Nesting statistics in the $O(n)$ loop model on random maps of any topology

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joint work with G. Borot

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January 6, 2017

Outline

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Objects of study

- A *map* M of genus g is a finite connected graph embedded into a closed orientable surface of genus g such that the connected components of the complement of the graph (called *faces*) are homeomorphic to an open disk.
- A *map with* k *boundaries* is a map with k pairwise distinct marked faces, labeled from 1 to k, and with a marked edge (called *root*) on every marked face.

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- A *loop* is an undirected simple close path on the dual map not visiting boundaries. A *loop configuration* is a collection of disjoint loops.

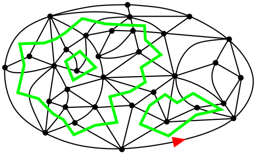

Figure : Planar triangulation with a boundary of length 8, endowed with a loop configuration.

Statistical weights...

In the $O(n)$ loop model on random maps, the Boltzmann weight of a configuration C is

$$w(C) = \frac{1}{|\mathrm{Aut}\, C|}\, n^{\mathcal{L}} \prod_{l \geq 3} g_l^{N_l} \prod_{\substack{\{l_1, l_2\} \\ l_1 + l_2 \geq 1}} g_{l_1, l_2}^{N_{l_1, l_2}},$$

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where

- $\mathcal{L}$ is the number of loops,
- N_l is the number of unvisited faces of degree l,
- N_{l_1,l_2} is the number of visited faces of degree $(l_1 + l_2 + 2)$ whose boundary consists, in cyclic order with an arbitrary orientation, of l_1 uncrossed edges, 1 crossed edge, l_2 uncrossed edges and 1 crossed edge.

...and generating series

The generating series of **configurations of the $O(n)$ model** with underlying map of genus g with k boundaries of lengths $\ell_1, \ell_2, \ldots, \ell_k \geq 1$ and k' marked points is

$$F^{(\mathrm{g},k,\bullet k')}_{\ell_1,\ldots,\ell_k} = \delta_{k,1}\delta_{\ell_1,0}\, u + \sum_C u^{|V(C)|} w(C),$$

where $|V(C)|$ denotes the number of vertices of the underlying map of C, also called *volume*.

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Usual maps (no loops):

- $w(C) = \frac{1}{|\mathrm{Aut}\, C|} \prod_{l \geq 1} g_l^{N_l}$.
- The generating series is denoted $\mathcal{F}^{(\mathrm{g},k,\bullet k')}_{\ell_1,\ldots,\ell_k}$.

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- The $O(n)$ loop model gives rise to two new universality classes, which depend continuously on n, called *dense* or *dilute*.

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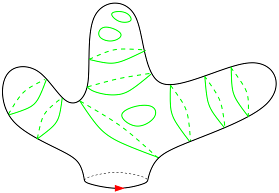
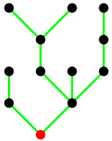
- Our approach for any topology: the substitution approach and the topological recursion.

Outline

Primary nesting graph Γ_0 of a map M

- Cut M along every loop $\rightsquigarrow$ Connected components $c_1, \ldots, c_N$.
- Vertices $(V(\Gamma_0))$: $\{c_i\}_i$. Edges $(E(\Gamma_0))$: $\{c_i, c_j\}$ if c_i and c_j have a common boundary.
- Save the genus h(v) of the connected component corresponding to every vertex v.
- $* : \{\text{Marked elements in } M\} \to V(\Gamma_0)$.





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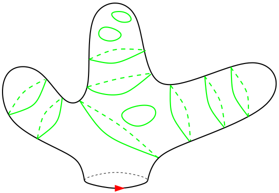
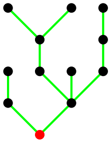


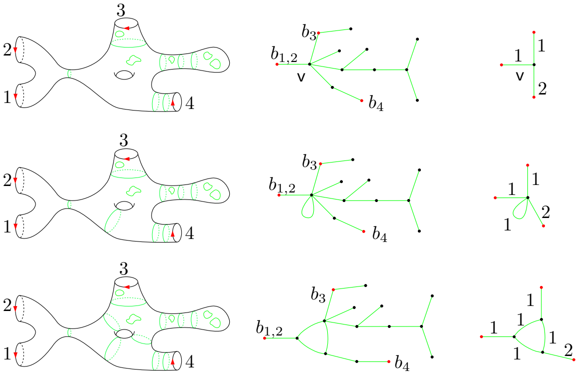


Definition

In a map M with a non empty set of marked elements P, a loop is *separating* if it is not contractible in $M \setminus P$.

Nesting graph Γ of a map M

- Erase univalent unmarked genus 0 vertices.
- Replace $v_0 - v_1 - \cdots - v_P$ with $P \geq 2$, where $(v_i)_{i=1}^{P-1}$ are bivalent unmarked genus 0 vertices, by a single edge $v_0 - v_P$ carrying a length P.



Substitution approach

{Disks M with a loop configuration} $\longleftrightarrow$ {Triples $(G, \mathcal{R}, M')$}

- G is a usual disk, called the *gasket* of M. $\rightsquigarrow$ Connected component containing the boundary in the complement all loops in M.
- $\mathcal{R}$ is a disjoint union of *annuli*, which are sequences of faces visited by a single loop and rooted on its outer boundary $\rightsquigarrow$ Collection of faces crossed by the outermost loops in M.
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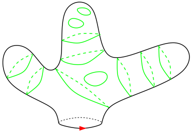

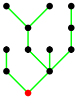

Functional relation:

$$F_\ell = \mathcal{F}_\ell(G_1, G_2, \ldots).$$

The *renormalized face weights* G_m satisfy

$$G_m = g_m + \sum_{r \geq 0} A_{m,r} \mathcal{F}_r(G_1, G_2, \ldots) = g_m + \sum_{\ell' \geq 1} A_{m,r}\, F_r,$$

where $A_{m,r}$ is the generating series of rooted annuli.

Notations

- For $\mathsf{e} \in E$, $\{\mathsf{e}_+, \mathsf{e}_-\}$ is its set of half-edges.
- For $\mathsf{v} \in V$, $\mathsf{e}(\mathsf{v}) := \{\text{half-edges incident to } \mathsf{v}\}$, $d(\mathsf{v})$ denotes the degree of v, $\partial(\mathsf{v}) := \{\text{boundaries of } \mathsf{v}\}$ and $|\partial(\mathsf{v})| := k(\mathsf{v})$.
- $V_{0,2} := \{\text{univalent vertices } \mathsf{v} \mid \mathsf{h}(\mathsf{v}) = 0, k(\mathsf{v}) = 1\}$, $\tilde{V} := V \setminus V_{0,2}$ and for $\mathsf{v} \in V_{0,2}$, $\mathsf{e}_+(\mathsf{v})$ denotes the incident half-edge.
- $E_{\text{un}} := \{\mathsf{e}(\mathsf{v}) \text{ for } \mathsf{v} \in V_{0,2}\}$, $\tilde{E} := E \setminus E_{\text{un}}$ and

$$E_{\text{glue}} := \bigcup_{\mathsf{e} \in \tilde{E}} \{\mathsf{e}_+, \mathsf{e}_-\} \cup \bigcup_{\mathsf{v} \in V_{0,2}} \mathsf{e}_+(\mathsf{v}).$$

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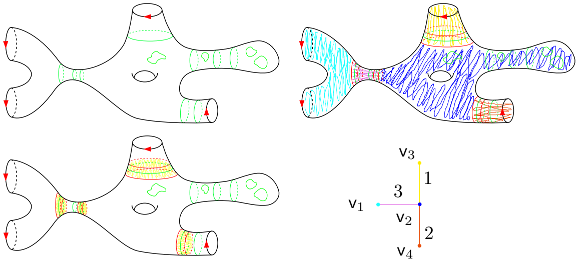
Generating series

- $\mathcal{F}^{(\mathsf{g},k,\bullet k')}_{\ell_1,\ldots,\ell_k} \rightsquigarrow$ usual maps evaluated at renormalized face weights.
- $F^{(2)}_{\ell_1,\ell_2}[s] \rightsquigarrow$ *refined* generating series of cylinders, where $w(C)$ has an extra factor s^P, with $P := |\{\text{separating loops}\}|$.
- $\hat{F}^{(2)}_{\ell_1,\ell_2}[s] = s \sum_{l \geq 0} R_{\ell_1,l} F^{(2)}_{l,\ell_2}[s] \rightsquigarrow$ cylinders with one annulus (with unrooted outer boundary) glued to one of the two boundaries.
- $\tilde{F}^{(2)}_{\ell_1,\ell_2}[s] = sR_{\ell_1,\ell_2} + s^2 \sum_{l,l' \geq 0} R_{\ell_1,l} F^{(2)}_{l,l'}[s]\, R_{l',\ell_2} \rightsquigarrow$ cylinders capped with two annuli with unrooted outer boundaries.

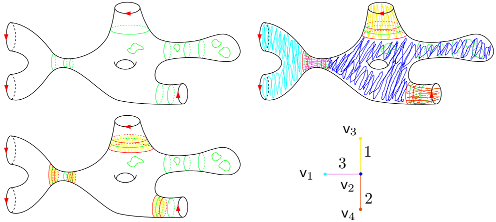
We can retrieve the original map from $(\Gamma, \star, \mathbf{P})$, by glueing together:

- $\forall$v $\in V_{0,2}(\Gamma)$, a cylinder with one annulus glued to one of the boundaries;
- $\forall$v $\in \tilde{V}(\Gamma)$ of valency d(v), a usual map (with renormalized weights) of genus h(v) with k(v) labeled boundaries and d(v) other unlabeled boundaries, and k'(v) marked points;
- $\forall$e $\in \tilde{E}(\Gamma)$ of length 1, an annulus;
- $\forall$e $\in \tilde{E}(\Gamma)$ of length P(e) ≥ 2, two annuli capping a cylinder with P(e) $- 2$ separating loops.



v$_1$, v$_2 \in \tilde{V}$: h(v$_1$) $= 0, k$(v$_1$) $= 2, d$(v$_1$) $= 1$, h(v$_2$) $= 1, k$(v$_2$) $= 0, d$(v$_2$) $= 3$.
v$_3$, v$_4 \in V_{0,2}$.

Combinatorial decomposition of maps



We can determine the refined generating series of maps with fixed associated nesting graph $(\Gamma, \star, \mathbf{P})$:

Proposition

$$\mathscr{F}^{(\mathsf{g},k,\bullet k')}_{\ell_1,\ldots,\ell_k}[\Gamma,\star,\mathbf{s}] = \sum_{l\,:\,E_{\mathrm{glue}}(\Gamma)\to\mathbb{N}} \prod_{\mathsf{v}\in\tilde{V}(\Gamma)} \frac{\mathcal{F}^{(\mathsf{h}(\mathsf{v}),k(\mathsf{v})+d(\mathsf{v}),\bullet k'(\mathsf{v}))}_{\ell(\partial(\mathsf{v})),l(\mathsf{e}(\mathsf{v}))}}{d(\mathsf{v})!} \prod_{\mathsf{e}\in\tilde{E}(\Gamma)} \tilde{F}^{(2)}_{l(\mathsf{e}_-),l(\mathsf{e}_+)}[s(\mathsf{e})] \prod_{\mathsf{v}\in V_{0,2}(\Gamma)} \hat{F}^{(2)}_{l(\mathsf{e}_+(\mathsf{v})),\ell(\partial(\mathsf{v}))}[s(\mathsf{e}_+(\mathsf{v}))],$$

where $\ell : \bigcup_{\mathsf{v}\in V(\Gamma)} \partial(\mathsf{v}) \to \mathbb{N}$ *is given by* $\ell_1,\ldots,\ell_k$.

Outline

Admissibility

We say that u and a sequence $(g_l)_{l\geq 1}$ of nonnegative real numbers are *admissible* if $\mathcal{F}^{\bullet}_{\ell} < \infty$ for any ℓ.

$$\boldsymbol{\mathcal{F}}(x) \coloneqq \sum_{\ell\geq 0} \frac{\mathcal{F}_\ell}{x^{\ell+1}} \in \mathbb{Q}[[x^{-1}]]$$

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is a well-defined Laurent series expansion at $x = \infty$ of a function denoted likewise which is

1. holomorphic for $x \in \mathbb{C} \setminus \gamma$, where $\gamma = [\gamma_-, \gamma_+] \subset \mathbb{R}$ depends on the vertex and face weights, and
2. uniformly bounded for $x \in \mathbb{C} \setminus \gamma$.
3. Its boundary values on the cut satisfy a functional relation,
4. $\boldsymbol{\mathcal{F}}(x) = u/x + O(1/x^2)$ when $x \to \infty$.

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These properties uniquely determine γ_-, γ_+ and $\boldsymbol{\mathcal{F}}(x)$. Analogously, we define $\boldsymbol{\mathcal{F}}^{(\mathrm{g},k)}(x_1, \ldots, x_k)$ which satisfies the properties analogous to 1 and 3. Regarding 2 and 4, we have that $\sigma(x_1)\sigma(x_2)\boldsymbol{\mathcal{F}}^{(2)}(x_1, x_2)$ remains uniformly bounded for $x_1, x_2 \in \mathbb{C} \setminus \gamma$ and $\boldsymbol{\mathcal{F}}^{(2)}(x_1, x_2) \in O(x_1^{-2} x_2^{-2})$ when $x_1, x_2 \to \infty$. $\boldsymbol{\mathcal{F}}^{(2)}(x_1, x_2)$ is also uniquely determined by these properties.

For $2\mathrm{g} - 2 + k > 0$, $\exists r(\mathrm{g}, k) > 0$ such that $\sigma(x_1)^{r(\mathrm{g},k)} \mathcal{F}^{(\mathrm{g},k)}(x_1, \ldots, x_k)$ remains bounded when x_1 approaches γ while $(x_i)_{i=2}^{k}$ are fixed away from γ and $\mathcal{F}^{(\mathrm{g},k)}(x_1, x_I) \in O(x_1^{-2})$ when $x_1 \to \infty$.

Definition

For the $O(n)$ model, we say that two sequences of real numbers $(g_l)_{l \geq 3}$ and $(A_{l_1,l_2})_{l_1,l_2}$ are *admissible* if the corresponding sequence of renormalized face weights $(G_1, G_2, \ldots)$ is admissible.

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Remark

Admissibility $\Rightarrow$

- $\mathbf{F}^{(\mathrm{g},k)}(x_1,\ldots,x_k)$ *satisfies analogous properties to those of* $\mathcal{F}^{(\mathrm{g},k)}$.
- *The annuli generating series* $\mathbf{R}(x,y)=\sum_{l+l'\geq 1}R_{l,l'}x^l y^{l'}$, *with* $R_{l,l'}=A_{l,l'}/l$ *(non-rooted) and* $\mathbf{A}(x,y)=\partial_x\mathbf{R}(x,y)$ *(1 boundary rooted) are holomorphic in a neighborhood of* $\gamma\times\gamma$.
- $\hat{\mathbf{F}}_s^{(2)}(x_1,x_2)=\sum_{\ell_1,\ell_2\geq 0}\hat{F}^{(2)}_{\ell_1,\ell_2}[s]\,\frac{x_1^{\ell_1}}{x_2^{\ell_2+1}}=s\oint_\gamma\frac{\mathrm{d}y}{2\mathrm{i}\pi}\,\mathbf{R}(x_1,y)\mathbf{F}_s^{(2)}(y,x_2)$ *is the series expansion when* $x_1\to 0$ *and* $x_2\to\infty$ *of a function which is holomorphic for* x_1 *in a neighborhood of* γ *and* x_2 *in* $\mathbb{C}\setminus\gamma$.

Decomposition of maps for a fixed nesting graph Γ

Remark

$$
\begin{aligned}
\tilde{\mathbf{F}}_s^{(2)}(x_1, x_2) &= \sum_{\ell_1, \ell_2 \geq 0} \tilde{F}^{(2)}_{\ell_1, \ell_2}[s]\, x_1^{\ell_1} x_2^{\ell_2} \\
&= s\,\mathbf{R}(x, y) + s^2 \oint_\gamma \frac{\mathrm{d}y_1}{2\mathrm{i}\pi} \frac{\mathrm{d}y_2}{2\mathrm{i}\pi} \mathbf{R}(x_1, y_1)\mathbf{F}_s^{(2)}(y_1, y_2)\mathbf{R}(y_2, x_2)
\end{aligned}
$$

is the series expansion at $x_i \to 0$ of a function denoted likewise, which is holomorphic for x_i in a neighborhood of γ.

Decomposition of maps for a fixed nesting graph Γ

Remark

$$\tilde{\mathbf{F}}^{(2)}_s(x_1,x_2) = \sum_{\ell_1,\ell_2\geq 0} \tilde{F}^{(2)}_{\ell_1,\ell_2}[s]\, x_1^{\ell_1} x_2^{\ell_2}$$
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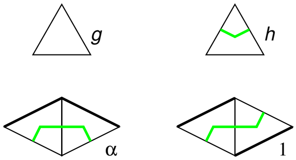
is the series expansion at $x_i \to 0$ of a function denoted likewise, which is holomorphic for x_i in a neighborhood of γ.

$$\mathscr{F}^{(\mathrm{g},k)}_{\Gamma,\star,\mathbf{s}}(x_1,\ldots,x_k) =$$
$$\oint_{\gamma^{E_{\mathrm{glue}}(\Gamma)}} \prod_{\mathsf{e}\in E_{\mathrm{glue}}(\Gamma)} \frac{\mathrm{d}y_{\mathsf{e}}}{2\mathrm{i}\pi} \prod_{\mathsf{v}\in\tilde{V}(\Gamma)} \frac{\boldsymbol{\mathcal{F}}^{(\mathsf{h}(\mathsf{v}),k(\mathsf{v})+d(\mathsf{v}),\bullet k'(\mathsf{v}))}(x_{\partial(\mathsf{v})},y_{\mathsf{e}(\mathsf{v})})}{d(\mathsf{v})!}$$
$$\prod_{\mathsf{e}\in\tilde{E}(\Gamma)} \tilde{\mathbf{F}}^{(2)}_{s(\mathsf{e})}(y_{\mathsf{e}_+},y_{\mathsf{e}_-}) \prod_{\mathsf{v}\in V_{0,2}(\Gamma)} \hat{\mathbf{F}}^{(2)}_{s(\mathsf{e}_+(\mathsf{v}))}(y_{\mathsf{e}_+(\mathsf{v})},x_{\partial(\mathsf{v})}).$$

Outline

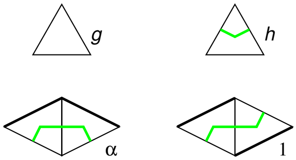
$O(n)$ loop model on random triangulations



Involution:

$$\varsigma(x) := \frac{1 - \alpha h x}{\alpha h + (1 - \alpha^2) h^2 x}.$$

$O(n)$ loop model on random triangulations



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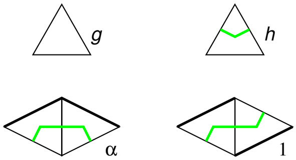
The annuli generating series $\mathbf{R}(x,y)$ and $\mathbf{A}(x,y)$ in this model are explicit

$$\mathbf{A}(x,z) = \partial_x \mathbf{R}(x,z) = n\left(\frac{\varsigma'(x)}{z-\varsigma(x)} + \frac{\varsigma''(x)}{2\varsigma'(x)}\right).$$

If f is holomorphic in $\mathbb{C}\setminus\gamma$ such that $f(x) \sim c_f/x$ when $x \to \infty$, then

$$\oint_\gamma \frac{\mathrm{d}y}{2\mathrm{i}\pi}\,\mathbf{A}(x,y)\,f(y) = -n\varsigma'(x)\,f(\varsigma(x)) + nc_f\,\frac{\varsigma''(x)}{2\varsigma'(x)}.$$

$O(n)$ loop model on random triangulations



Involution:

$$\varsigma(x) := \frac{1 - \alpha h x}{\alpha h + (1-\alpha^2)h^2 x}.$$

The annuli generating series $\mathbf{R}(x,y)$ and $\mathbf{A}(x,y)$ in this model are explicit

$$\mathbf{A}(x,z) = \partial_x \mathbf{R}(x,z) = n\left(\frac{\varsigma'(x)}{z - \varsigma(x)} + \frac{\varsigma''(x)}{2\varsigma'(x)}\right).$$

If f is holomorphic in $\mathbb{C} \setminus \gamma$ such that $f(x) \sim c_f/x$ when $x \to \infty$, then

$$\oint_\gamma \frac{\mathrm{d}y}{2\mathrm{i}\pi}\,\mathbf{A}(x,y)\,f(y) = -n\varsigma'(x)\,f(\varsigma(x)) + nc_f\,\frac{\varsigma''(x)}{2\varsigma'(x)}.$$

Therefore, the following linear equation can be solved explicitly:

$$f(x+\mathrm{i}0) + f(x-\mathrm{i}0) + s\oint_\gamma \frac{\mathrm{d}y}{2\mathrm{i}\pi}\,\mathbf{A}(x,y)\,f(y) = \varphi(x), \qquad \forall x \in (\gamma_-,\gamma_+).$$

Elliptic parametrization

How to solve the homogeneous equation

$$f(x+\mathrm{i}0) + f(x-\mathrm{i}0) - ns\,\varsigma'(x) f(\varsigma(x)) = 0?$$

Key: Use $v : \mathbb{C} \setminus \big(\gamma \cup \varsigma(\gamma)\big) \longrightarrow \big\{v \in \mathbb{C} \mid 0 < \mathrm{Re}\, v < 1/2, |\mathrm{Im}\, v| < T\big\}$.

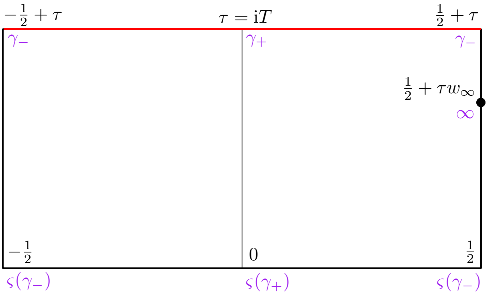


Figure : In purple: Special values of $x(v)$ at the corners.

The function $v \mapsto x(v)$ is analytically continued for $v \in \mathbb{C}$ by

$$x(-v) = x(v+1) = x(v+2\tau) = x(v).$$

Elliptic parametrization

$$v(\varsigma(x)) = \tau - v(x)$$

Our functional equation turns into

$\tilde{f}(v+2\tau)+\tilde{f}(v)-n\,\tilde{f}(v-\tau) = 0$, with $\tilde{f}(v) = \tilde{f}(v+1) = -\tilde{f}(-v), \forall v \in \mathbb{C}$,

for the analytic continuation of the function $\tilde{f}(v) = f(x(v))x'(v)$.

$$b := \frac{\arccos(n/2)}{\pi}$$

b ranges from $\frac{1}{2}$ to 0 when n ranges from 0 to 2.

Remark

There are explicit expressions for $\mathbf{F}(x(v))$, $\mathbf{F}_s^{\bullet}(x(v))$ *and* $\mathbf{F}_s^{(2)}(x(v_1), x(v_2))$.

Topological recursion

Meromorphic function $\mathbf{G}^{(\mathrm{g},k)}(v_1,\ldots,v_k) \rightsquigarrow$ analytical continuation of

$$\mathbf{F}^{(\mathrm{g},k)}(x(v_1),\ldots,x(v_k))\prod_{i=1}^{k}x'(v_i).$$

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Recursion kernel
$\varepsilon \in \{0,1/2\}$ $\rightsquigarrow$

$$\mathbf{K}_{\varepsilon}(v_0,v) = -\frac{\mathrm{d}v}{2}\,\frac{\int_{2(\tau+\varepsilon)-v}^{v}\mathrm{d}v'\,\mathbf{G}^{(2)}(v',v_0)}{\mathbf{G}(v)+\mathbf{G}(2\tau-v)}.$$

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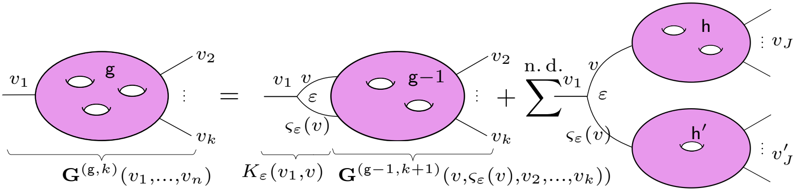
Theorem (Borot, Eynard 2011)

Let $I=\{2,\ldots,k\}$. For $2\mathrm{g}-2+k>0$, we have

$$\mathbf{G}^{(\mathrm{g},k)}(v_1,v_I) = \sum_{\varepsilon\in\{0,1/2\}}\operatorname*{Res}_{v\to\tau+\varepsilon}\mathbf{K}_\varepsilon(v_1,v)\Bigg[\mathbf{G}^{(\mathrm{g}-1,k+1)}(v,2(\tau+\varepsilon)-v,v_I) + \sum_{\substack{h+h'=g\\ J\sqcup J'=I}}^{\text{no disks}}\mathbf{G}^{(\mathrm{h},1+|J|)}(v,v_J)\mathbf{G}^{(\mathrm{h}',1+|J'|)}(2(\tau+\varepsilon)-v,v_{J'})\Bigg],$$

where "no disks" means that we exclude the terms containing disk generating series, that is (h,J) or (h',J') equal to $(0,\emptyset)$.

Decomposition of the TR invariants



Elementary blocks

$$\varepsilon \in \{0, \tfrac{1}{2}\}, \qquad \mathbf{B}_{\varepsilon,l}(v) = \frac{\partial^{2l}}{\partial v_2^{2l}} \mathbf{G}^{(2)}(v, v_2)\Big|_{v_2=\tau+\varepsilon}.$$

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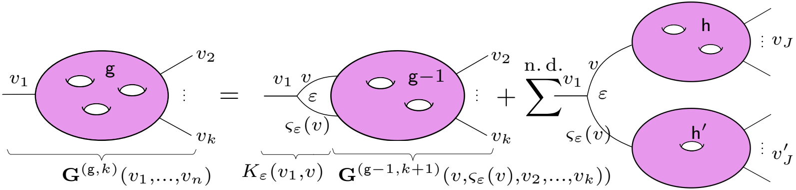

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Proposition

For $2\mathrm{g} - 2 + k > 0$, we have a decomposition

$$\mathbf{G}^{(\mathrm{g},k)}(v_1,\ldots,v_k) = \sum_{\substack{l_1,\ldots,l_k \geq 0 \\ \varepsilon_1,\ldots,\varepsilon_k \in \{0,\frac{1}{2}\}}} \mathsf{C}^{(\mathrm{g},k)}\begin{bmatrix} l_1 & \cdots & l_k \\ \varepsilon_1 & \cdots & \varepsilon_k \end{bmatrix} \prod_{i=1}^{k} \mathbf{B}_{\varepsilon_i,l_i}(v_i),$$

where the sum contains only finitely many non-zero terms.

Sketch (key idea: diagrammatic representation)

- The coefficients $\mathsf{C}^{(\mathrm{g},k)}\left[\begin{smallmatrix} l_I \\ \varepsilon_I \end{smallmatrix}\right]$ satisfy a recursion relation with two kinds of coefficients we denote K and $\tilde{K}$.

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- Diagrammatic representation by trivalent vertices with different properties of their incident edges of K, $\tilde{K}$, and the initial cases $\mathsf{C}^{(0,3)}$ and $\mathsf{C}^{(1,1)}$.
- Expression for $\mathsf{C}^{(\mathrm{g},k)}$'s as a sum over graphs composed by the previous four kinds of pieces.
- Critical behavior of the four kinds of pieces and of the elementary blocks.

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- Critical behavior of the four kinds of pieces and of the elementary blocks.
- Fixed the coloring of the k legs $\varepsilon_1, \ldots, \varepsilon_k$, determine which graph and coloring (of the graph) give the leading contribution to $\mathsf{C}^{(g,k)}$ in the critical regime.
- Critical behavior of $\mathbf{F}^{(g,k)}$ and $\mathcal{F}^{(g,k)}$, obtained summing all these contributions over the possible colorings of the legs.

Outline

Phase diagram

For fixed values (n, α, g, h), we introduce

$$u_c := \sup\{u \geq 0 \; : \; F_\ell^\bullet < \infty\}.$$

If $u_c = 1$ (resp. $u_c < 1$, $u_c > 1$), we say that the model is at a *critical* (resp. *subcritical*, *supercritical*) point.

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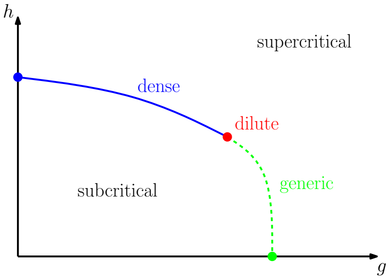

Figure : Qualitatively insensitive to the value of $n \in (0, 2)$ and α not too large.

At a critical point, $\mathcal{F}(x) = \mathbf{F}(x)$ has a singularity when $u \to 1^-$.

Small and large boundaries

- A non-generic critical point occurs when γ_+ approaches the fixed point of ς: $\gamma_+^* = \varsigma(\gamma_+^*) = \frac{1}{h(\alpha+1)}$

$$\Leftrightarrow T \to 0 \Leftrightarrow q = e^{-\frac{\pi}{T}} \to 0.$$

- (g, h) non-generic critical for $u = 1 \Rightarrow$

$$q \sim \left(\frac{1-u}{q_*}\right)^c, \text{ for } u \to 1^-, \text{ with } c = \begin{cases} \frac{1}{1-b} & \text{dense,} \\ 1 & \text{dilute.} \end{cases}$$

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- General principle: Study large maps ($V \to \infty$) $\leftrightarrow$ Study generating series close to critical points ($u \to 1$).
- Fixing lengths
$$\begin{cases} \ell_i \text{ finite} & \rightsquigarrow \text{ contour for } x_i \text{ around } \infty, x_i = x(\frac{1}{2} + \tau w_i), \\ & (x_i \text{ remains finite and away from } [\gamma_-^*, \gamma_+^*]), \\ \ell_i \to \infty & \rightsquigarrow \text{ contour for } x_i \text{ around } \gamma_+ \to \gamma_+^*, x_i = x(\tau w_i), \\ & (x_i \text{ scales with } q \to 0 \text{ such that } x_i - \gamma_+ \in O(q^{\frac{1}{2}})). \end{cases}$$

Generating series of maps

Let $k = k_0 + k_{1/2} \geq 1$ and g ≥ 0 such that $2\mathrm{g} - 2 + k > 0$.
Let $x_j = x(\varepsilon_j + \tau\varphi_j)$ for $j = 1, \ldots, k$.

$$\mathfrak{d} := \begin{cases} 1, & \text{dense,} \\ -1, & \text{dilute.} \end{cases}$$

$k_0 \rightsquigarrow$ number of large boundaries ($\varepsilon_j = 0$),
$k_{1/2} \rightsquigarrow$ number of small boundaries ($\varepsilon_j = 1/2$).

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Theorem (Borot, G-F 2016)

We have in the critical regime $q \to 0$:

$$\mathbf{F}^{(\mathrm{g},k)}(x_1, \ldots, x_k) \doteq q^{(2\mathrm{g}-2+k)(\mathfrak{d}\frac{b}{2}-1)-\frac{k}{2}+\frac{b+1}{2}k_{1/2}},$$

and for usual maps with renormalized face weights:

$$\boldsymbol{\mathcal{F}}^{(\mathrm{g},k)}(x_1, \ldots, x_k) \doteq q^{\widetilde{\beta}(\mathrm{g},k,k_{1/2})},$$

with $\widetilde{\beta}(\mathrm{g}, k, k_{1/2}) = (2\mathrm{g} - 2 + k)(\mathfrak{d}\frac{b}{2} - 1) - \frac{k}{2} + \frac{3}{4}k_{1/2}$.

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$V_{0,2}^{1/2}(\Gamma) \rightsquigarrow$ vertices in $V_{0,2}(\Gamma)$ with a small boundary, $k_{1/2}^{(0,2)} := |V_{0,2}^{1/2}(\Gamma)|$.

Fixed nesting graph Γ

We study the critical behavior of the generating series of capped cylinders $\hat{F}^{(2)}_{\ell_1,\ell_2}[s]$ and $\tilde{F}^{(2)}_{\ell_1,\ell_2}[s]$.

Theorem (Borot, G-F 2016)

When $q \to 0$, we have for the singular part with respect to u and x_i's:

$$\mathscr{F}^{(\mathrm{g},k)}_{\Gamma,\star,\mathbf{s}=\mathbf{1}}(x_1,\ldots,x_k) \doteq\sim q^{\varkappa(\mathrm{g},k,k_{1/2},k^{(0,2)}_{1/2}|b)},$$

where

$$\varkappa(\mathrm{g},k,k_{1/2},k^{(0,2)}_{1/2}|B) = \widetilde{\beta}(\mathrm{g},k,k_{1/2}) + (\tfrac{B}{2} - \tfrac{1}{4})k^{(0,2)}_{1/2}.$$

And, for the singular part with respect to s, u and x_i's:

$$\mathscr{F}^{(\mathrm{g},k)}_{\Gamma,\star,\mathbf{s}}(x_1,\ldots,x_k) \doteq\sim q^{\underline{\varkappa}(\mathrm{g},k,k_{1/2},k^{(0,2)}_{1/2}|\mathbf{s})},$$

where

$$\underline{\varkappa}(\mathrm{g},k,k_{1/2},k^{(0,2)}_{1/2}|\mathbf{s}) = \varkappa(\mathrm{g},k,k_{1/2},k^{(0,2)}_{1/2}|0) + \sum_{\mathrm{e}\in\tilde{E}} b[s(\mathrm{e})]$$

$$+ \sum_{\mathrm{v}\in V^{0}_{0,2}} b[s(\mathrm{e}_+(\mathrm{v}))] + \sum_{\mathrm{v}\in V^{1/2}_{0,2}} \tfrac{1}{2}b[s(\mathrm{e}_+(\mathrm{v}))].$$

Qualitative conclusions: Most probable nesting graphs

Remember $b \in \left(0, \frac{1}{2}\right)$ and

$$\varkappa(\mathrm{g}, k, k_{1/2}, k_{1/2}^{(0,2)} | b) = (2\mathrm{g} - 2 + k)(\mathfrak{d}\,\tfrac{b}{2} - 1) - \tfrac{k}{2} + \tfrac{3}{4} k_{1/2} + (\tfrac{b}{2} - \tfrac{1}{4}) k_{1/2}^{(0,2)}.$$

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- For $\mathcal{F}^{(\mathrm{g},k)}$, the result does not depend on the details of the map. Fixing a topology (g, k), it only depends on the number of large k_0 and small boundaries $k_{1/2}$. If $k_{1/2} = 0$, all nesting graphs for a given topology have comparable probabilities to be realized.

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- The greater the number of large boundaries k_0, the bigger the contribution.
- If we fix $(k_0, k_{1/2})$, the biggest possible $k_{1/2}^{(0,2)}$ contributes the most.

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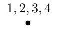
- The greater the number of large boundaries k_0, the bigger the contribution.
- If we fix $(k_0, k_{1/2})$, the biggest possible $k_{1/2}^{(0,2)}$ contributes the most.
- Biggest contributions for the contour integral come from gluing along large loops.

Outline

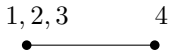
Corollary

Take (g,h) on the non-generic critical line and assume $2\mathsf{g}-2+k>0$. The generating series of connected maps of volume V with $k_{1/2}$ boundaries of finite perimeter $L_i=\ell_i$ and k_0 boundaries of perimeters $L_i=\ell_i V^{c/2}$ – for fixed positive $\boldsymbol{\ell}=(\ell_i)_{i=1}^k$ – behaves when $V\to\infty$ as

$$\Big[u^V\prod_{i=1}^{k}x_i^{-(L_i+1)}\Big]\mathscr{F}^{(\mathsf{g},k)}_{\Gamma,\star,\mathbf{1}}\ \dot\sim\ V^{[-1+c((2\mathsf{g}-2+k)(1-\eth\frac{b}{2})-\frac{1}{4}k_{1/2}+(\frac{1}{4}-\frac{b}{2})k^{(0,2)}_{1/2})]}.$$



$V^{-1+c(1-\eth b)}$



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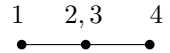

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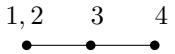

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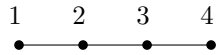

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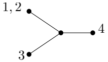

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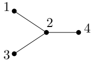
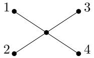


$V^{-1+c(1-\eth b+\frac{3}{4}-\frac{3b}{2})}$



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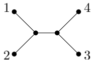


$V^{-1+c(1-\eth b+1-2b)}$

$$\mathbb{P}^{(\mathsf{g},k)}\left[\mathbf{P}|\Gamma,\star,V,\mathbf{L}\right] := \frac{\left[u^V \prod_{\mathsf{e}\in E(\Gamma)} s(\mathsf{e})^{P(\mathsf{e})} \prod_{i=1}^{k} x_i^{-(L_i+1)}\right] \mathscr{F}^{(\mathsf{g},k)}_{\Gamma,\star,\mathbf{s}}(x_1,\ldots,x_k)}{\left[u^V \prod_{i=1}^{k} x_i^{-(L_i+1)}\right] \mathscr{F}^{(\mathsf{g},k)}_{\Gamma,\star,\mathbf{1}}(x_1,\ldots,x_k)}$$

Corollary

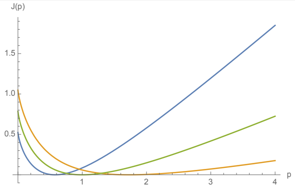
Fix positive $\mathbf{p} = \big(p(\mathsf{e})\big)_{\mathsf{e}\in E(\Gamma)}$ *such that* $p(\mathsf{e}) \ll \ln V$. *We consider the regime*

$$P(\mathsf{e}) = \frac{c\ln V\, p(\mathsf{e})}{\jmath(\mathsf{e})\pi}, \qquad \jmath(\mathsf{e}) = \begin{cases} 2 & \text{if } \mathsf{e} \text{ is incident to a vertex in } V_{0,2}^{1/2}, \\ 1 & \text{otherwise,} \end{cases}$$

In the limit $V \to \infty$, *we have*

$$\mathbb{P}^{(\mathsf{g},k)}\left[\mathbf{P}|\Gamma,\star,V,\mathbf{L}\right] \dot{\sim} \prod_{\mathsf{e}\in E(\Gamma)} (\ln V)^{-\frac{1}{2}}\, V^{-\frac{c}{\jmath(\mathsf{e})\pi} J[p(\mathsf{e})]}.$$

$$J(p) = \sup_{s\in[0,2/n]} \Big\{p\ln(s) + \arccos(ns/2) - \arccos(n/2)\Big\}$$



Qualitative conclusions: Most probable configurations

For a given nesting graph, the arm lengths typically behave like independent random variables of order $\ln V$, with large deviation function proportional to $J(p)$, which is universal (up to a factor of 2 when there is a small boundary involved).

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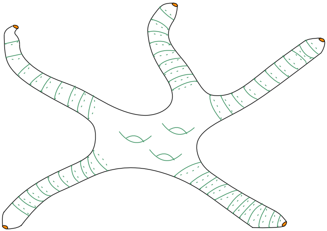

Figure : A typical map of the $O(n)$ model with small boundaries. These are most likely to be incident to distinct long arms (with $O(\ln V)$ separating loops).

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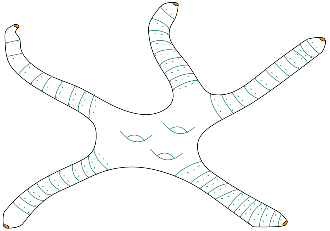

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Marked points behave as small boundaries:

$$\mathcal{F}^{(\mathrm{g},k,\bullet k')}(x_1,\ldots,x_k) \dot{\sim} q^{\widetilde{\beta}(\mathrm{g},k+k',k_{1/2}+k')}.$$

THE END

Thanks for your attention!