\delta^2$, $\forall \gamma \in \Gamma$. $\square$

(B) Consider now a measure-preserving action of a group Γ with property (T) on a standard measure space (X, μ) and let $F = E_\Gamma^X$ be the associated equivalence relation and $E \subseteq F$ a subequivalence relation. Applying the preceding to φ_E, we obtain:

Corollary 5.4. *Let Γ be an infinite group with property* (T) *and (Q, ε) a Kazhdan pair for Γ. If Γ acts in a measure-preserving way on (X, μ) with associated equivalence relation $F = E_\Gamma^X$ and $E \subseteq F$ is a subequivalence relation, then:*

(i) *For any $\delta > 0$, $\min_{\gamma \in Q} \varphi_E(\gamma) \geq 1 - \frac{\delta^2\varepsilon^2}{2}$ implies that $\varphi_E^0 = \inf_{\gamma \in \Gamma} \varphi_E(\gamma) \geq 1 - 2\delta^2$.*

(ii) *If $\min_{\gamma \in Q} \varphi_E(\gamma) > 1 - \frac{\varepsilon^2}{2}$, then there is an E-invariant set of positive measure A such that $[F|A : E|A] < \infty$. If moreover $\min_{\gamma \in Q} \varphi_E(\gamma) > 1 - \frac{\varepsilon^2}{4}$, then $\varphi_E^0 > 0$ and $[F|A : E|A] \leq \frac{1}{\varphi_E^0}$. If $\min_{\gamma \in Q} \varphi_E(\gamma) > 1 - \frac{\varepsilon^2}{8}$, then $\varphi_E^0 > \frac{1}{2}$ and $F|A = E|A$. Finally, if $\min_{\gamma \in Q} \varphi_E(\gamma) > 1 - \frac{\varepsilon^2}{16}$, then $\varphi_E^0 > \frac{3}{4}$, and we can find such an A with $\mu(A) \geq 4\varphi_E^0 - 3$.*

(iii) *Let the action of Γ be free and let E be induced by a free action of Δ. Then if $\min_{\gamma \in Q} \varphi_E(\gamma) > 1 - \frac{\varepsilon^2}{2}$, it follows that Γ and Δ are ME and so Δ has property* (T).

Proof. (i) follows from 5.3. For (ii) first notice that if $\min_{\gamma \in Q} \varphi_E(\gamma) > 1 - \frac{\varepsilon^2}{2}$, then for the representation τ discussed in Section 2 (A) and letting $\tau(\gamma)(\xi) = \gamma \cdot \xi$, we have $\|\gamma \cdot \xi_0 - \xi_0\|^2 = 2(1 - \langle \gamma \cdot \xi_0, \xi_0 \rangle) = 2(1 - \varphi_E(\gamma)) < \varepsilon^2$ for all $\gamma \in Q$. Hence τ has an invariant non-0 vector, thus, by 2.3, there is an E-invariant set $A \subseteq X$ of positive measure for which $[F|A : E|A] < \infty$. If $\min_{\gamma \in Q} \varphi_E(\gamma) > 1 - \frac{\varepsilon^2}{4}$, then, by 5.3, $\varphi_E^0 > 0$, so by 2.5 we can find such an A with $[F|A : E|A] \leq \frac{1}{\varphi_E^0}$. If $\min_{\gamma \in Q} \varphi_E(\gamma) > 1 - \frac{\varepsilon^2}{8}$, then again by 5.3, $\varphi_E^0 > \frac{1}{2}$, so such an A can be found with $E|A = F|A$. Finally if $\min_{\gamma \in Q} \varphi_E(\gamma) > 1 - \frac{\varepsilon^2}{16}$, then $\varphi_E^0 > \frac{3}{4}$ and such an A can be found with $\mu(A) > 4\varphi_E^0 - 3$, using 2.15. Clearly (iii) follows from the above and 2.8. $\qquad \square$

We next note the following quantitative version of 3.4 for groups with property (T).

Proposition 5.5. *In the notation of Section 3 (A), let Γ have property (T) and let (Q, ε) be a Kazhdan pair for Γ. If $\min_{\gamma \in Q} \varphi_E(\gamma) > 1 - \frac{\varepsilon^2}{2}$, then for any $a \in A(\Delta, Y, \nu)$: if $b = \mathrm{CInd}(a_0, b_0)_\Delta^\Gamma(a)$, then*

$$b \text{ is ergodic} \implies a \text{ is ergodic.}$$

Proof. Assume that a is not ergodic and repeat the proof of 3.4, with $k = n = 0$. Then for $\xi = f^{(0)}$, $\langle \gamma \cdot \xi, \xi \rangle = \varphi_E(\gamma^{-1}) = \varphi_E(\gamma)$, $\forall \gamma \in Q$. So $\|\gamma \cdot \xi - \xi\|^2 = 2(1 - \langle \gamma \cdot \xi, \xi \rangle) < \varepsilon^2$, $\forall \gamma \in Q$, thus there is a non-0 Γ-invariant vector, so b is not ergodic. $\qquad \square$

(C) We next consider some consequences concerning percolation on Cayley graphs of property (T) groups.

Theorem 5.6. *Let Γ be an infinite group with property (T), (Q, ε) a Kazhdan pair, and $\boldsymbol{P}$ an invariant, ergodic, insertion-tolerant bond percolation on $\mathcal{G}_Q$. Then if the survival probability $\boldsymbol{P}(\{\omega : \omega(e) = 1\})$ of each edge e is $> 1 - \frac{\varepsilon^2}{2}$, it follows that ω has a unique infinite cluster, $\boldsymbol{P}$-a.s.*

Proof. In the context of Section 4, and keeping its notation, take the free action of Γ on (Y, ν) to be weakly mixing, so that the Γ-action on $(X, \mu) = (\Omega_Q \times Y, \boldsymbol{P} \times \nu)$ is free and ergodic. If $Q = \{\gamma_1, \dots, \gamma_n\}$, then we have $\boldsymbol{P}(\{\omega : \omega(\{1, \gamma_i\}) = 1\}) \leq \tau(1, \gamma_i) = \varphi_E(\gamma_i)$, so $\min_{\gamma \in Q} \varphi_E(\gamma) > 1 - \frac{\varepsilon^2}{2}$, thus, by 5.4 (ii), there is an E-invariant set $A \subseteq X_\infty$ of positive measure with $[F|A : E|A] < \infty$. Since $\boldsymbol{P}$ is insertion-tolerant, by Section 4 (C), $E|X_\infty$ is ergodic, so $A = X_\infty$, which again by Section 4 (C) implies that $[F|A : E|A] = 1$, i.e., $F|X_\infty = E|X_\infty$. This means that ω has a unique infinite cluster, $\boldsymbol{P}$-a.s. $\qquad \square$

In the case of Bernoulli bond percolation P_p, $p \in (0, 1)$, on $\mathcal{G}_Q$, let $p_u = p_u(Q)$ be the critical probability for uniqueness defined by

$$p_u = \inf\{p : \text{there is a unique infinite cluster, } P_p\text{-a.s.}\}.$$

In Lyons–Schramm [LS] the authors show that for property (T) groups Γ and any finite set of generators Q one has $p_u(Q) < 1$. From the preceding result one has a quantitative version.

Corollary 5.7. *Let Γ be an infinite group with property* (T) *and (Q, ε) a Kazhdan pair. Then $p_u(Q) \leq 1 - \frac{\varepsilon^2}{2}$.*

We also have the following

Corollary 5.8. *For each $\rho > 0$ and every infinite group Γ with property* (T)*, there is a finite set of generators Q for Γ such that for any invariant, ergodic, insertion-tolerant bond percolation P on $\mathcal{G}_Q$ the following holds: If the survival probability of each edge is $\geq \rho$, then ω has a unique infinite cluster, P-a.s.*

Proof. By 5.1, $\sup_Q \varepsilon_Q = \sqrt{2}$, where the sup is taken over all finite generating sets Q for Γ.

So fix $\rho > 0$, Γ an infinite group with property (T) and Q a finite generating set of Γ such that $\varepsilon = \varepsilon_Q > \sqrt{2(1 - \rho)}$. Then for P as in the statement of the present corollary, the survival probability of each edge is bigger than $1 - \frac{\varepsilon^2}{2}$, so ω has a unique infinite cluster, P-a.s., by 5.6. $\square$

There is also a version of 5.8 for arbitrary invariant, ergodic bond percolations.

Corollary 5.9. *Let Γ be an infinite group with property* (T) *and (Q, ε) a Kazhdan pair. Then for any invariant, ergodic bond percolation P on $\mathcal{G}_Q$, if the probability of survival of every edge is $> 1 - \frac{\varepsilon^2}{2}$, there is $n \geq 1$ and a Γ-invariant Borel map $\mathcal{C}_0 \colon \Omega_Q \to [2^\Gamma]^n$ (= the space of n-element subsets of the power set of Γ) such that $\mathcal{C}_0(\omega)$ is a set of n infinite clusters of ω, P-a.s.*

In particular, for every $\rho > 0$ and every infinite group Γ with property (T) *there is a finite set of generators Q for Γ such that for any invariant, ergodic bond percolation P on $\mathcal{G}_Q$: If the survival probability of each edge is $\geq \rho$, then we can assign in a Γ-invariant Borel way a finite set (of fixed size) of infinite clusters to each ω, P-a.s.*

Proof. We follow the proof of 5.6, whose notation and that of Section 4 we use below. Let $Q = \{\gamma_1, \ldots, \gamma_n\}$. Then $\mu(A_i) > 1 - \frac{\varepsilon^2}{2}$, so if E is the equivalence relation induced by $\gamma_1|A_1, \ldots, \gamma_n|A_n$, we have $\varphi_E(\gamma_i) > 1 - \frac{\varepsilon^2}{2}$. So, by 5.4, there is an E-invariant set $A \subseteq X_\infty$ of positive measure such that $[F|A : E|A] = k < \infty$. For each $x = (\omega, y)$, let $f(x) = \{\gamma : \gamma \cdot x \in A\} \in 2^\Gamma$. Clearly for $\delta \in \Gamma$,

$f(\delta \cdot x) = f(x)\delta^{-1} = \delta \cdot f(x)$, where Γ acts on the set of subsets 2^Γ of Γ by right multiplication. Moreover $f(x)$ is the union of k infinite clusters of ω. Thus the map $f_\omega \colon Y \to [\mathcal{C}(\omega)]^k$ (= the set of k-element subsets of $\mathcal{C}(\omega)$) induces a measure $(f_\omega)_* \nu = \nu_\omega$ on $[\mathcal{C}(\omega)]^k$. Here $\mathcal{C}(\omega)$ = the set of infinite clusters of ω given by $f_\omega(y)$ = the set of clusters contained in $f(\omega, y)$. Moreover, $\gamma \cdot \nu_\omega = \nu_{\gamma \cdot \omega}$, where $\gamma \cdot \nu_\omega(B) = \nu_\omega(\gamma^{-1} \cdot B)$ for $B \subseteq [\mathcal{C}(\omega)]^k$. But $(f_\omega)_* \nu$ is a countably additive measure on the countable set $[\mathcal{C}(\omega)]^k$, thus can be viewed as given by a weight function $W(\omega, \bar{C})$, $\bar{C} \in [\mathcal{C}(\omega)]^k$, where $0 \leq W(\omega, \bar{C}) \leq 1$ and $\sum_{\bar{C} \in [\mathcal{C}(\omega)]^k} W(\omega, \bar{C}) = 1$. Moreover, $W(\omega, \bar{C}) = W(\gamma \cdot \omega, \gamma \cdot \bar{C})$. Let $\{\bar{C}_1(\omega), \ldots, \bar{C}_{n(\omega)}(\omega)\}$ be the set of k-element subsets of $C(\omega)$ of maximal weight. Again $\omega \mapsto \{\bar{C}_1(\omega), \ldots, \bar{C}_{n(\omega)}(\omega)\}$ is Γ-invariant. It follows from the ergodicity of P that $n(\omega) = n_0$, P-a.s. Let $\mathcal{C}_0(\omega) = \bar{C}_1(\omega) \cup \cdots \cup \bar{C}_{n_0}(\omega) \in [\mathcal{C}(\omega)]^{<\mathbb{N}}$ = the set of finite subsets of $\mathcal{C}(\omega)$. Again $\mathcal{C}_0(\omega)$ is Γ-invariant, so for some n, $\mathcal{C}_0(\omega) \in [C(\omega)]^n$, P-a.s., and the proof is complete. $\qquad\square$

(D) Next we derive some upper bounds for the cost of a group with property (T). Below $C(\Gamma)$ denotes the cost of a group Γ. If Γ is infinite and has property (T) and Q is a finite set of generators with $\mathrm{card}(Q) = n$, then it is well known that $1 \leq C(\Gamma) < n$ (the strict inequality follows from Gaboriau [G1], since no free measure-preserving action of Γ is treeable; see Adams–Spatzier [AS]). We prove below some improvements on this upper bound. It should be pointed out however that at this time no property (T) groups Γ with $C(\Gamma) > 1$ are known to exist.

The next result is obtained by a combination of Lyons–Peres–Schramm [LPS] and 5.7.

Theorem 5.10. *Let Γ be an infinite group with property* (T). *Let (Q, ε) be a Kazhdan pair for Γ. If $n = \mathrm{card}(Q)$, then*

$$C(\Gamma) \leq n\left(1 - \frac{\varepsilon^2}{2}\right) + \frac{n-1}{2n-1}.$$

Proof. For the Bernoulli bond percolation P_p, $p \in (0, 1)$ on $\mathcal{G}_Q$, let $p_c = p_c(Q)$ be the critical probability for infinite clusters defined by

$$p_c = \sup\{p : \text{all clusters are finite, } P_p\text{-a.s.}\}.$$

If d is the degree of $\mathcal{G}_Q$, so that $d \leq 2n$, it is well known that $p_c \geq \frac{1}{d-1}$; see Lyons–Peres [LP]. Consider the probability space

$$\widetilde{\Omega}_Q = \{\widetilde{\omega} \in [0, 1]^{E_Q} : \text{all edge labels are distinct}\}$$

equipped with the product measure of the Lebesgue measure on $[0, 1]$. Consider the random variable $\Im \colon \widetilde{\Omega}_Q \to 2^{E_Q}$ corresponding to the free minimal spanning forest as defined in Lyons–Peres–Schramm [LPS].

Then we have

$$C(\Gamma) \leq \frac{1}{2} \boldsymbol{E}(\deg_1 \mathfrak{I}) \leq \frac{1}{2}\left(2 + d \int_{p_c}^{p_u} \theta(p)^2 \, dp\right), \tag{3}$$

where $\deg_1 \mathfrak{I}$ is the degree of the identity of Γ in the forest. Here $\theta(p)$ is the probability that the cluster of 1 is infinite in Bernoulli p-percolation. The first inequality, due to Lyons, follows from the following observation. Fix a positive number $\bar{\varepsilon}$ and consider the probability space 2^{E_Q} equipped with the Bernoulli measure $\boldsymbol{P}_{\bar{\varepsilon}}$. Consider also the diagonal action of Γ on $\tilde{\Omega}_Q \times 2^{E_Q}$ with associated product measure μ. Define the graphing $\mathscr{G}$ of the orbit equivalence relation as follows:

$$\{(\tilde{\omega}_1, \omega_1), (\tilde{\omega}_2, \omega_2)\} \in \mathscr{G} \iff \exists \gamma \in Q[\gamma \cdot (\tilde{\omega}_1, \omega_1) = (\tilde{\omega}_2, \omega_2)$$
$$\text{and } (\{1, \gamma\} \in \mathfrak{I}(\tilde{\omega}_1) \text{ or } \{1, \gamma\} \in \omega_1)].$$

That $\mathscr{G}$ spans the equivalence relation follows from [LPS], Theorem 3.22 and for the cost, we have

$$C_\mu(\mathscr{G}) \leq \frac{1}{2}(\boldsymbol{E}(\deg_1 \mathfrak{I}(\tilde{\omega})) + \boldsymbol{E}(\deg_1 \omega)) = \frac{1}{2}(\boldsymbol{E}(\deg_1 \mathfrak{I}) + \bar{\varepsilon}d).$$

Since $\bar{\varepsilon}$ was arbitrary, we obtain the desired inequality. The second inequality in (3) is [LPS], Corollary 3.24.

Thus we have, using 5.7,

$$C(\Gamma) \leq 1 + \frac{d}{2}(p_u - p_c)$$
$$\leq 1 + \frac{d}{2}\left(\left(1 - \frac{\varepsilon^2}{2}\right) - \frac{1}{d-1}\right)$$
$$= \frac{d}{2}\left(1 - \frac{\varepsilon^2}{2}\right) + \frac{d-2}{2(d-1)} \leq n\left(1 - \frac{\varepsilon^2}{2}\right) + \left(\frac{n-1}{2n-1}\right). \qquad \square$$

The first inequality in this calculation comes from Gaboriau [G2] and is contained in the proof of Proposition 8.7 (p. 40) attributed to Lyons.

When Γ is torsion-free we also obtain some additional estimates.

Theorem 5.11. *Let Γ be an infinite group with property* (T) *and* (Q, ε) *a Kazhdan pair for Γ. Let* $\mathrm{card}(Q) = n$. *If Q contains an element of infinite order, then*

$$C(\Gamma) \leq n - (n-1)\frac{\varepsilon^2}{8}.$$

Proof. Consider a free, ergodic action of Γ on (X, μ) with associated equivalence relation $F = E_\Gamma^X$. Let $Q = \{\gamma_1, \ldots, \gamma_n\}$, where γ_1 has infinite order. Fix $\delta < \frac{1}{2}$ and let $A \subseteq X$ have measure $\mu(A) = 1 - \frac{\delta^2 \varepsilon^2}{2}$. Let $E = \gamma_1 \vee \gamma_2|A \vee \cdots \vee \gamma_n|A$ be the

equivalence relation generated by $\gamma_1, \gamma_2|A, \ldots, \gamma_n|A$, so that E is aperiodic. Then $C_\mu(E) \leq 1 + (n-1)(1 - \frac{\delta^2 \varepsilon^2}{2})$. Also $\varphi_E(\gamma) \geq 1 - \frac{\delta^2 \varepsilon^2}{2}$, $\forall \gamma \in Q$, so, by 5.4 (i), $\varphi_E^0 \geq 1 - 2\delta^2 > \frac{1}{2}$. Then, by 2.13, $C_\mu(F) \leq C_\mu(E) \leq 1 + (n-1)(1 - \frac{\delta^2 \varepsilon^2}{2})$. Taking $\delta \to \frac{1}{2}$ we are done. $\qquad\square$

Theorem 5.12. *Let Γ be an infinite group with property* (T) *and let (Q, ε) be a Kazhdan pair for Γ with Q containing an element of infinite order. Let* $\mathrm{card}(Q) = n$. *Then*

$$C(\Gamma) \leq n - \frac{\varepsilon^2}{2}.$$

Proof. Let b_0 be a free, mixing action of Γ on (X, μ) and put $F = E_\Gamma^X$. Let $Q = \{\gamma_1, \ldots, \gamma_n\}$, where γ_1 has infinite order. Consider the graphing of F given by $\gamma_1, \ldots, \gamma_n$. Applying the argument in Kechris–Miller [KM], 28.11, and (independently) Pichot [P], we obtain a treeing of a subequivalence relation $E \subseteq F$, generated by $\gamma_1, \gamma_2|A_2, \ldots, \gamma_n|A_n$, for some Borel sets $A_2, \ldots, A_n$, such that $C_\mu(E) \geq C_\mu(F)$. Thus $C_\mu(F) \leq 1 + \sum_{i=2}^n \mu(A_i)$. Now E is treeable and ergodic, so by Hjorth [H4], E is induced by a free action a_0 of a group Δ. If we had $\mu(A_i) > 1 - \frac{\varepsilon^2}{2}$ for all $i = 2, \ldots, n$, then $\min_{\gamma \in Q} \varphi_E(\gamma) > 1 - \frac{\varepsilon^2}{2}$, so, by 5.4 (iii), Δ has property (T) a contradiction (since by Adams–Spatzier [AS] no free, measure-preserving action of an infinite group with property (T) can be treeable). So $\mu(A_i) \leq 1 - \frac{\varepsilon^2}{2}$ for some $i = 2, \ldots, n$, therefore $C_\mu(E) \leq 1 + (n-2) + (1 - \frac{\varepsilon^2}{2}) = n - \frac{\varepsilon^2}{2}$. $\qquad\square$

(E) Finally, we note that there is an analog of 5.4 (iii), when Γ does not have the HAP.

Proposition 5.13. *Let Γ be an infinite group without the HAP. Then there exists $\varepsilon > 0$ and a finite set $Q \subseteq \Gamma$ with the following property:*

Let Δ be a group and consider two free, measure-preserving actions of Γ and Δ on (X, μ) such that $E = E_\Delta^X \subseteq F = E_\Gamma^X$. If $\min_{\gamma \in Q} \varphi_E(\gamma) > 1 - \varepsilon$, then Δ does not have the HAP.

Proof. Since Γ does not have the HAP, we can find $\varepsilon > 0$ and $Q \subseteq \Gamma$ finite such that if $\varphi \colon \Gamma \to \mathbb{C}$ is a positive-definite function with $\varphi(1) = 1$ and $\varphi \in c_0(\Gamma)$, then $\min_{\gamma \in Q} \varphi(\gamma) \leq 1 - \varepsilon$.

Now let Δ be as above and assume that it has the HAP. Let $\psi_n \colon \Delta \to \mathbb{C}$ be positive-definite functions such that $\lim_{n \to \infty} \psi_n(\delta) = 1$ for all $\delta \in \Delta$, and $\psi_n \in c_0(\Delta)$ for all n. If $A_{\gamma, \delta} = \{x \in X : \gamma \cdot x = \delta \cdot x\}$, then the formula $\varphi_n(\gamma) = \sum_{\delta \in \Delta} \psi_n(\delta) \mu(A_{\gamma, \delta})$ defines a sequence of positive-definite functions on Γ.

Next we have that

$$\lim_{n \to \infty} \varphi_n(\gamma) = \sum_{\delta \in \Delta} \mu(A_{\gamma, \delta}) = \mu(\{x \in X : \gamma \cdot x \in \Delta \cdot x\}) = \varphi_E(\gamma), \ \forall \gamma \in \Gamma.$$

Thus, to get a contradiction to the non-HAP assumption, it suffices to show that $\varphi_n \in c_0(\Gamma)$ for all n. This is clear, since for a fixed n we have that $\lim_{\delta \to \infty} \psi_n(\delta) = 0$, $\lim_{\gamma \to \infty} \mu(A_{\gamma,\delta}) = 0$ for all $\delta \in \Delta$, and $\sum_\delta \mu(A_{\gamma,\delta}) \leq 1$ for all $\gamma \in \Gamma$. $\qquad\square$

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