Perturbative methods in Algebraic QFT with applications to Thermal Field Theory

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07.04.2017

- The AQFT approach is based on the identification of a $*$-algebra $\mathcal{A}$ of physical observables. [Haag & Kastler '64]
- For free theories this construction is well under control. [Brunetti, Duetsch & Fredenhagen '09, F. & Rejzner '12-'14]
- Interacting theories are treated perturbatively and $\mathcal{A}$ is identified up to renormalization freedom. Physical requirements give restrictions on the possible choices. [Brunetti & Fredenhagen '00, Hollands & Wald '01-'02-'05, B., Duetsch & F. '09]
- The Principle of the Perturbative Agreement (PPA) [Hollands & Wald '05, Zahn '13] provides an example of such a requirement:

$$\Box_g \varphi + m^2 \varphi = \Box_g \varphi + m^2 \varphi \,.$$

- The generalized PPA [D., T.-P. Hack, N. Pinamonti, '16] provides a generalization to PPA in case of higher order polynomial interactions:

$$\Box_g \varphi + m^2 \varphi + \lambda \varphi^3 = \Box_g \varphi + m^2 \varphi + \lambda \varphi^3 \,.$$

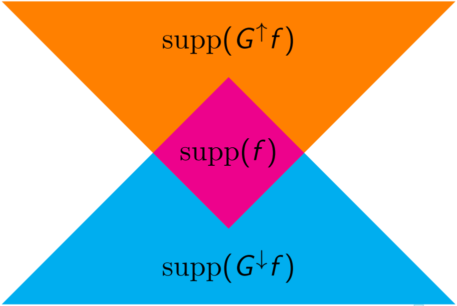
Globally hyperbolic spacetimes

- M globally hyperbolic: $\mathsf{M} \simeq \mathbb{R} \times \Sigma \qquad g = -\beta \mathrm{d}t^2 + h_t\,.$
- $P = \Box_g + m^2$ is normally hyperbolic.
- $\exists! \mathsf{G}^{\uparrow/\downarrow}$ retarded/advanced propagators that is

$$G^{\uparrow/\downarrow} \colon C^\infty_{\mathrm{c}}(\mathsf{M}) \to C^\infty(\mathsf{M})$$

$$PG^{\uparrow/\downarrow} = G^{\uparrow/\downarrow}P = \mathrm{id}_{C^\infty_{\mathrm{c}}(\mathsf{M})}\,, \qquad \mathrm{supp}(G^{\uparrow/\downarrow}f) \subseteq J^{\uparrow/\downarrow}\mathrm{supp}(f)\,.$$

- $\mathsf{G} = G^{\uparrow} - \mathsf{G}^{\downarrow}$ is the causal propagator.



Local Functionals

Generators of $\mathcal{A}$ are $F \in \mathcal{F}_{\mathrm{loc}}$ i.e. $F\colon \mathcal{E} := C^\infty(\mathrm{M}) \to \mathbb{C}$ such that:

(i) Smooth:

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} F(\varphi + \lambda\psi)\Big|_{\lambda=0} = F^{(n)}[\varphi](\psi^{\odot n});$$

(ii) Local:

$$\begin{aligned} \mathrm{supp}(F) &= \overline{\bigcup_{\varphi\in\mathcal{E}} \mathrm{supp}(F^{(1)}[\varphi])} \quad \text{is compact} \\ \mathrm{supp}(F^{(n)}[\varphi]) &\subseteq \{(x,\dots,x) \in \mathrm{M}^n\}. \end{aligned}$$

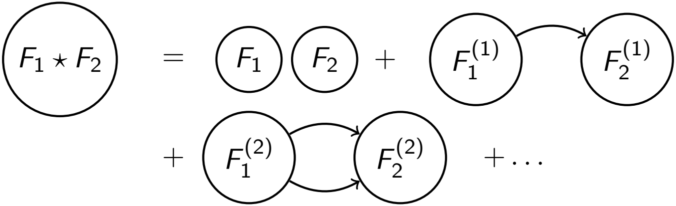
Example

$$F(\varphi) := \int f\varphi^k = \int_{\mathrm{M}^k} f(x_1)\delta(x_1,\dots,x_k)\varphi(x_1)\dots\varphi(x_k).$$

$\star$-Product

$$
\begin{aligned}
(F_1 \star F_2)(\varphi) &:= \color{magenta}{F_1(\varphi)F_2(\varphi)} + \hbar \mathrm{G}^+ \left(F_1^{(1)}[\varphi], F_2^{(1)}[\varphi] \right) + \dots \\
&= \sum_{n \geq 0} \frac{\hbar^n}{n!} (\mathrm{G}^+)^{\otimes n} \left(F_1^{(n)}[\varphi], F_2^{(n)}[\varphi] \right) .
\end{aligned}
$$

where G^+ is an Hadamard distribution.



$\surd$ $\quad [F_f, F_h]_\star = i\hbar \mathrm{G}(f,h) \quad$ for $F_f(\varphi) = \int f\varphi$

$\star$-Product

$$(F_1 \star F_2)(\varphi) := F_1(\varphi)F_2(\varphi) + \hbar \mathrm{G}^+ \left(F_1^{(1)}[\varphi], F_2^{(1)}[\varphi] \right) + \dots$$

$$= \sum_{n \geq 0} \frac{\hbar^n}{n!} (\mathrm{G}^+)^{\otimes n} \left(F_1^{(n)}[\varphi], F_2^{(n)}[\varphi] \right) .$$

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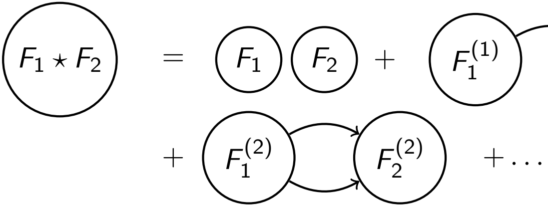


$\mathcal{A} = \mathrm{Alg}(\mathcal{F}_{\mathrm{loc}}, \star, *)$ is the free $*$-algebra associated to the free φ-theory.

- $\mathsf{G}^+ \in C_c^\infty(\mathsf{M}^2)'$ is such that

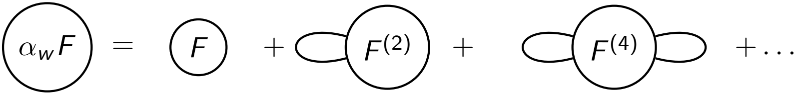
$$G^+(f_1, f_2) - G^+(f_2, f_1) = iG(f_1, f_2)\,, \qquad G^+(\bar{f}, f) \geq 0\,,$$

$$\textcolor{red}{\text{Hadamard}} \quad \mathrm{WF}(G^+) = \{(\xi, \xi') \in T^*\mathsf{M}^2 \setminus \{0\} | \quad \xi \sim \xi' \quad \xi \triangleright 0\}\,.$$

- G^+ is not unique: different choices $\mathsf{G}^+, \widehat{\mathsf{G}}^+$ give to $*$-isomorphic algebras

$$\alpha_w \colon \widehat{\mathcal{A}} \to \mathcal{A}, \quad w \quad := \quad \mathsf{G}^+ - \widehat{\mathsf{G}}^+ \in C^\infty(\mathsf{M})$$

$$\alpha_w F(\varphi) \quad := \quad F(\varphi) + \sum_{n \geq 1} \frac{\hbar^n}{n!} \left(\frac{w}{2}\right)^{\otimes n} \left(F^{(2n)}[\varphi]\right)\,,$$



- $\mathcal{F}_{\mathrm{reg}} := \{F \colon \mathcal{E} \to \mathbb{C} | \quad \mathrm{WF}(F^{(n)}[\varphi]) = \emptyset \quad \forall n, \forall \varphi\}$

$$\mathcal{A}^{\mathrm{reg}} := \mathrm{Alg}(\mathcal{F}_{\mathrm{reg}} \cap \mathcal{F}_{\mathrm{loc}}, \star, *) \subset \mathcal{A}\,,$$

is the $*$-algebra generated by regular functionals.

Quantum Møller map: ideal world

Interacting dynamics: $P_V\varphi = P\varphi + V^{(1)}[\varphi]\,, \quad V \in \mathfrak{F}_{\mathrm{loc}}.$

What about $\widetilde{\mathcal{A}}_V$?

- Exploit the quantum Møller map

$$\mathsf{R}^{\hbar}_V : \widetilde{\mathcal{A}}_V \to \mathcal{A}\,.$$

- Operations on $\widetilde{\mathcal{A}}_V$ are implicitly defined as

$$\mathsf{R}^{\hbar}_V(F_1 \star_V F_2) := \mathsf{R}^{\hbar}_V(F_1) \star \mathsf{R}^{\hbar}_V(F_2), \qquad \mathsf{R}^{\hbar}_V(F^{*_V}) := \mathsf{R}^{\hbar}_V(F)^*\,.$$

(i) Time-ordered product: symmetric product $\cdot_T$ such that

$$F_1 \cdot_T F_2 = F_1 \star F_2 \quad \text{if} \quad \mathrm{supp}(F_1) \cap J^{\downarrow}\mathrm{supp}(F_2) = \emptyset\,.$$

(ii) For $F_1, F_2 \in \mathcal{F}_{\mathrm{reg}}$

$$(F_1 \cdot_T F_2)(\varphi) = F_1(\varphi)F_2(\varphi) + \sum_{n\geq 1} \frac{\hbar^n}{n!} \mathsf{G}^{F\otimes n}\left(F_1^{(n)}[\varphi], F_2^{(n)}[\varphi]\right)\,.$$

with $\mathsf{G}^F := \mathsf{G}^+ + i\mathsf{G}^{\downarrow}$ is the Feynman propagator.

(iii) $\cdot_T$ can be extended as a map

$$T\colon \mathcal{F}_{\mathrm{mreg}} \subset \mathcal{F}_{\mathrm{mloc}} \to \mathcal{A}, \qquad \mathcal{F}_{\mathrm{m}\sharp} := \bigoplus_n \mathcal{F}_{\sharp}^{\otimes n}$$

$$F_1 \cdot_T \ldots \cdot_T F_n \;:=\; T\left(T^{-1}(F_1) \otimes \cdots \otimes T^{-1}(F_n)\right)\,.$$

Non-uniqueness is controlled by renormalization freedom.

(iv) $\mathsf{R}_V^{\hbar}(F) := \left(\exp_T\left[\frac{i}{\hbar}\lambda V\right]\right)^{-1} \star \left(\exp_T\left[\frac{i}{\hbar}\lambda V\right] \cdot_T F\right) \in \mathcal{A}[[\lambda]]\,.$

Extension of T

Criteria for the extension of $T : \mathcal{F}_{\rm mreg} \subset \mathcal{F}_{\rm mloc} \to \mathcal{A}$:

- T is symmetric and $T(F) = F$ on linear functionals;
- Causality $T(F_1, \ldots, F_n) = T(F_1, \ldots, F_k) \star T(F_{k+1}, \ldots, F_n)$;
- Field independence $T(F_1, \ldots, F_n)^{(1)} = \sum_j T(F_1, \ldots, F_j^{(1)}, \ldots, F_n)$;
- Covariance i.e. $(\mathrm{M}, g) \mapsto T(\mathrm{M}, g)$ is functorial;
- Microlocal spectrum condition: bound on $\mathrm{WF}(\omega_{T,N})$ where $\omega_{T,N}(f_1, \ldots, f_n) := T(F_1, \ldots, F_N)|_{\varphi=0}$, $F_j(\varphi) = \int f_j \varphi$.
- Unitarity $T(F_1, F_2)^* = T(F_1^*) \star T(F_2^*) - T(F_1^*, F_2^*)$;
- Leibniz rule (Action Ward Identity): $T(F) = 0$ if $F(\varphi) = \int \mathrm{d}B(\phi)$.
- Suitable scaling properties and a suitable smooth or analytic dependence on the metric;
- The Principle of Perturbative Agreement (see later).

Non-uniqueness is controlled by renormalization freedom.

Quantum Møller map: reality

Interacting dynamics: $P_V\varphi = P\varphi + V^{(1)}[\varphi]\,, \quad V \in \mathcal{F}_{\mathrm{loc}}.$

What about $\widetilde{\mathcal{A}}_V$?

- Exploit the quantum Møller map

$$\mathsf{R}^\hbar_V\colon \widetilde{\mathcal{A}}_V \to \mathcal{A}[[\lambda]]\,.$$

- Operations on $\widetilde{\mathcal{A}}_V$ are implicitly defined as

$$\mathsf{R}^\hbar_V(F_1 \star_V F_2) := \mathsf{R}^\hbar_V(F_1)\star \mathsf{R}^\hbar_V(F_2), \qquad \mathsf{R}^\hbar_V(F^{*_V}) := \mathsf{R}^\hbar_V(F)^*\,.$$

- An explicit description of $\widetilde{\mathcal{A}}_V(\mathsf{M})$ is not at disposal.
  For all practical purposes, one refers to a concrete realization of $\widetilde{\mathcal{A}}_V$

$$\mathcal{A}_V := \mathrm{Alg}(\mathsf{R}^\hbar_V(\mathcal{F}_{\mathrm{loc}}),\star,*) \subset \mathcal{A}[[\lambda]]\,.$$

Principle of Perturbative Agreement

$$P_1\varphi + Q^{(1)}[\varphi] = \Box\varphi + m_1^2\varphi + (m_2^2 - m_1^2)\varphi = \Box\varphi + m_2^2\varphi = P_{1+Q}\varphi\,.$$

$\mathcal{A}_{1+Q}$, $\mathcal{A}_{1,Q}$ provide the same physical information.

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Principle of Perturbative Agreement

$\mathcal{A}_{1,Q} \simeq \mathcal{A}_{1+Q}\,.$

$$\begin{tikzcd}
\mathcal{A}_{1+Q} \arrow[rr, "\mathsf{R}_{1,Q}"] & & \mathcal{A}_1[[\lambda]] \\
& \widetilde{\mathcal{A}}_{1,Q} \arrow[ul, dashed, "\gamma_{Q,1} := \mathsf{R}_{1,Q}^{-1} \circ \mathsf{R}^{\hbar}_{1,Q}"] \arrow[ur, dashed, "\mathsf{R}^{\hbar}_{1,Q}"'] &
\end{tikzcd}$$

Principle of Perturbative Agreement

$$P_1\varphi + Q^{(1)}[\varphi] = \Box\varphi + m_1^2\varphi + (m_2^2 - m_1^2)\varphi = \Box\varphi + m_2^2\varphi = P_{1+Q}\varphi\,.$$

$\mathcal{A}_{1+Q}$, $\mathcal{A}_{1,Q}$ provide the same physical information.

Principle of Perturbative Agreement (Hollands & Wald '05)

$T_{1+Q} = \gamma_{Q,1} \circ T_1$ on $\mathcal{F}_{\mathrm{mloc}}$ defines a time-ordered map for $\mathcal{A}_{1+Q}$.

$$\begin{tikzcd}
\mathcal{F}_{\mathrm{mloc}} \arrow[rr, "\mathrm{R}_{1,Q}"] & & \mathcal{F}_{\mathrm{mloc}}[[\lambda]] \\
& \mathcal{F}_{\mathrm{mloc}} \arrow[ul, "\gamma_{Q,1} := \mathrm{R}_{1,Q}^{-1} \circ \mathrm{R}_{1,Q}^{\hbar}"] \arrow[ur, "\mathrm{R}_{1,Q}^{\hbar}"'] &
\end{tikzcd}$$

Classical Møller operator

The classical Møller operator intertwines the dynamics of $\mathcal{A}_{1+Q}$ and $\mathcal{A}_1$ by identifying them in the past.

$$\mathrm{R}_{1,Q}F(\varphi) \doteq F(r_{1,Q}\varphi)\,, \quad P_{1+Q} \circ r_{1,Q} = P_1\,, \quad r_{1,Q}f\big|_{\mathrm{M}\setminus J^{\downarrow}\mathrm{supp}(Q)} = f\,.$$

Theorem (D., Hack, Pinamonti, 2016)

- $\mathrm{G}^+_{1+Q}(f,h) := G^+_1(R^\dagger_{1,Q}f, R^\dagger_{1,Q}h)$ is Hadamard for $\mathcal{A}_{1+Q}$;
- $\mathrm{R}_{1,Q}\colon \mathcal{A}_{1+Q} \to \mathcal{A}_1$ is a $*$-isomorphism.

We consider the $\star_{1+Q}$-product on $\mathcal{A}_{1+Q}$ induced by $\star_1$ via $\mathrm{R}_{1,Q}$.

PPA for $\mathcal{F}_{\mathrm{reg}}$

$$
\begin{tikzcd}
\mathcal{A}_{1+Q} \arrow[rr, "\mathsf{R}_{1,Q}"] & & \mathcal{A}_1[[\lambda]] \\
& \widetilde{\mathcal{A}}_{1,Q} \arrow[ul, dashed, "\gamma_{Q,1} := \mathsf{R}_{1,Q}^{-1} \circ \mathsf{R}_{1,Q}^{\hbar}"] \arrow[ur, dashed, "\mathsf{R}_{1,Q}^{\hbar}"'] &
\end{tikzcd}
$$

$$\begin{tikzcd}
\mathcal{A}^{\text{reg}}_{1+Q} \arrow[rr, "\mathsf{R}_{1,Q}"] & & \mathcal{A}^{\text{reg}}_{1}[[\lambda]] \\
& \widetilde{\mathcal{A}}^{\text{reg}}_{1,Q} \arrow[ul, "\gamma_{Q,1} := \mathsf{R}^{-1}_{1,Q} \circ \mathsf{R}^{\hbar}_{1,Q}"] \arrow[ur, "\mathsf{R}^{\hbar}_{1,Q}"'] &
\end{tikzcd}$$

Proposition (D., Hack, Pinamonti, 2016)

- $\widetilde{\mathcal{A}}^{\text{reg}}_{1,Q} := \text{Alg}(\mathcal{F}_{\text{reg}}, \star_{1,Q}, *_{1,Q})$;
- $\gamma_{Q,1} : \widetilde{\mathcal{A}}^{\text{reg}}_{1,Q} \to \mathcal{A}^{\text{reg}}_{1+Q}$ is a $*$-isomorphism.

Characterization of $\gamma_{Q,1}$ on $\mathrm{F}_{\mathrm{reg}}$

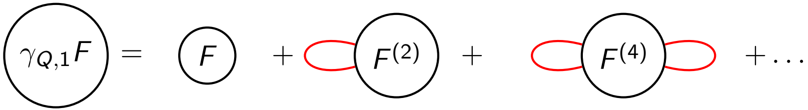
$F_f(\varphi) := \int f\varphi\,, \quad \gamma_{Q,1} \colon \widetilde{\mathcal{A}}^{\mathrm{reg}}_{1,Q} \to \mathcal{A}^{\mathrm{reg}}_{1+Q}.$

Theorem (D., Hack, Pinamonti, 2016)

- $\mathrm{R}^{\hbar}_{1,Q} F_f = \mathrm{R}_{1,Q} F_f \iff \gamma_{Q,1} F_f = F_f$;
- $[F_f, F_h]_{1,Q} = \gamma^{-1}_{Q,1}[\gamma_{Q,1}F_f, \gamma_{Q,1}F_h]_{1+Q} = i\hbar \mathrm{G}_{1+Q}(f,h)$.
- The $\star_{1,Q}$-product on $\widetilde{\mathcal{A}}^{\mathrm{reg}}_{1,Q}$ is given by an exponential formula with

$$\mathrm{G}^{+}_{1,Q} \doteq \mathrm{G}^{+}_{1+Q} + \mathrm{G}^{F}_{1} - \mathrm{G}^{F}_{1+Q}\,.$$

- $\gamma_{Q,1} = \alpha_{D_{1,Q}}$ where $D_{1,Q} = \mathrm{G}^{F}_{1+Q} - \mathrm{G}^{F}_{1}$.
- The Principle of Perturbative Agreement holds on $\mathcal{F}_{\mathrm{reg}}$.



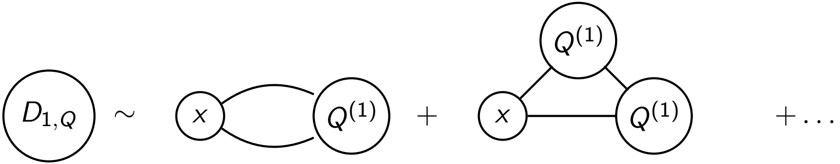
Extension on $\mathcal{F}_{\mathrm{mloc}}$

$$\gamma_{Q,1} F = \alpha_{D_{1,Q}} F \qquad \text{for } F \in \mathcal{F}_{\mathrm{loc}}?$$

$$D_{1,Q}(x,y) = (\mathsf{G}^F_{1+Q} - \mathsf{G}^F_1)(x,y) \to_{x\to y} \infty \quad \text{logarithmically divergent}\,.$$

Perturbative expansion:

$$\begin{aligned} \mathsf{R}_{1,Q} &= (\mathrm{id} + \lambda \mathsf{G}^{\uparrow}_1 Q^{(1)})^{-1} = \sum (-\lambda \mathsf{G}^{\uparrow}_1 Q^{(1)})^n \\ \mathsf{G}^F_{1+Q} - \mathsf{G}^F_1 &\sim \sum i^n \mathsf{G}^F_1 (Q^{(1)} \mathsf{G}^F_1)^n + \text{smooth terms}\,. \end{aligned}$$



It is enough to renormalize G^F_1.

Extension on $\mathcal{F}_{\mathrm{mloc}}$

Theorem (D., Hack, Pinamonti, 2016)

- $\gamma_{Q,1} : \mathcal{F}_{\mathrm{mloc}} \to \mathcal{F}_{\mathrm{mloc}}$ is the deformation $\alpha_{D_{1,Q}}$.
- $T_{1+Q} \doteq \gamma_{Q,1} \circ T_1$ defines a time ordered map for $\mathcal{A}_{1+Q}$.
- It holds the cocycle condition

$$\gamma_{Q_3,1} = \gamma_{Q_3,1+Q_2} \circ \gamma_{Q_2,1} \,.$$

- Fixing T_1 the map

$$T(Q) \doteq \gamma_{Q,1} \circ T_1 \,.$$

satisfies the Perturbative Agreement for mass/curvature variation.

The perturbative construction is "exact".

Generalized PPA

$$P_1\varphi + Q^{(1)}[\varphi] + V^{(1)}[\varphi] = P_1\varphi + Q^{(1)}[\varphi] + V^{(1)}[\varphi]\,.$$

$$\begin{tikzcd}
\mathcal{A}_{1+Q}[[\lambda]] \arrow[rr, "\mathsf{R}_{1,Q}"] & & \mathcal{A}_1[[\lambda]] \\
& \widetilde{\mathcal{A}}_{1,Q} \arrow[ul, dashed, "\gamma_{Q,1}"'] \arrow[ur, dashed, "\mathsf{R}^{\hbar}_{1,Q}"] & \\
\widetilde{\mathcal{A}}_{1+Q,V} \arrow[uu, dashed, "\mathsf{R}^{\hbar}_{1+Q,V}"] & & \widetilde{\mathcal{A}}_{1,Q+V} \arrow[ll, dashed, "?"'] \arrow[uu, dashed, "\mathsf{R}^{\hbar}_{1,Q+V}"']
\end{tikzcd}$$

Generalized PPA

$$P_1\varphi + Q^{(1)}[\varphi] + V^{(1)}[\varphi] = P_1\varphi + Q^{(1)}[\varphi] + V^{(1)}[\varphi]\,.$$

$$\begin{tikzcd}
\mathcal{F}_{\mathrm{mloc}}[[\lambda]] \arrow[rr, "\mathsf{R}_{1,Q}"] & & \mathcal{F}_{\mathrm{mloc}}[[\lambda]] \\
& \mathcal{F}_{\mathrm{mloc}} \arrow[ul, "\gamma_{Q,1}"'] \arrow[ur, "\mathsf{R}^{\hbar}_{1,Q}"] & \\
\mathcal{F}_{\mathrm{mloc}} \arrow[uu, "\mathsf{R}^{\hbar}_{1+Q,V}"] & & \mathcal{F}_{\mathrm{mloc}} \arrow[ll, "\gamma_{Q,1}"'] \arrow[uu, "\mathsf{R}^{\hbar}_{1,Q+V}"']
\end{tikzcd}$$

Theorem (D., Hack, Pinamonti 2016)

$$\mathsf{R}^{\hbar}_{1,Q+V} = \mathsf{R}_{1,Q} \circ \mathsf{R}^{\hbar}_{1+Q,V} \circ \gamma_{Q,1} \qquad \text{on } \mathcal{F}_{\mathrm{mloc}}\,.$$

Applications: thermal mass (in a nutshell)

- On Minkowski spacetime the $\star$-product is usually induced by the so-called vacuum G_0^+.
- States for massive theories ($\Box\varphi + m^2\varphi = 0$) are more suitable for perturbative constructions of interacting theories ($\Box\varphi + m^2\varphi + \lambda\varphi^3 = 0$).
- Constructions for interacting massless theories ($\Box\varphi + \lambda\varphi^3 = 0$) can be achieved in presence of temperature τ:

$$\alpha_{w_\tau} : \mathcal{A}_0 \to \mathcal{A}_\tau \qquad w_\tau := G_\tau^+ - G_0^+$$
$$\Box\varphi + \lambda\varphi^3 = 0 \longrightarrow \Box\varphi + m_\tau^2\varphi + \lambda\varphi^3 = 0\,.$$

- $m_\tau^2 \simeq \tau^2$ is the so-called thermal mass.
- The gPPA provides a solid mathematical background for the folklore

"In presence of temperature a massless theory can be treated as a massive one".