Scattering theory of the Hodge Laplacian under a conformal perturbation

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- Let (M, g) be an arbitrary smooth Riemannian manifold of dimension m.
- An important problem in the setting of geometric analysis is the determination of the spectrum of the Hodge-Laplace operator $\Delta_g^{(j)}$ acting on differential j-forms.
- If M is compact, then the spectrum $\sigma(\Delta_g^{(j)})$ of $\Delta_g^{(j)}$ is given by a sequence of eigenvalues with finite multiplicity, $\{\lambda_k\}$, such that

$$\lambda_k \sim 4\pi \left(\frac{\mathrm{vol}(M)}{\Gamma(n/2)+1} \right)^{\frac{2}{m}} k^{\frac{2}{m}}$$

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- If M is noncompact, then $\sigma(\Delta_g^{(j)})$ will typically contain some continuous part which is impossible to control in general.

In this case at least two decompositions occur for $\sigma(\Delta_g^{(j)})$:

$$\sigma(\Delta_g^{(j)}) = \sigma_{disc}(\Delta_g^{(j)}) \cup \sigma_{ess}(\Delta_g^{(j)})$$

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$$\sigma(\Delta_g^{(j)}) = \sigma_{pp}(\Delta_g^{(j)}) \cup \sigma_{ac}(\Delta_g^{(j)}) \cup \sigma_{sing}(\Delta_g^{(j)})$$

where

- $\sigma_{disc}(\Delta_g^{(j)})$ is the discrete spectrum,
- $\sigma_{ess}(\Delta_g^{(j)})$ is essential spectrum,
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In the non compact setting an interesting problem is to investigate the stability of the continuous part the spectrum.

In particular here we are interested in a "perturbative" way to control the stability of the absolutely continuous part $\sigma_{\mathrm{ac}}(\Delta_g^{(j)})$ of $\sigma(\Delta_g^{(j)})$ in the based on the wave operators.

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Assume that there is a quasi-isometric metric $\tilde{g}$ on M such that we have some good information about the absolutely continuous part $(\Delta^{(j)}_{\tilde{g}})_{\mathrm{ac}}$ of $\Delta^{(j)}_{\tilde{g}}$. The wave operators are defined as

$$W_{\pm}(\Delta^{(j)}_{g}, \Delta^{(j)}_{\tilde{g}}, I) := s - \lim_{t \to \pm\infty} e^{it\Delta^{(j)}_{g}} I e^{it\Delta^{(j)}_{\tilde{g}}} P_{ac}(\Delta^{(j)}_{\tilde{g}})$$

Then once we can show that the wave operators $W_{\pm}(\Delta^{(j)}_{g}, \Delta^{(j)}_{\tilde{g}}, I)$ exist and are complete, they induce unitary equivalences

$$(\Delta^{(j)}_{\tilde{g}})_{\mathrm{ac}} \sim (\Delta^{(j)}_{g})_{\mathrm{ac}}, \text{ so that } \sigma_{\mathrm{ac}}(\Delta^{(j)}_{\tilde{g}}) = \sigma_{\mathrm{ac}}(\Delta^{(j)}_{g}).$$

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- $\tilde{g}$ and g are quasi-isometric. This means that $c_1 g \leq \tilde{g} \leq c_2 g$ for some positive constants c_1 and c_2.
- Existence of the wave operators: the strong limit

$$s - \lim_{t \to \pm\infty} e^{it\Delta_g^{(j)}} I e^{it\Delta_{\tilde{g}}^{(j)}} P_{ac}(\Delta_{\tilde{g}}^{(j)})$$

exists.
- Completeness of the wave operators:

$$\ker(W_\pm(\Delta_g^{(j)}, \Delta_{\tilde{g}}^{(j)}, I))^\perp = \mathrm{im}(P_{ac}(\Delta_{\tilde{g}}^{(j)})) \text{ in } \Omega_{L^2}^{(j)}(M, \tilde{g})$$

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The question we address here is:

In what sense do $\tilde{g}$ and g have to be close to each other to ensure that $W_{\pm}\big(\Delta_g, \Delta_{\tilde{g}}, I\big)$ exist and are complete?

From calculating $\Delta_{\tilde{g}} - \Delta_g$ in the (particulary important) case where one metric arises as a conformal perturbation of the other, we expect the correct assumption to be a *first order* control in the deviation of g and $\tilde{g}$.

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- One of the first result for functions without assuming special structures for the metrics has been established by W. Müller/G. Salomonsen. We can sketch their result as follows:

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- One of the first result for functions without assuming special structures for the metrics has been established by W. Müller/G. Salomonsen. We can sketch their result as follows:

Theorem (Müller/Salomonsen 2007, JFA 253)

Let g, $\tilde{g}$ be complete metrics on M with $|\mathrm{sec}_g|, |\mathrm{sec}_{\tilde{g}}| \leq L$, such that the covariant C^2-deviation $x \mapsto {}^2|g - \tilde{g}|_g(x)$ of g from $\tilde{g}$ is bounded from above by some $\beta : M \to (0, \infty)$ of moderate decay, in a way such that for appropriate constants a, b, c, C one has

$$\beta^a \in \mathrm{L}^1(M, g), \ \left|\beta^b(x)\widetilde{\mathrm{inj}}_g(x)^c\right| \leq C \ \text{ for all } x,$$

where $\widetilde{\mathrm{inj}}_g(x) := \min\left(\frac{\pi}{12\sqrt{L}}, \mathrm{inj}_g(x)\right)$. Then $W_\pm\left(\Delta_g^{(0)}, \Delta_{\tilde{g}}^{(0)}, I^{(0)}\right)$ exist and are complete.

Note that the required control is of second order in the deviation of g and $\tilde{g}$.

Indeed, using the harmonic radius function $x \mapsto r_g(x)$ with a certain Sobolev control, one can prove a stronger result:

Theorem (Hempel/Weder/Post 2013, JFA 266)

Let g, $\tilde{g}$ be complete quasi-isometric metrics on M with

$$\int_M \mathrm{d}(g,\tilde{g})(x)h^{-(m+2)}(x)\mu_g(\mathrm{d}x) < \infty, \tag{1}$$

where $\mathrm{d}(g,\tilde{g}) : M \to (0,\infty)$ is a certain function (later...) which measures a zero order deviation of the metrics, and where $h : M \to (0,1]$ is an arbitrary lower bound on

$$M \ni x \longmapsto \max\big(\min(r_g(x),1), \min(r_{\tilde{g}}(x),1)\big) \in (0,1].$$

Then $W_\pm(\Delta_g^{(0)}, \Delta_{\tilde{g}}^{(0)}, I^{(0)})$ exist and are complete.

This result should be the state of the art for *functions*.

We were interested in extending the latter result to differential forms.

For some entirely algebraic reasons we have restricted ourselves to conformal perturbations.

Given a Riemannian metric g we have

- ∇_g: the Levi-Civita connection
- $\mathrm{Q}_g : \wedge^2 \mathrm{T}M \to \wedge^2 \mathrm{T}M$ the s.a. curvature endomorphism
- for a smooth 1-form α, $\mathrm{int}_g(\alpha) = \mathrm{ext}^{\dagger_g} : \wedge \mathrm{T}^* M \to \wedge \mathrm{T}^* M$ is interior multiplication with α
- the codifferential $\delta_g := \mathrm{d}^{\dagger_g} : \Omega_{\mathsf{C}^\infty}(M) \to \Omega_{\mathsf{C}^\infty}(M)$
- the Dirac type operator $D_g := \mathrm{d} + \delta_g : \Omega_{\mathsf{C}^\infty}(M) \to \Omega_{\mathsf{C}^\infty}(M)$
- the Hodge-Laplacian $\Delta_g := D_g^2 : \Omega_{\mathsf{C}^\infty}(M) \to \Omega_{\mathsf{C}^\infty}(M)$
- the Friedrichs realization H_g of Δ_g in $\Omega_{\mathsf{L}^2}(M, g)$
- the resolvents $R_{g,\lambda} := (H_g + \lambda)^{-1}$, $\lambda > 0$.
- everything filters through the form degree; notation: $\Omega_{\mathsf{L}^2}(M) = \bigoplus_{j=0}^m \Omega_{\mathsf{L}^2}^{(j)}(M, g)$, $H_g = \bigoplus_{j=0}^m H_g^{(j)}$ etc.

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One of our main technical tool, as in the work of H-P-W, will be harmonic coordinates with Sobolev control:

Definition (Cheeger/Anderson)

Let $p \in (m, \infty)$, $q \in (1, \infty)$, $x \in M$. Then the $W^{1,p}_g$*-harmonic radius at* x *with Euclidean distortion* q, $r_g(x, p, q) \in (0, \infty]$, is defined to be supremum of all $r > 0$ such that there is a $\Delta^{(0)}_g$-harmonic chart $\Phi : \mathrm{B}_g(x, r) \to U \subset \mathbb{R}^m$ such that, with respect to the Φ-coordinates, the following estimates hold:

$$q^{-1}(\delta_{ij}) \leq (g_{ij}) \leq q(\delta_{ij}) \text{ as symmetric bilinear forms,} \tag{2a}$$

$$r^{1-\frac{m}{p}}\Big(\int_U |\partial_k g_{ij}(y)|^p \mathrm{d}y\Big)^{1/p} \leq q - 1 \text{ for all } i, j, k \in \{1, \ldots, m\}. \tag{2b}$$

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One has the following elementary fact:

Proposition (B/G/M)

For any fixed p, q, the function $x \mapsto \min(1, r_g(x,p,q))$ is 1-Lipschitz w.r.t. g.

Moreover the fact that $r_g(x,p,q) > 0$ can be seen as a consequence of the following results near x:

Proposition (Cheeger/Anderson)

Assume that $\mathrm{Ric}_g(x) \geq -\frac{1}{\beta^2}$ *and* $\mathrm{inj}_g(x) \geq \tilde{h}(x)$, *where* $\beta > 0$ *is a constant and* $\tilde{h} : M \to (0, \infty)$ *is a continuous. Then:*

a) *If* $\tilde{h}$ *is* g*-Lipschitz, then for any* p, q *there is* $C = C(m, p, q) > 0$ *such that for all* $x \in M$ *one has*

$$\min(r_g(x, p, q), 1) \geq C \min\left(1, \ \frac{\tilde{h}(x)}{1 + \|\mathrm{d}\tilde{h}\|_{\infty,g}}, \ \beta\right).$$

b) *If there is a* $x_0 \in M$, *and* $c_1 > 0$, $c_2 \geq 0$ *such that* $\tilde{h} \geq c_1 \mathrm{e}^{-c_2 \mathrm{d}_g(\cdot, x_0)}$, *then for any* p, q *there is* $C = C(m, p, q) > 0$ *such that for all* $x \in M$ *one has*

$$\min(r_g(x, p, q), 1) \geq C \min\left(1, \ \frac{c_1}{\mathrm{e}^{c_2}} \mathrm{e}^{-c_2 \mathrm{d}_g(x, x_0)}, \ \beta\right).$$

Setting of the theorem

The importance of Sobolev harmonic coordinates: By embedding theorems, we get a Hölder control on g_{ij}.

To make an effective use of this observation in the form-case, we add:

Definition

For any $K > 0$ and any function $h : M \to (0, 1]$, let

$$\mathscr{M}_{K,h}(M) := \Big\{ \tilde{g} \;\Big|\; \tilde{g} \textit{ is a complete metric on } M \textit{ with } \mathrm{Q}_{\tilde{g}} \geq -K$$

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- Given a smooth function $\psi : M \to \mathbb{R}$ let g_ψ denote the conformally equivalent metric $g_\psi := \mathrm{e}^{2\psi} g$. Then g and g_ψ are quasi-isometric if and only if ψ is bounded and then we have the canonical identification operator $I = I_{g,g_\psi} : \Omega_{\mathrm{L}^2}(M,g) \to \Omega_{\mathrm{L}^2}(M,g_\psi)$.
- Given a Borel function $h : M \to (0,\infty)$ and a smooth function $\psi : M \to \mathbb{R}$ define

$$\mathrm{d}(g,\psi)(x) := \max\{\sinh(2|\psi(x)|), |\mathrm{d}\psi(x)|_g\}, \quad x \in M,$$

$$\mathrm{d}_h(g,\psi) := \int_M \mathrm{d}(g,\psi)(x) h(x)^{-(m+2)}\, \mu_g(\mathrm{d}x) \quad \in [0,\infty].$$

We call ψ a h-scattering perturbation of g, if one has $\mathrm{d}_h(g,\psi) < \infty$.

- Given a smooth function $\psi : M \to \mathbb{R}$ let g_ψ denote the conformally equivalent metric $g_\psi := \mathrm{e}^{2\psi} g$. Then g and g_ψ are quasi-isometric if and only if ψ is bounded and then we have the canonical identification operator $I = I_{g,g_\psi} : \Omega_{\mathrm{L}^2}(M,g) \to \Omega_{\mathrm{L}^2}(M,g_\psi)$.
- Given a Borel function $h : M \to (0,\infty)$ and a smooth function $\psi : M \to \mathbb{R}$ define

$$\mathrm{d}(g,\psi)(x) := \max\{\sinh(2|\psi(x)|), |\mathrm{d}\psi(x)|_g\}, \quad x \in M,$$

$$\mathrm{d}_h(g,\psi) := \int_M \mathrm{d}(g,\psi)(x) h(x)^{-(m+2)}\, \mu_g(\mathrm{d}x) \quad \in [0,\infty].$$

We call ψ a h-scattering perturbation of g, if one has $\mathrm{d}_h(g,\psi) < \infty$.

Now we can formulate our main result for forms:

Theorem (B/G/M)

Let $\psi : M \to \mathbb{R}$ be smooth with ψ, $|\mathrm{d}\psi|_g$ bounded, and assume that $g, g_\psi \in \mathscr{M}_{K,h}(M)$ for some pair (K, h), in a way such that ψ is a h-scattering perturbation of g. Then the wave operators

$$W_\pm(H_{g_\psi}, H_g, I) = \underset{t\to\pm\infty}{s\text{–}\lim}\, \mathrm{e}^{\mathrm{i}tH_{g_\psi}} I \mathrm{e}^{-\mathrm{i}tH_g} P_{\mathrm{ac}}(H_g)$$

exist and are complete, and everything filters (a posteriori...$\rightsquigarrow$ total forms and Dirac type operators!) through the form degree.

Some steps in the proof of our main result...

- The strategy is to show that the assumptions of the Belopol'skii-Birman's theorem are satisfied.
- Estimates for the integral kernel of the resolvent.
- A decomposition formula (the algebra of which forced us to restrict ourselves to the conformal case) efficiently with harmonic coordinates

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Theorem (Belopol'skii-Birman)

For $k = 1, 2$, let H_k be a self-adjoint operator in a Hilbert space $\mathscr{H}_k$, where E_{H_k} denotes the operator valued spectral measure, $\mathcal{Q}_k$ the sesqui-linear form, and $P_{\mathrm{ac}}(H_k)$ the projection onto the absolutely continuous subspace of $\mathscr{H}_k$ corresponding to H_k. Assume that $I : \mathscr{H}_1 \to \mathscr{H}_2$ is bounded operator which satisfies

- *I has a two-sided bounded inverse*
- *For any bounded interval $S \subset \mathbb{R}$ one has*

$$E_{H_2}(S)(H_2 I - I H_1)E_{H_1}(S) \in \mathscr{J}^1(\mathscr{H}_1, \mathscr{H}_2),$$
$$(I^* I - 1)E_{H_1}(S) \in \mathscr{J}^\infty(\mathscr{H}_1)$$

- *either $I(dom(Q_1)) = dom(Q_2)$, or $I(dom(H_1)) = dom(H_2)$.*

Then the wave operators

Theorem (Belopol'skii-Birman)

$$W_\pm(H_2, H_1, I) = \underset{t\to\pm\infty}{s\text{–}\lim}\, \mathrm{e}^{\mathrm{i}tH_2} I \mathrm{e}^{-\mathrm{i}tH_1} P_{\mathrm{ac}}(H_1)$$

exist and are complete, where completeness means that

$$(\mathrm{Ker}\, W_\pm(H_2, H_1, I))^\perp = \mathrm{Im}\, P_{\mathrm{ac}}(H_1),$$

$$\overline{\mathrm{Im}\, W_\pm(H_2, H_1, I)} = \mathrm{Im}\, P_{\mathrm{ac}}(H_2).$$

Moreover, $W_\pm\big(H_2, H_1, I\big)$ *are partial isometries with inital space* $\mathrm{Im}\, P_{\mathrm{ac}}(H_1)$ *and final space* $\mathrm{Im}\, P_{\mathrm{ac}}(H_2)$.

Estimates for the resolvent

Theorem (B/G/M)

Assume that $g \in \mathscr{M}_{K,h}(M)$ for some pair (K, h). Then for all $n \in \mathbb{N}$ with $n \geq m/4 + 2$ there is a $C = C(m, n) > 0$, such that for all $\lambda > K \max_{j=0,\dots,m} j(m-j) + 1$, the operator $R^n_{g,\lambda}$ is an integral operator, with a Borel integral kernel

$$M \times M \ni (x, y) \longmapsto R^n_{g,\lambda}(x, y) \in \mathrm{Hom}\left(\wedge \mathrm{T}^*_y M, \wedge \mathrm{T}^*_x M\right)$$

which satisfies the estimate

$$\int_M \left| R^n_{g,\lambda}(x, y) \right|^2_{\mathscr{J}^2} \mu_g(\mathrm{d}y) \leq C h(x)^{-m} \text{ for all } x \in M.$$

The proof is quite involved. The key steps are:

- $V_g^{(j)} := \Delta_g^{(j)} - \nabla_{g,j}^{\dagger}\nabla_{g,j}$ is zeroth order and s.a. by Weitzenböck's formula
- The Gallot-Meyer estimate states that under $\mathrm{Q}_g \geq -K$ one has $V_g^{(j)} \geq -K \cdot j(m-j)$
- Now one can use the Kato-Simon inequality for covariant Schrödinger semigroups $\mathrm{e}^{-t(\nabla^{\dagger}\nabla+V)}$ to control $R_{g,\lambda}^{(j),n}$ by $R_{g,1}^{(0),n}$. The latter can be controlled by $\min(1, r_g(\cdot, p, q))$ and finally by $Ch(x)^{-m}$.

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Decomposition formula

Proposition

Assume that ψ and $|\mathrm{d}\psi|_g$ are bounded, let $\lambda > 0$, $n \geq 1$ and let g (and thus g_ψ) be complete. Then the bounded operator

$$R^n_{g_\psi,\lambda}(H_{g_\psi} I - I H_g) R^n_{g,\lambda} : \Omega_{\mathsf{L}^2}(M, g) \longrightarrow \Omega_{\mathsf{L}^2}(M, g_\psi)$$

can be decomposed as

$$\begin{aligned} &R^n_{g_\psi,\lambda}(H_{g_\psi} I - I H_g) R^n_{g,\lambda} = \\ &R^n_{g_\psi,\lambda}\Big(D_{g_\psi} \cdot 2\sinh(2\psi) I D_g + D_{g_\psi} I (1 - \mathrm{e}^{-2\psi})\mathrm{d} - \mathrm{d} \circ (1 - \mathrm{e}^{2\psi}) I D_g \\ &\qquad + D_{g_\psi} \operatorname{int}_{g_\psi}(\mathrm{d}\psi)\, \tau\, I - \tau \operatorname{int}_g(\mathrm{d}\psi) D_g \Big) R^n_{g,\lambda}. \qquad (3) \end{aligned}$$

Here $\tau := \bigoplus_{i=0}^m (m-2i) 1_{\wedge^i \mathrm{T}^* M} : \wedge \mathrm{T}^* M \longrightarrow \wedge \mathrm{T}^* M.$

- Now we combine the latter decomposition formula with our resolvent estimate and the commutator relations $[A, R^n_{g,\lambda}] = 0$, where $A \in \{D_g, \mathrm{d}, \delta_g\}$, to get that for large n

$$R^n_{g_\psi,\lambda}(H_{g_\psi} I - I H_g) R^n_{g,\lambda} \text{ is trace class}$$

and that

$$(I^* I - 1) R^n_{g,\lambda} \text{ is compact}$$

- The assumptions of Belopol'skii-Birman's theorem are satisfied: this follows by the fact that for all bounded intervals $S \subset \mathbb{R}$, $\ell \in \mathbb{R}$, $r > 0$, one has

$$E_{H_g}(S)(H_g + r)^\ell = (H_g + r)^\ell E_{H_g}(S) \in \mathscr{L}(\Omega_{\mathrm{L}^2}(M, g))$$

and analogously

$$E_{H_{g_\psi}}(S)(H_{g_\psi} + r)^\ell = (H_{g_\psi} + r)^\ell E_{H_{g_\psi}}(S) \in \mathscr{L}(\Omega_{\mathrm{L}^2}(M, g_\psi))$$

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- The decomposition formula heavily requires that the operators are of the form L^*L. That is why we work with *total forms and the Dirac type operator* D_g and $H_g = D_g^2 = D_g^* D_g$ instead of on a fixed form degree. On functions, all of this is very simple as $\Delta^{(0)} = \mathrm{d}^{\dagger_g}\mathrm{d}$ where the differential d does not depend on the metric (and this leads to a zeroth order condition in this case).

Some applications

Corollary

Assume that g is complete with $\mathrm{Q}_g \geq -K$ for some $K > 0$ and that $\tilde{g}$ is a metric on M which is conformally equivalent to g and which coincides with g at infinity. Then the assumptions of our main result are satisfied.

Indeed, since ψ is compactly supported by assumption, we can take

$$h(x) := \min(1, r_g(x, p, q), r_{g_\psi}(x, p, q)) \text{ for all } p > m,\ 1 < q < \sqrt{2},$$

which is a positive continuous function, to make ψ a h-scattering perturbation of g.

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which is a positive continuous function, to make ψ a h-scattering perturbation of g.

Corollary

Assume that $\psi : M \to \mathbb{R}$ is smooth and bounded, g is complete such that $|\mathrm{sec}_g|, |\mathrm{sec}_{g_\psi}| \leq L$ for some $L > 0$, that there is some $\beta : [0,\infty) \to (0,\infty)$ exponentially bounded from below, and a point $x_0 \in M$ such that with $\beta(x) := \beta(1 + \mathrm{d}_g(x,x_0))$ one has:

(i) *There are constants $b \in (0,1)$ with $\beta^b \in \mathsf{L}^1(M,g)$, and $C_1 > 0$ such that for all $x \in M$,*

$$\widetilde{\mathrm{inj}}_g(x) := \min\left(\frac{\pi}{12\sqrt{L}}, \, \mathrm{inj}_g(x)\right) \geq C_1 \cdot \beta(x)^{\frac{1-b}{m+2}}. \qquad (4)$$

(ii) *For some constant $C > 0$ one has*

$$^1|g - g_\psi| := |g - g_\psi|_g + |\nabla_g - \nabla_{g_\psi}|_g \leq C \cdot \beta \qquad (5)$$

Then the assumptions of our main result are satisfied.

Corollary

Let g be such that $|\mathrm{Sec}_g|$ is bounded and that g has a positive injecitivity radius (in particular, g is complete). Assume that $\psi : M \to \mathbb{R}$ is smooth with $\max\{\psi, |\mathrm{d}\psi|_g, |\mathrm{Hess}_g(\psi)|_g\}$ *bounded, and*

$$\int_M \max\{\sinh(2|\psi(x)|), |\mathrm{d}\psi(x)|_g\}\mu_g(\mathrm{d}x) < \infty.$$

Then the wave operators $W_\pm(H_{g_\psi}, H_g, I)$ exist and are complete.

Let M be a smooth connected manifold (without boundary) $\dim(M) = n+1$, let $U \subset M$ be a smooth compact submanifold with boundary and $\dim(N) = n$. Let us label by $N := \partial U$ the boundary of U and by U' the interior of U. Assume that there exists a smooth diffeomorphism:

$$F : M \setminus U' \to [1,\infty) \times N.$$

Finally consider a smooth metric g on M such that

$$(F^{-1})^*(g|_{M\setminus U'}) = h^2 \mathrm{d}r^2 + f^2 g_N$$

where $f : [1,\infty) \to [0,\infty)$ and $h : [1,\infty) \to [0,\infty)$ are smooth and g_N is a smooth metric on N.

Corollary

Assume in the above warped product situation that $|\mathrm{Sec}_g|$ is bounded, that g has a positive injectivity radius and that $\beta : [1,\infty) \to (0,\infty)$ is a bounded Borel function with $\beta \in \mathsf{L}^1([1,\infty), h(r)f^n(r)\mathrm{d}r)$. Then for any bounded smooth function $\psi : M \to \mathbb{R}$ with bounded g-Hessian and

$$\max\{\sinh(|2\psi|), |\mathrm{d}\psi|_g\}|_{F^{-1}(r,q)} \leq \beta(r) \;\; \textit{for all } (r,q) \in [1,\infty) \times N,$$

the wave operators $W_\pm(H_{g_\psi}, H_g, I)$ exist and are complete.

Proposition

Let M, N, U, U' and $F : M \setminus U' \to [1, \infty) \times N$ be as before. Let g be a warped product metric such that for some b with $0 \leq b \leq 1$ one has

$$(F^{-1})^*(g|_{M \setminus U'}) = \mathrm{d}r^2 + r^{2b} g_N$$

Let $\beta : [1, \infty) \to (0, \infty)$ is a bounded Borel function with $\beta \in \mathsf{L}^1([1, \infty), r^{bn}(r)\mathrm{d}r)$. Then for any bounded smooth function $\psi : M \to \mathbb{R}$ with bounded g-Hessian and

$$\max\{\sinh(|2\psi|), |\mathrm{d}\psi|_g\}|_{F^{-1}(r,q)} \leq \beta(r)$$

for all $(r, q) \in [1, \infty) \times N$ one has

Proposition

a) If $b = 0$ then for every $j = 0, ..., n+1$ we have

$$\sigma_{\mathrm{ac}}(H^{(j)}_{\tilde{g}}) \subset \bigcup_{k \in \mathbb{N}} \left([\lambda^{(j)}_k, \infty) \cup [\lambda^{(j-1)}_k, \infty) \right) \tag{6}$$

with $(\lambda^{(j)}_k)_{k \in \mathbb{N}}$ (resp. $(\lambda^{(j-1)}_k)_{k \in \mathbb{N}}$) the eigenvalues of the Hodge-Laplacian $H^{(j)}_{g_N}$ (resp. $H^{(j-1)}_{g_N}$) acting on N.

b) Assume now that U' is diffeomorphic to the open Euclidean ball $B(0,1) \subset \mathbb{R}^{n+1}$.

If $b = 0$ then for every $j = 0, ..., n+1$ we have

$$\sigma_{ac}(H^{(j)}_{\tilde{g}}) = [\overline{\lambda^{(j)}}, \infty), \tag{7}$$

Proposition

where $\overline{\lambda^{(j)}} := \min\{\lambda_0^{(j)}, \lambda_0^{(j-1)}\}$ *is the minimum of the lowest eigenvalue* $\lambda_0^{(j)}$ *of* $H_{g_{\mathbb{S}^n}}^{(j)}$ *and the lowest eigenvalue* $\lambda_0^{(j-1)}$ *of* $H_{g_{\mathbb{S}^n}}^{(j-1)}$*, with* $g_{\mathbb{S}^n}$ *the standard metric on the unit sphere* $\mathbb{S}^n$*.*
Finally, if $0 < b \leq 1$ *then for every* $j = 0, ..., n+1$ *we have*

$$\sigma_{\mathrm{ess}}(H_{\tilde{g}}^{(j)}) = \sigma_{\mathrm{ac}}(H_{\tilde{g}}^{(j)}) = [0, \infty). \tag{8}$$

Some final remarks

- We believe that the curvature assumptions $\mathrm{Q}_g \geq -K$ are not necessary.
- What can we do in the non-conformal perturbations? A problem is given by the fact that it is not easy to calculate (or even to estimate) $\delta_{\tilde{g}}$ in term of δ_g. In the conformal case, there are somewhat accessible perturbative formulae.

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Thank you for listening!