Duality between 2d Ising model & 3d Quantum Gravity

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Work with F. Costantino & V. Bonzom - arXiv:1504.02822 [math-ph]

& work in progress with V. Bonzom & J. Ben Geloun

Duality between 2d Ising and 3d Quantum Gravity

For a planar 3-valent graph:

Generating function for spin network evaluation (chromatic evaluation) = Partition function for 2d Ising model (with variable edge couplings)

3d quantum gravity amplitude for 2d boundary state

Duality between 2d Ising & 3d QG

What's in for 3d Quantum Gravity?

Ising model very well-studied & understood

↓

- Coarse-graining methods and renormalisation for 3d quantum gravity on the boundary
- Phase diagram and phase transitions ?
- Continuum limit as a 2d CFT (coset WZW theory), interpretation as example of AdS/CFT (away from criticality)
- More interplay between quantum gravity & condensed matter models

What's in it for the 2d Ising model?

Quantum gravity is about geometry

↓

- A new realization of the Ising partition function, offering a new perspective on the model
- Relation between coarse-graining of Ising model & topological invariance of 3d quantum gravity ?
- Geometrical interpretation for Fisher zeroes and critical couplings ?

Duality between 2d Ising and 3d Quantum Gravity

Outline:

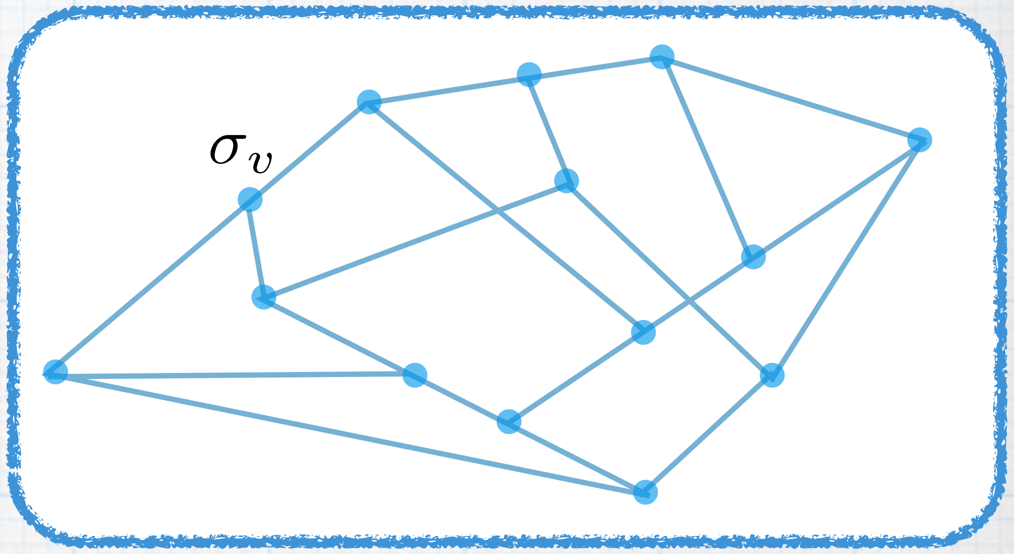
1. Ising partition function: loop expansion & fermionic integral
2. 3d QG Ponzano-Regge amplitudes as spin network evaluations
3. Generating function for spin networks
4. Westbury theorem realized through Supersymmetry
5. Large spin asymptotic & geometric formula for Fisher zeroes

The Ising Model Partition Function

Let's start with the Ising model

The Ising Model Partition Function

On same graph, put « spins » on vertices: $\sigma_v = \pm 1 \in \mathbb{Z}_2$



$$Z_\Gamma^{Ising}(\{y_e\}) = \sum_\sigma \exp\left(\sum_e y_e \sigma_{s(e)}\sigma_{t(e)}\right)$$

Can define high temperature expansion...

$$Z_\Gamma^{Ising}(\{y_e\}) = (\prod_e \cosh(y_e)) \sum_\sigma \prod_e (1 + \tanh(y_e)\sigma_{s(e)}\sigma_{t(e)})$$

... as sum over loops:

$$Z_\Gamma^{Ising}(\{y_e\}) = 2^V (\prod_e \cosh(y_e)) \sum_{\gamma \in \mathcal{G}} \prod_{e \in \gamma} Y_e \quad \text{with } Y_e = \tanh y_e$$

The Ising Model as a Fermion Path Integral

Two-level system naturally represented in terms of fermions.

Here explicitly: $$Z_\Gamma^{Ising}(\{y_e\}) = 2^V \prod_e \cosh(y_e)\, Z_f(\{X_\alpha\})$$

$$Z_f(\{X_\alpha\}) = \int \prod_{ev} d\psi_{ev}\, \exp\left(\sum_e \psi_{s(e)}\psi_{t(e)} + \sum_\alpha X_\alpha\, \psi_{s(\alpha)}\psi_{t(\alpha)}\right)$$

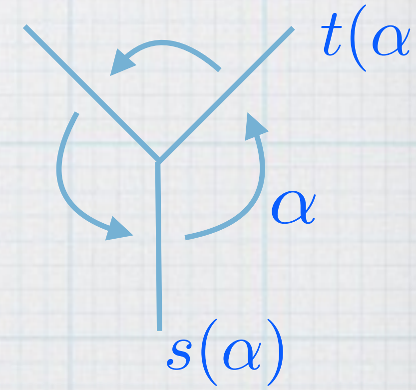
with angle couplings: $$X_\alpha = \sqrt{Y_{s(\alpha)}Y_{t(\alpha)}}$$

Choose edge orientations:

Kasteleyn orientation (with odd number of clockwise edges around each face)

Choose cyclic orientation around each vertex:

anti-clockwise cyclic ordering



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- In terms of odd-Grassman variables, one psi for each half-edge
- Proof through expansion of exp and careful tracking of signs due commuting psi's

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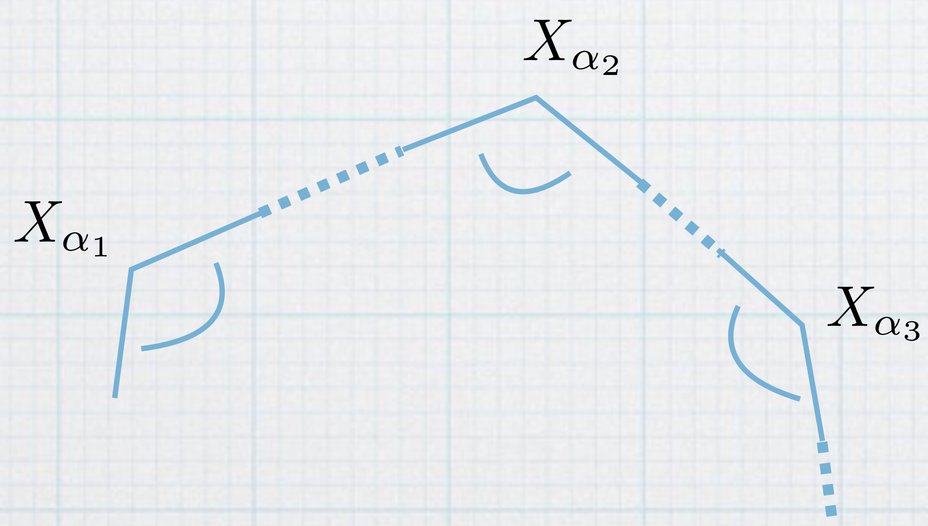
→ We glue angles along edges to form loops



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→ We glue angles along edges to form loops, or vice-versa

And for our purpose, we look at squared partition function :

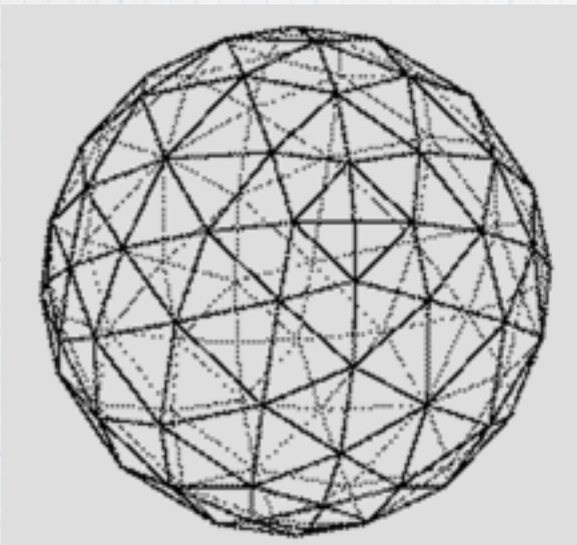
$$(Z_f)^2 = \int \prod_{ev} [d\psi d\eta d\bar{\psi} d\bar{\eta}]_e^v\, e^{\sum_{e,v} \psi_e^v \bar{\eta}_e^v + \bar{\psi}_e^v \eta_e^v}$$

$$e^{-\sum_e \bar{\psi}_{s(e)}\bar{\psi}_{t(e)} + \bar{\eta}_{s(e)}\bar{\eta}_{t(e)}}\, e^{\sum_\alpha X_\alpha (\psi_{s(\alpha)}\psi_{t(\alpha)} + \eta_{s(\alpha)}\eta_{t(\alpha)})}$$

Let's turn to quantum gravity side

3d Quantum Gravity: Spinfoams & Spin Networks

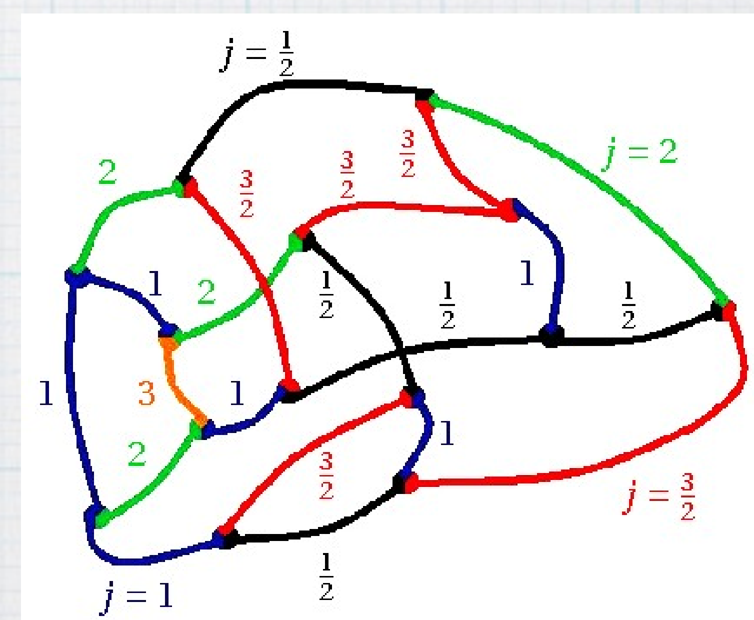
3d gravity as a TQFT can be discretized & exactly quantized:



- 3d triangulation for the bulk
- Half-integer spins on edges j_e interpreted as quantized edge lengths
- Amplitude as product of 6j-symbols

$$\mathcal{A}_\Delta = \sum_{\{j_e\}} \prod_e (2j_e + 1) \prod_T \{6j\}$$

- Boundary 2d triangulated surface or dual 3-valent graph
- Spins on boundary edges or dual links: **boundary spin network**



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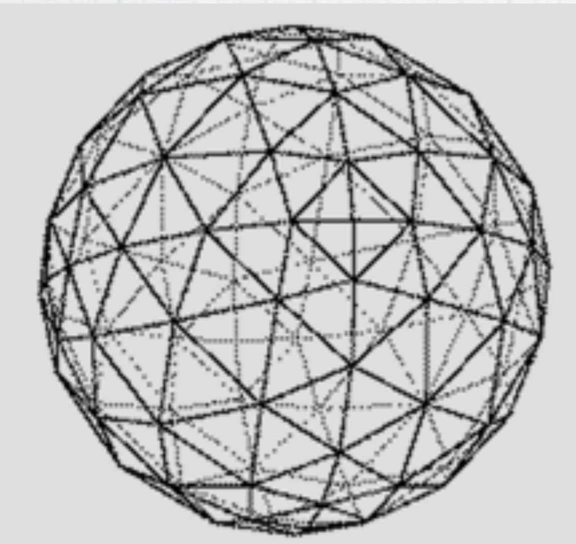
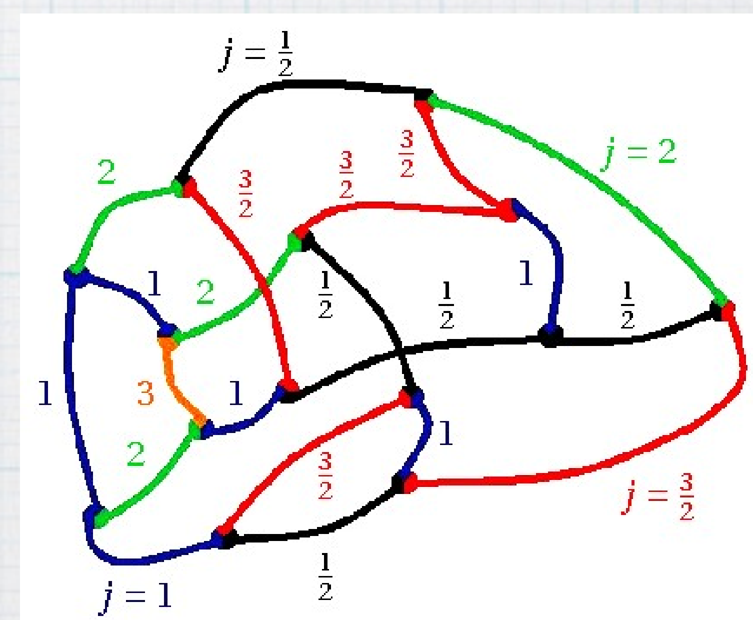


- Assume trivial topology (3-ball)
- Theory is topological, with no local degree of freedom, so everything gets projected onto the boundary

For a trivial topology, 3d QG amplitude expressed explicitly in terms of boundary data:

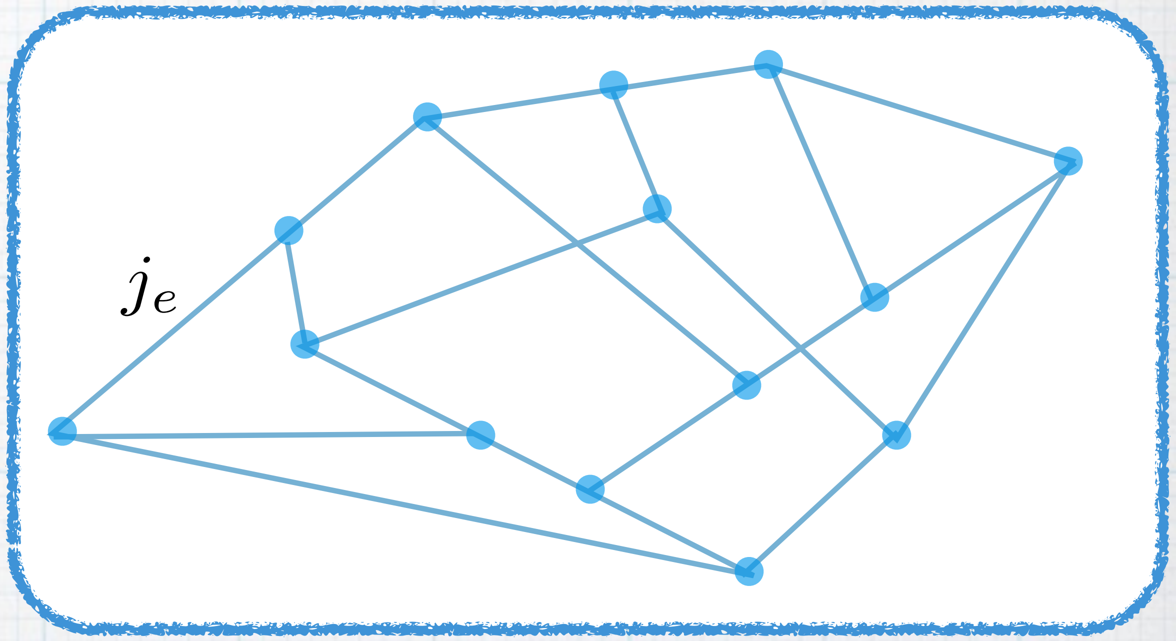
evaluation of boundary spin network



Spin Networks Evaluations

Consider 3-valent planar connected oriented boundary graph

Spin network evaluation is a 3nj symbol, obtained by gluing Clebsh-Gordan coefficients:



$$s^\Gamma(\{j_e\}) = \psi^\Gamma_{\{j_e\}}(\mathbb{1}) = \sum_{\{m_e\}} \prod_e (-1)^{j_e - m_e} \prod_v \begin{pmatrix} j_{e_1^v} & j_{e_2^v} & j_{e_3^v} \\ \epsilon^v_{e_1} m_{e_1^v} & \epsilon^v_{e_2} m_{e_2^v} & \epsilon^v_{e_3} m_{e_3^v} \end{pmatrix}$$

Choose Kasteleyn orientation on planar graph to fix signs:
show evaluation is independent of choice of orientation & matches standard normalizations (chromatic evaluation, unitary evaluation, ...)

Generating Function for Spin Network Evaluations

Consider **3-valent planar connected oriented boundary graph**

Define generating function for 3nj's using specific combinatorial weights:

$$Z_\Gamma^{Spin}(\{Y_e\}) = \sum_{\{j_e\}} \sqrt{\frac{\prod_v (J_v+1)!}{\prod_{ev}(J_v - 2j_e)!}}\, s^\Gamma(\{j_e\}) \prod_e Y_e^{2j_e}$$

That's a specific choice of **boundary state with superposition of spins**

Usually, spins = length of edges of triangulation dual to graph

→ Semi-classical **coherent states** **peaked on what triangulations (determined by the couplings Y) ?**

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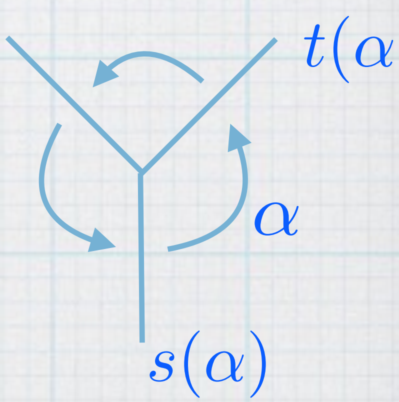
Get it from gluing the 3j-symbol generating functions using Gaussian weights:

$$\sum_{j_e,m_e} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \sqrt{(J+1)!} \prod_e \frac{Y_e^{j_e} z_e^{j_e+m_e} w_e^{j_e-m_e}}{\sqrt{(J-2j_e)!(j_e-m_e)!(j_e+m_e)!}}$$

$$= \exp \sum_\alpha X_\alpha (z_{s(\alpha)} w_{t(\alpha)} - w_{s(\alpha)} z_{t(\alpha)})$$



$$X_\alpha = \sqrt{Y_{s(\alpha)} Y_{t(\alpha)}}$$

Generating Function for Spin Network Evaluations

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Get it from gluing the 3j-symbol generating functions using Gaussian weights:

$$Z_\Gamma^{Spin}(\{Y_e\}) = \int \prod_{ev} \frac{d^2 z_{ev} d^2 w_{ev}}{\pi^2}\, e^{-\sum_{ev}(|z_{ev}|^2+|w_{ev}|^2)}$$

$$e^{-\sum_e(\bar{z}_{s(e)}\bar{w}_{t(e)}-\bar{w}_{s(e)}\bar{z}_{t(e)})+\sum_\alpha X_\alpha(z_{s(\alpha)}w_{t(\alpha)}-w_{s(\alpha)}z_{t(\alpha)})}$$

It's a Gaussian integral

« Simply » have to compute the determinant …

Matching Loop Expansions

The Hessian can be computed explicitly to prove:

$$(Z_f)^2\, Z_\Gamma^{Spin} = 1 \qquad Z_f = \sum_{\gamma\in\mathcal{G}} \prod_{\alpha\in\gamma} X_\alpha = \sum_{\gamma\in\mathcal{G}} \prod_{e\in\gamma} Y_e$$

$$\boxed{(Z^{Ising})^2\, Z^{Spin} = 2^{2V} \prod_e \cosh(y_e)^2}$$

→ **Duality between Ising model & Spin Evaluations**

That's Westbury theorem !

Duality through Supersymmetry

We can introduce a meta-theory combining

- Ising model ⟷ Fermions
- Spin networks ⟷ Bosons

$$\mathcal{Z}_\Gamma = (Z_f)^2\, Z^{Spin} = \int dz\, dw\, d\psi\, d\eta\, e^{S[\{z,w,\psi,\eta\}_{ev}]}$$

$$S = \sum_{e,v} \lambda_{e,v} K_{e,v} + \sum_e \mu_e S_e - \sum_\alpha X_\alpha S_\alpha$$

$$\left|\begin{array}{lcl} K_{e,v} & = & |z_e^v|^2 + |w_e^v|^2 - \psi_e^v \bar{\eta}_e^v - \bar{\psi}_e^v \eta_e^v \\ S_e & = & \bar{z}_{s(e)}\bar{w}_{t(e)} - \bar{w}_{s(e)}\bar{z}_{t(e)} + \bar{\psi}_{s(e)}\bar{\psi}_{t(e)} + \bar{\eta}_{s(e)}\bar{\eta}_{t(e)} \\ S_\alpha & = & z_{s(\alpha)}w_{t(\alpha)} - w_{s(\alpha)}z_{t(\alpha)} + \psi_{s(\alpha)}\psi_{t(\alpha)} + \eta_{s(\alpha)}\eta_{t(\alpha)} \end{array}\right.$$

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We define a supersymmetry generator, acting on each half-edge $i = (ev)$:

$$\left| \begin{array}{ccr} Qz_i & = & \psi_i \\ Qw_i & = & \eta_i \\ Q\psi_i & = & w_i \\ Q\eta_i & = & -z_i \end{array} \right.$$

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$$S = \sum_{e,v} \lambda_{e,v} K_{e,v} + \sum_e \mu_e S_e - \sum_\alpha X_\alpha S_\alpha$$

All terms are both Q-closed & Q-exact: $QK_{e,v} = QS_e = QS_\alpha = 0$

$$\left|\begin{array}{lcl} K_{e,v} & = & Q\,(\psi\bar{w} - \eta\bar{z}) \\ S_e & = & Q\,(\overline{z\psi} + \overline{w\eta}) \\ S_\alpha & = & Q\,(z\psi + w\eta) \end{array}\right. \longrightarrow \frac{\partial \mathcal{Z}_\Gamma}{\partial \lambda_{e,v}} = \frac{\partial \mathcal{Z}_\Gamma}{\partial \mu_e} = \frac{\partial \mathcal{Z}_\Gamma}{\partial X_\alpha} = 0$$

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$$S = \sum_{e,v} \lambda_{e,v} K_{e,v} + \sum_e \mu_e S_e - \sum_\alpha X_\alpha S_\alpha$$

It is trivial due to a supersymmetry !

$$\longrightarrow \quad \frac{\partial \mathcal{Z}_\Gamma}{\partial \lambda_{e,v}} = \frac{\partial \mathcal{Z}_\Gamma}{\partial \mu_e} = \frac{\partial \mathcal{Z}_\Gamma}{\partial X_\alpha} = 0$$

Critical Ising & Spin Network Saddle Points

Let's see how to use this relation !

$$\left(Z^{Ising}\right)^{-2} = Z^{Spin} = \sum_{\{j_e\}} \dots$$

- Poles of spin network generating function give Zeroes of Ising
- Possible to look at saddle approximation for the sum over spins, i.e. study spin network evaluations in the large spin asymptotic

Critical Ising & Spin Network Saddle Points

Let's come back to the combinatorial definition of the generating function of spin network evaluations:

$$Z_\Gamma^{Spin}(\{Y_e\}) = \sum_{\{j_e\}} \sqrt{\frac{\prod_v (J_v+1)!}{\prod_{ev}(J_v-2j_e)!}}\, s^\Gamma(\{j_e\}) \prod_e Y_e^{2j_e}$$

Spin distribution defined by statistical weight ?

$$\rho(\{j_e\}) = \sqrt{\frac{\prod_v (J_v+1)!}{\prod_{ev}(J_v-2j_e)!}} \prod_e Y_e^{2j_e}$$

Saddle point? Geometrical Interpretation?

Critical Ising & Spin Network Saddle Points

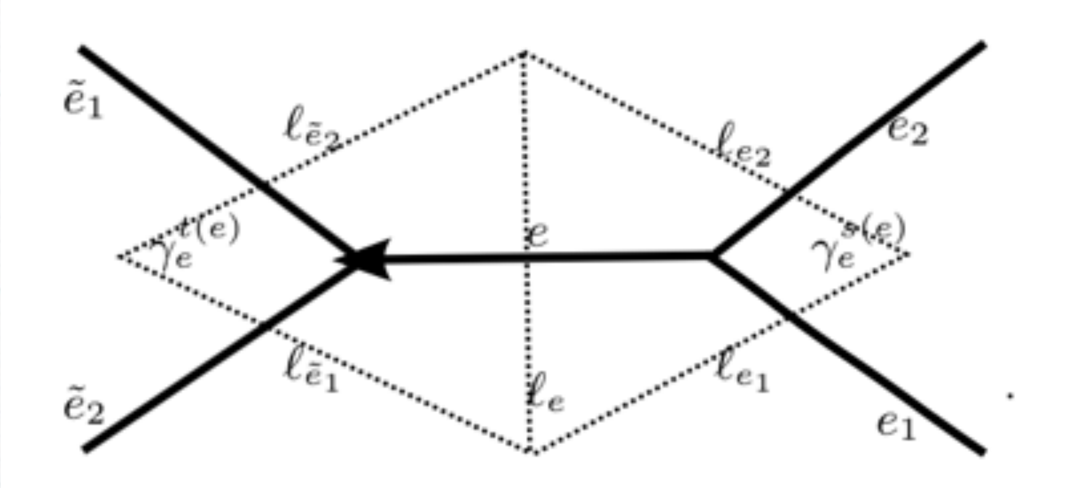
We proceed as usual in quantum gravity:

- Large spin approx, Stirling formula
- Look for stationary point(s)
- Interpret spins as lengths

Stationary points are when spins j_e are the edge lengths of a triangulation determined by the edge couplings Y_e

We have a condition between the edge couplings Y_e and the triangulation angles:

$$Y_e^2 = \tan\frac{\gamma_e^{s(e)}}{2}\,\tan\frac{\gamma_e^{t(e)}}{2}$$



Critical Ising & Spin Network Saddle Points

We get a stationary point when spins j_e are length of a triangulation if the edge couplings Y_e are determined by the condition in terms of the triangulation angles:

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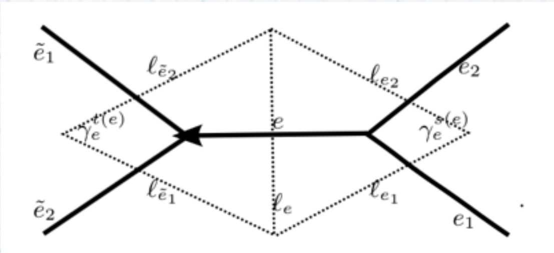
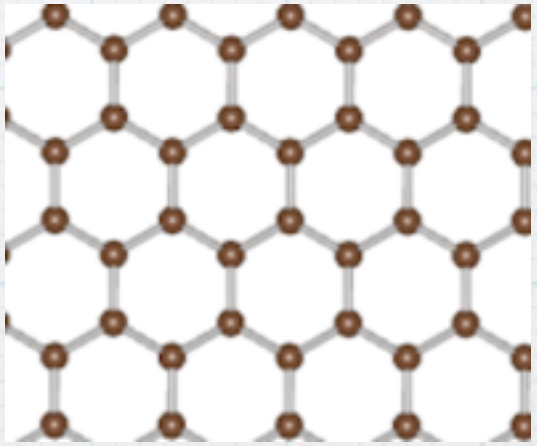


- Regular honeycomb network

$$Y = \frac{1}{\sqrt{3}} = Y^{critical}$$



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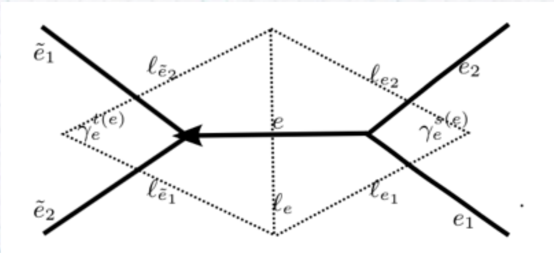
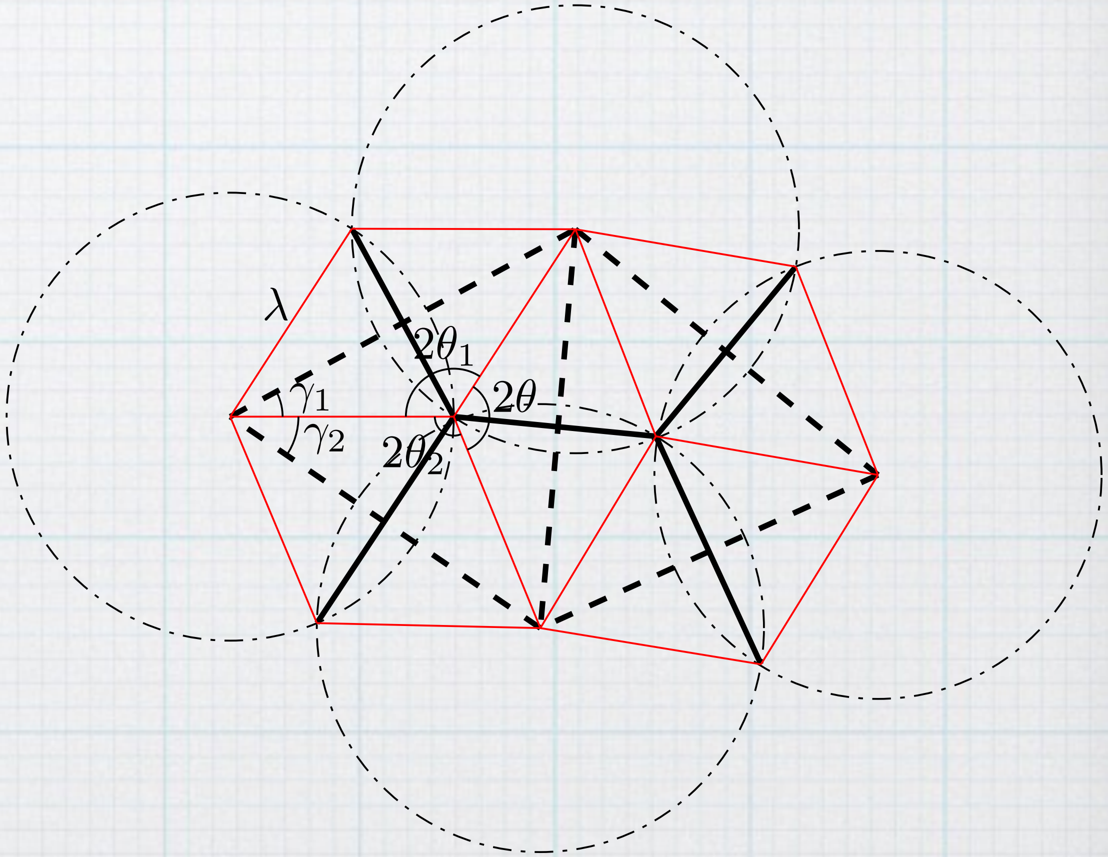


- Regular honeycomb network

$$Y = \frac{1}{\sqrt{3}} = Y^{critical}$$

- Also tested on isoradial graphs

$$Y_e^c = \tan\frac{\gamma_e}{2} = \tan\frac{\theta_e}{2}$$



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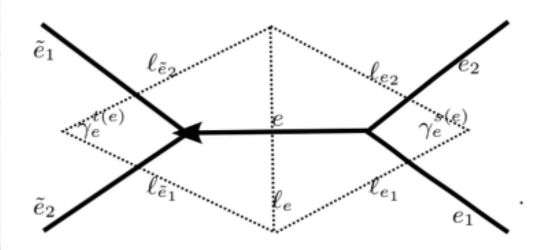

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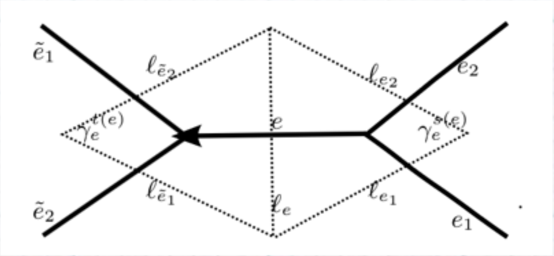
$$Y = \frac{1}{\sqrt{3}} = Y^{critical}$$

- Also isoradial graphs !

$$Y_e^c = \tan \frac{\gamma_e}{2} = \tan \frac{\theta_e}{2}$$

So what's happening?

$$Y_e^2 = \tan\frac{\gamma_e^{s(e)}}{2}\,\tan\frac{\gamma_e^{t(e)}}{2}$$



Admissible geometric couplings Y_e^{geom}

Scale invariant saddle points in spins j_e

Pole in the generating function $Z^{Spin} \to \infty$

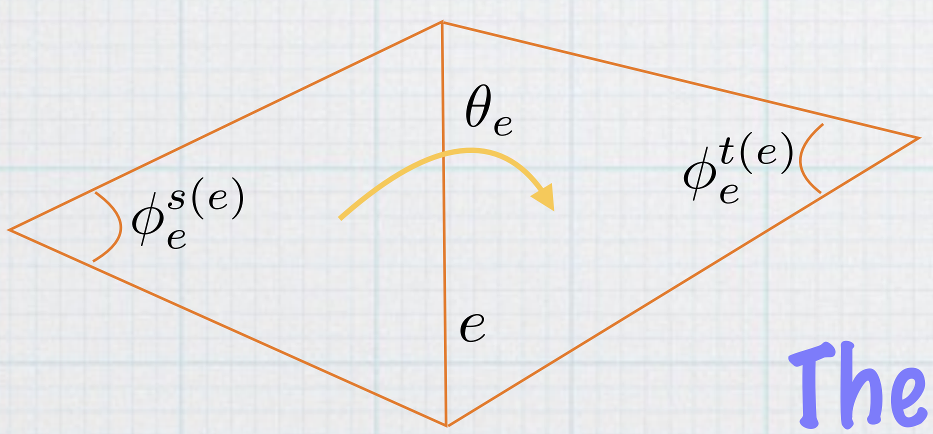
Zero of Ising partition function i.e. critical couplings Y_e^c

Critical Ising & Spin Network Saddle Points

Let's do this cleanly !

- Fisher zeroes are usually complex
- And we need to take into account the large spin asymptotic of the spin network evaluations (given by Regge action)

$$Z_\Gamma^{Spin}(\{Y_e\}) = \sum_{\{j_e\}} \sqrt{\frac{\prod_v (J_v+1)!}{\prod_{ev}(J_v-2j_e)!}}\, s^\Gamma(\{j_e\}) \prod_e Y_e^{2j_e}$$



$$s^\Gamma(\{j_e\}) \propto \cos\left(\sum_e j_e \theta_e\right)$$

The triangulation is planar but not flat !!

Critical Ising & Spin Network Saddle Points

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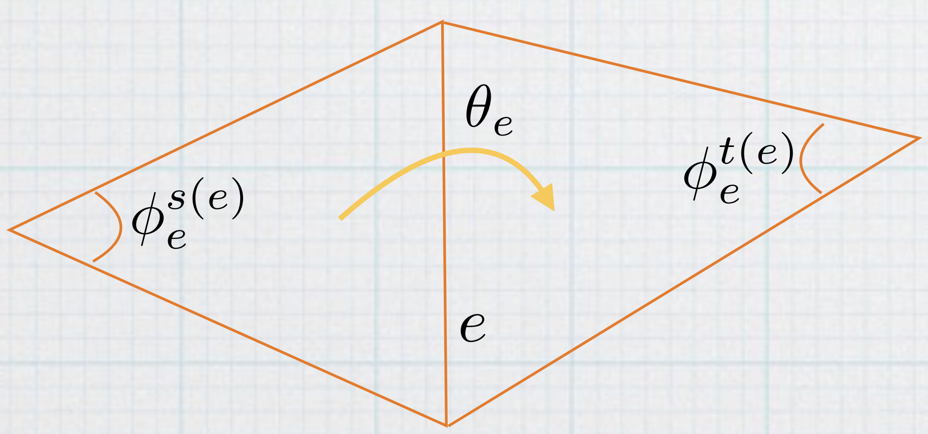
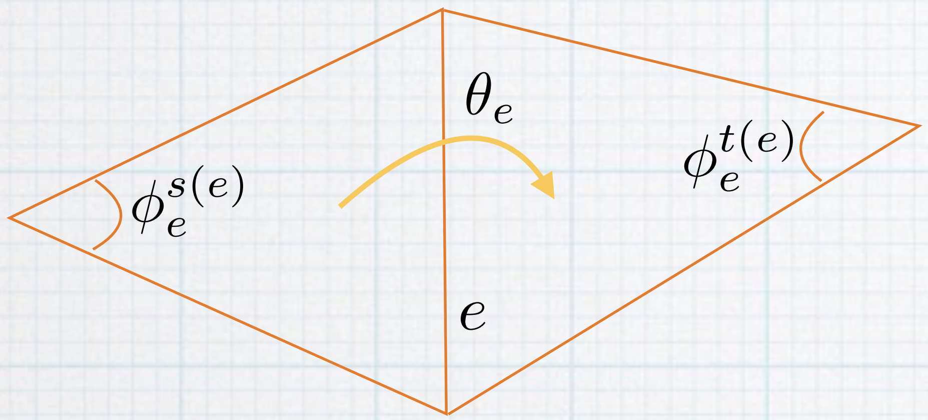


$$Y_e^c = e^{\epsilon \frac{i}{2}\theta_e} \sqrt{\tan\frac{\phi_e^{s(e)}}{2} \tan\frac{\phi_e^{t(e)}}{2}}$$

$\epsilon = \pm$ global sign

Critical Ising & Spin Network Saddle Points



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$\epsilon = \pm$ **global sign**

- Phase represents extrinsic curvature of surface in 3d space
- Real zeroes corresponds to special case of flat triangulation (usually happens in thermodynamical limit)

The 3d embedding is important!

The Tetrahedron & the 6j Symbol

Test all this on the Tetrahedron !

- Look at generating function for 6j symbols
- Study saddle points of combining both weight & 6j symbol with Regge action at large spins
- Provide geometrical interpretation for Fisher zeroes on tetrahedron graph

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \underset{j\to\infty}{\sim} \frac{1}{\sqrt{12\pi V}} \cos\left(\sum_{k=1}^{6}\left(j_k+\frac{1}{2}\right)\theta_k+\frac{\pi}{4}\right)$$

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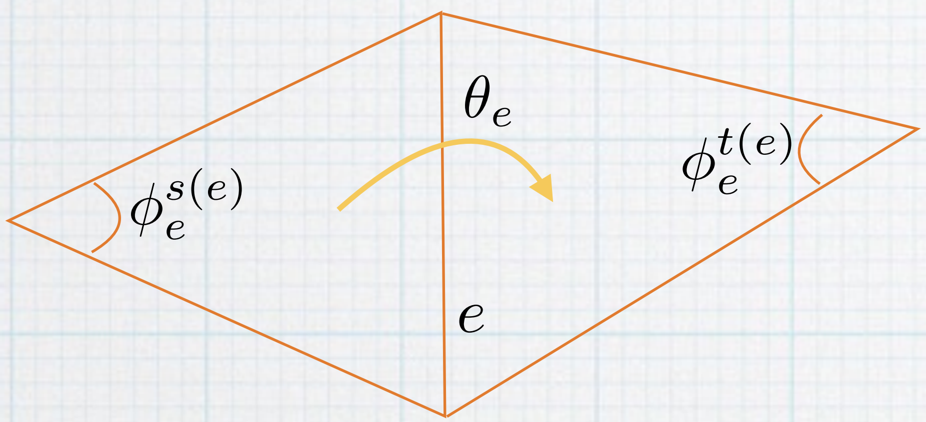
Work with V. Bonzom
to appear soon

The Tetrahedron & the 6j Symbol

Critical couplings for Ising are complex, with phase given by 3d dihedral angles and modulus given by 2d triangle angles



$$Y_e^c = e^{\epsilon \frac{i}{2}\theta_e} \sqrt{\tan\frac{\phi_e^{s(e)}}{2}\tan\frac{\phi_e^{t(e)}}{2}}$$

These are roots of the tetrahedron loop polynomial :

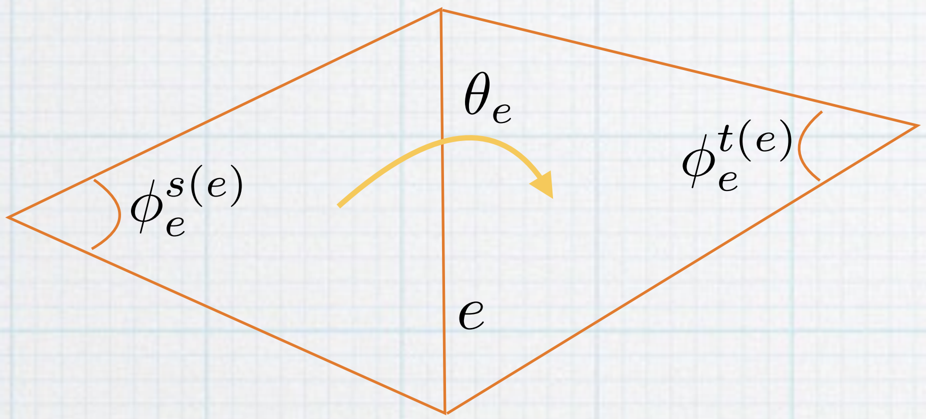
$$P[Y_e] = 1 + Y_1Y_2Y_6 + Y_1Y_3Y_5 + Y_2Y_3Y_4 + Y_4Y_5Y_6 + Y_1Y_4Y_2Y_5 + Y_2Y_5Y_3Y_6 + Y_1Y_4Y_3Y_6$$

Direct proof from spherical trigonometry

and this only gives a 5d manifold within the 10d space of solutions

The Tetrahedron & the 6j Symbol

Critical couplings for Ising are complex, with phase given by 3d dihedral angles and modulus given by 2d triangle angles



$$Y_e^c = e^{\epsilon \frac{i}{2}\theta_e} \sqrt{\tan\frac{\phi_e^{s(e)}}{2} \tan\frac{\phi_e^{t(e)}}{2}}$$

These are roots of the tetrahedron loop polynomial :

$$P[Y_e] = 1 + Y_1Y_2Y_6 + Y_1Y_3Y_5 + Y_2Y_3Y_4 + Y_4Y_5Y_6 + Y_1Y_4Y_2Y_5 + Y_2Y_5Y_3Y_6 + Y_1Y_4Y_3Y_6$$

Direct proof from spherical trigonometry

and this only gives a 5d manifold within the 10d space of solutions

Have to go to complex tetrahedra ! Work in progress

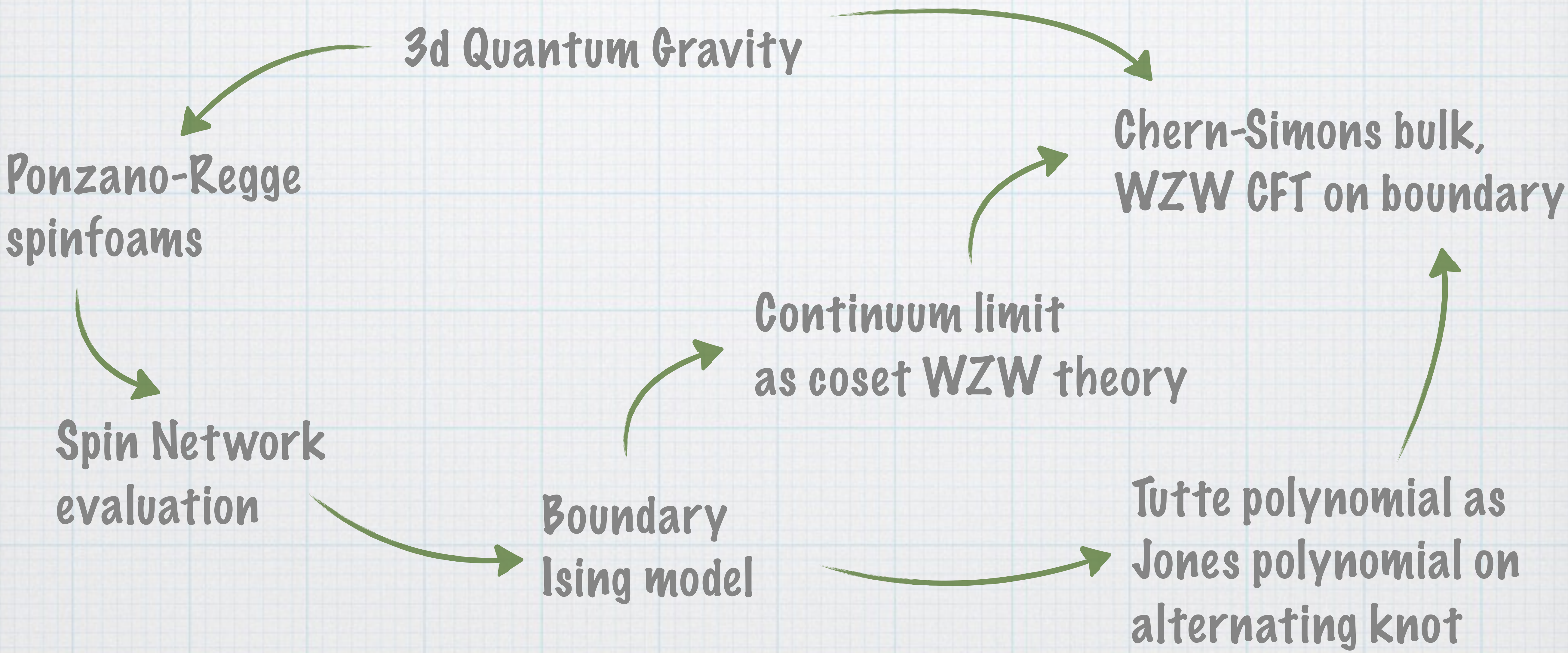
Ising-QG Duality: Extensions & Prospects

- Technical improvements: arbitrary valence, non-planar graphs, q-deformation, ... ?
- Generalization beyond the Ising model: add magnetic field (fugacity), work out dual to Potts model
- Application to Spin glasses ?
- More on the supersymmetry, higher order localized integrals
- Full geometrical characterization of all complex Fisher zeroes
- Boundary CFT for 3d spin foams from continuum limit of Ising model as WZW field theory

Continuum limit and boundary CFT

Use known continuum limit of Ising models to derive boundary CFT description of Ponzano-Regge spinfoam models at critical point

Let's try to close the loop :



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- Relation between coarse-graining Ising & topological invariance of Ponzano-Regge models

From Coarse-Graining Ising to boundary Pachner moves

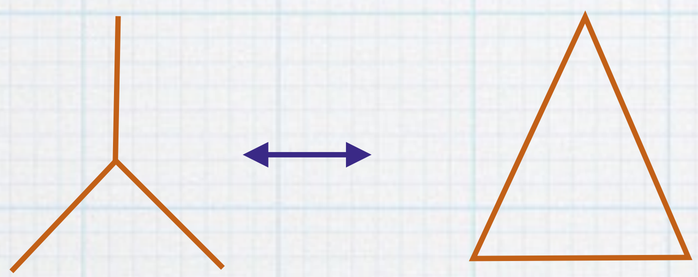
Natural application of duality between Ising models & spin networks :

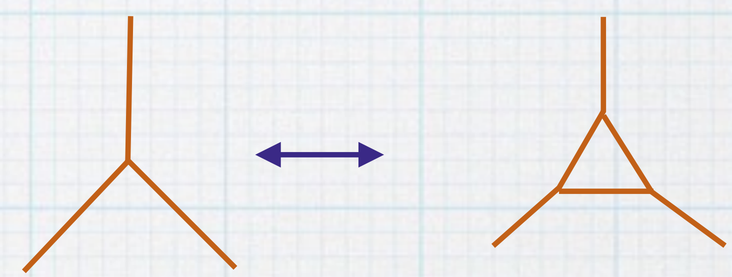
COARSE-GRAINING

Work with J. Ben Geloun
to appear hopefully soon

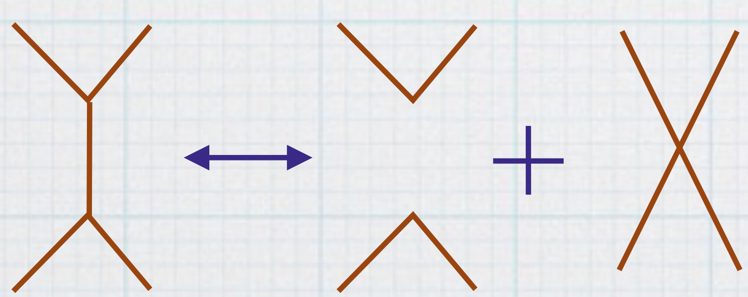
Star-Triangle relation → **3-1 Pachner move**

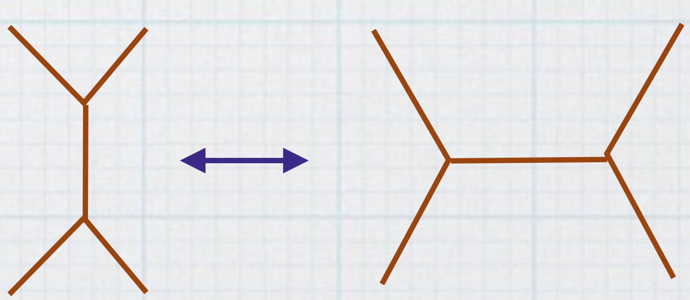
Related to
Yang-Baxter
equation





(multivariate) Tutte recursion relation → **2-2 Pachner move**





Ising-QG Duality: Extensions & Prospects

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- Generalization beyond the Ising model: add magnetic field (fugacity), work out dual to Potts model
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- More on the supersymmetry, higher order localized integrals
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- Relation between coarse-graining Ising & topological invariance of Ponzano-Regge models
- Ising duality as a realization of AdS/CFT correspondance

Thank you for your attention !!

3d Quantum Gravity: Spinfoams & Spin Networks

3d gravity as a TQFT:

$$S[A,e]=\int_{\mathcal{M}}\mathrm{Tr}\, e\wedge F[A]=\int_{\mathcal{M}}\delta_{ij}\epsilon^{abc}\, e^i_a\, F^j_{bc}[A]$$

- Triad e 1-form with value in $\mathfrak{su}(2)$ Lie algebra
- $\mathrm{SU}(2)$ Connection A with curvature $F[A]=dA+A\wedge A$

Topological field theory with no local degrees of freedom

- SU(2) Gauge invariant & Diffeomorphism invariant
- Theory of a pure flat connection F[A]=0

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Topological field theory with no local degrees of freedom

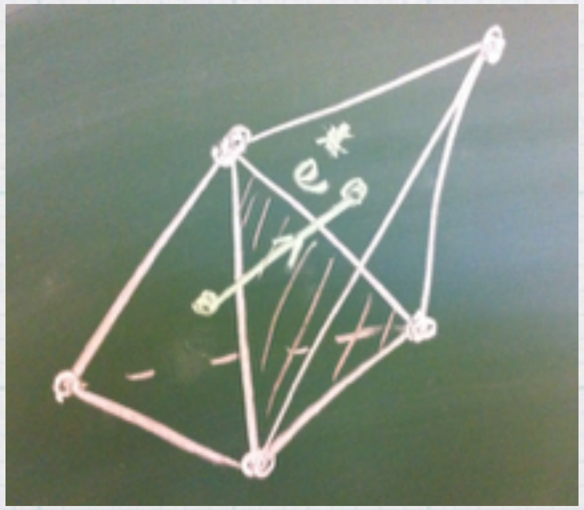
- SU(2) Gauge invariant & Diffeomorphism invariant
- Theory of a pure flat connection F[A]=0
- If add volume term, equivalent to Chern-Simons theory

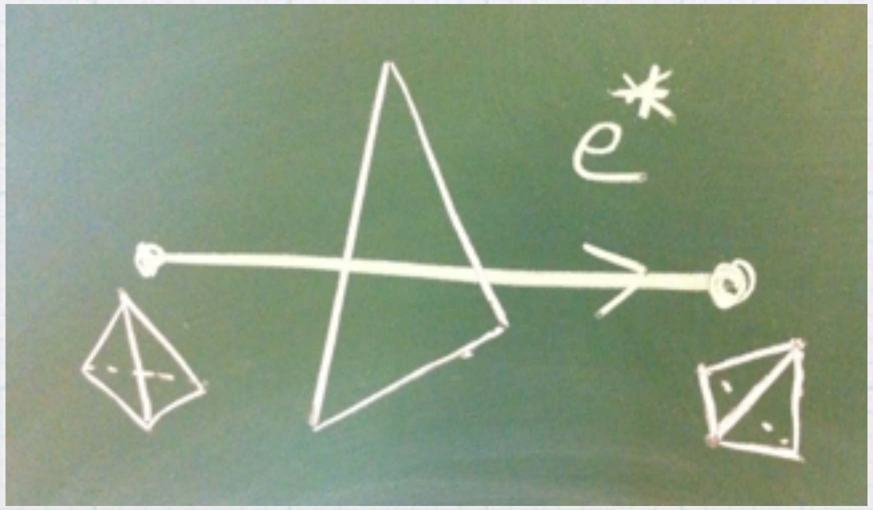
3d Quantum Gravity: Spinfoams & Spin Networks

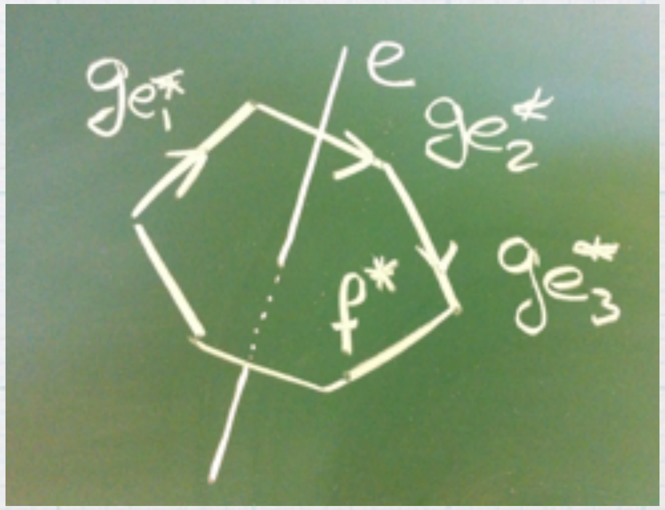
3d gravity as a TQFT can be exactly spinfoam quantized:

Topological field theory → Can be discretized exactly

1. Choose a 3d triangulation (cellular decomposition works too)
2. Define dual 2-complex, the **spinfoam**
3. Discretize connection along dual edges $g_{e^*} \in \mathrm{SU}(2)$
4. Discretize triad along edges $X_e \in \mathfrak{su}(2)$







3d Quantum Gravity: Spinfoams & Spin Networks

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Topological field theory ⟶ Can be discretized exactly

- Connection along dual edges $g_{e^*} \in \mathrm{SU}(2)$
- Triad along edges $X_e \in \mathfrak{su}(2)$

X's are Lagrange multipliers imposing flatness of connection around dual faces (i.e around edges)

$$G_e = G_{f^*} = \overrightarrow{\prod_{e^*\in\partial f^*}} g_{e^*}$$

$$Z = \int \mathrm{d}e\mathrm{d}A\, e^{iS[e,A]} = \int \mathrm{d}A\, \delta(F[A]) = \int \prod_{e^*} \mathrm{d}g_{e^*} \prod_e \delta(G_e)$$

3d Quantum Gravity: Spinfoams & Spin Networks

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We decompose onto irreps of SU(2) i.e **spins** :

$$Z = \int \prod_{e^*} \mathrm{d}g_{e^*} \sum_{\{j_e \in \frac{\mathbb{N}}{2}\}} \prod_{e} (2j_e+1)\chi_{j_e}(G_e)$$

and we integrate over all group elements,
leaving us with spin recoupling symbols

3d Quantum Gravity: Spinfoams & Spin Networks

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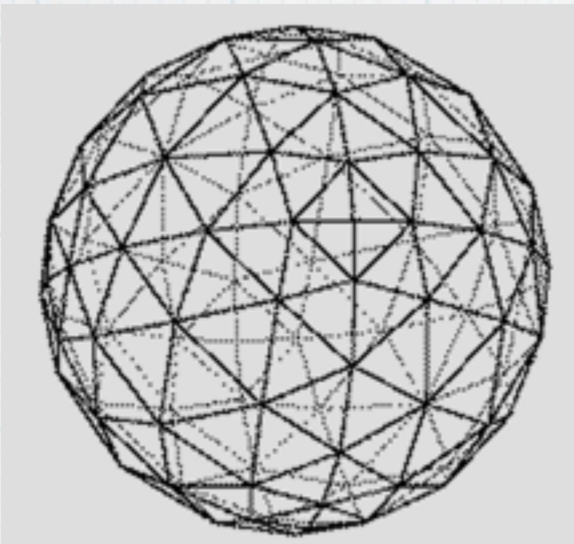
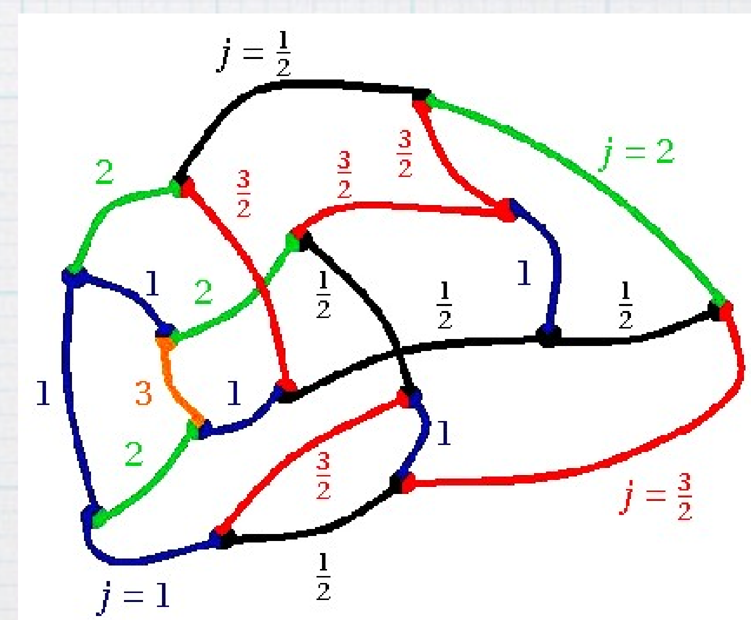


- 3d bulk triangulations or dual 2-complex
- Spins on edges j_e
- Amplitude as product of 6j-symbols

$$\mathcal{A}_\Delta = \sum_{\{j_e\}} \prod_e (2j_e+1) \prod_T \{6j\}$$

- Boundary 2d triangulated surface or dual 3-valent graph
- Spins on boundary edges or dual links: **boundary spin network**



3d Quantum Gravity: Spinfoams & Spin Networks

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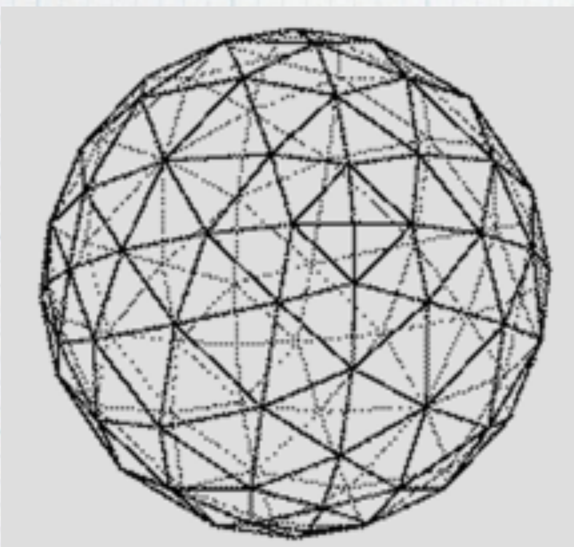
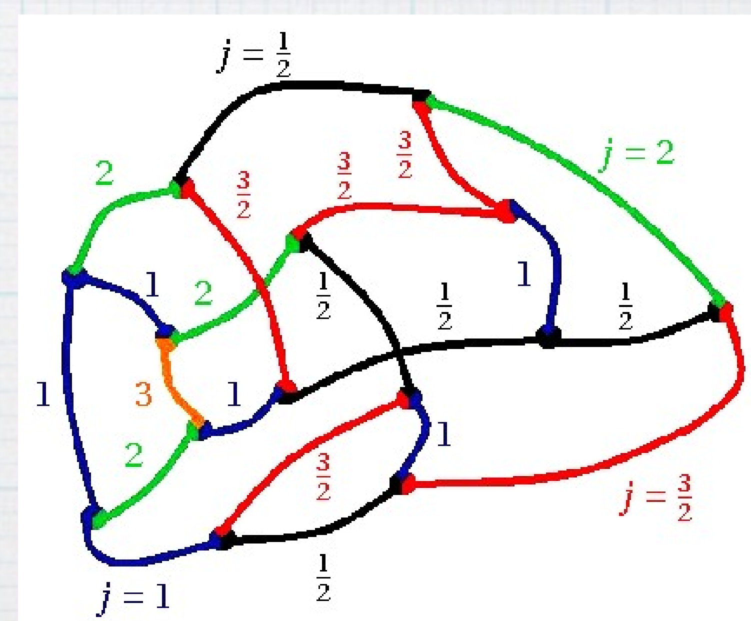


- Assume trivial spherical topology
- Use topological invariance to gauge fix bulk
- PR amplitude becomes projector on flat connection

$$\mathcal{A}_\Delta = \mathcal{A}_{\partial\Delta} = \langle \mathbb{1} | \psi \rangle = \psi(\mathbb{1})$$

For a trivial topology, amplitude expressed explicitly in terms of boundary data:

evaluation of boundary spin network



Mapping Spin Averages to Ising correlations

Compare spin insertions in both partition functions :

$$\langle \sigma_{v_1}\, \sigma_{v_2} \cdots \sigma_{v_n} \rangle = \frac{1}{Z^{Ising}} \sum_{\sigma} \sigma_{v_1}\, \sigma_{v_2} \cdots \sigma_{v_n}\; e^{\sum_e y_e \sigma_s(e) \sigma_t(e)}$$

$$\langle j_{e_1}^{n_1} j_{e_2}^{n_2} \cdots j_{e_k}^{n_k} \rangle = \frac{1}{Z^{Spin}} \sum_{\{j_e\}} j_{e_1}^{n_1} j_{e_2}^{n_2} \cdots j_{e_k}^{n_k}\; s(\Gamma, \{j_e\}) \mathcal{W}(\{j_e\}) \prod_e (\tanh y_e)^{2j_e}$$

Can get general relation :

$$\langle j_e \rangle = \sinh y_e \big(\sinh y_e - \cosh y_e \, \langle \sigma_{s(e)} \sigma_{t(e)} \rangle \big)$$

$$\langle \sigma_v \sigma_w \rangle_c^{(\mathcal{P})} = \frac{-2^{n-1}}{\prod_{e\in\mathcal{P}} \sinh(2j_e)} \langle \prod_{e\in\mathcal{P}} (2j_e) \rangle_c^{(\mathcal{P})}$$

Mapping Spin Averages to Ising correlations

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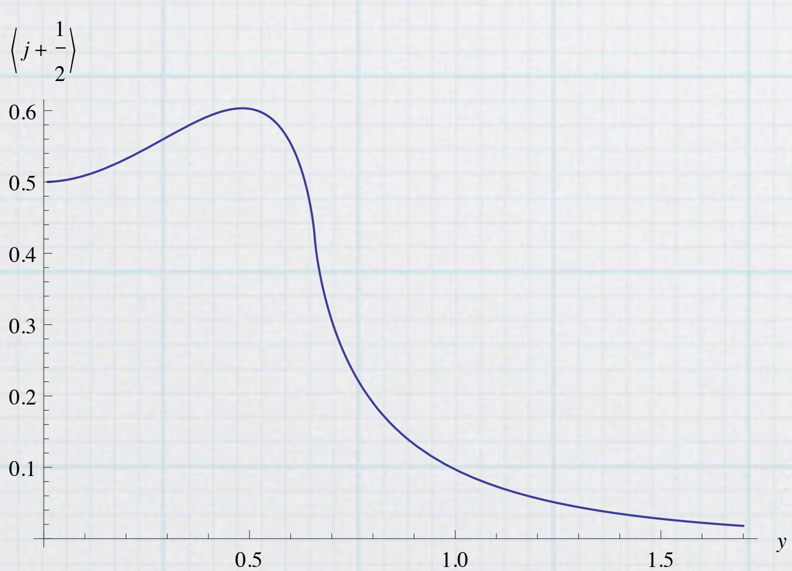
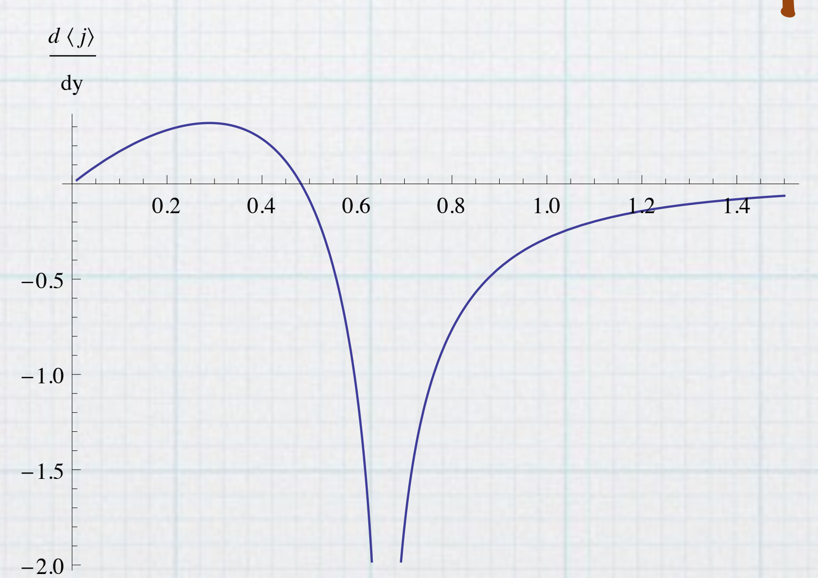
$$\langle j_{e_1}^{n_1} j_{e_2}^{n_2}\cdots j_{e_k}^{n_k}\rangle = \frac{1}{Z^{Spin}}\sum_{\{j_e\}} j_{e_1}^{n_1} j_{e_2}^{n_2}\cdots j_{e_k}^{n_k}\; s(\Gamma,\{j_e\})\mathcal{W}(\{j_e\})\prod_e(\tanh y_e)^{2j_e}$$

Get exact formula for spin average :

Phase Transition !!





Higher order Supersymmetric Theories and Integrals

We can go **beyond Gaussian integrals** with a quadratic action !

Up to now, we have decoupled Ising & Spin networks....

So we introduce **higher order susy interaction terms** !

→ $K^n_{e,v},\ S^n_e,\ S^n_\alpha$ terms still supersymmetric

Adding higher order angle terms ⇒

- affects the spin network distribution & modifies saddle point (geometry background)
- geometric-dependent coupling for Ising
- couples the two Ising models

What's the physical meaning of those theories ?

Can have deeper on tetrahedron with high T/low T duality

Use loop expansion of 2d Ising to show duality identity on the partition function :

High T loop expansion:

$$Z_\Gamma(y_e) = \sum_{\{\sigma_{v=\pm 1}\}} e^{\sum_e y_e \sigma_{s(e)} \sigma_{t(e)}} = 2^V \prod_e \cosh y_e \sum_{C\subset\Gamma} \prod_{e\in C} \tanh y_e$$

Low T cluster expansion:

$$Z_\Gamma(y_e) = 2 \prod_e e^{y_e} \sum_{C^*\subset\Gamma^*} \prod_{e\in C^*} e^{-2y_e}$$

Can have more fun on tetrahedron with high T/low T duality

Use loop expansion of 2d Ising to show duality identity on the partition function :

$$Z_\Gamma(y_e) = \frac{2\prod_e e^{y_e}}{2^{V^*}\prod_e \cosh\tilde{y}_e} Z_{\Gamma^*}(\tilde{y}_e)$$

with dual couplings $Y_e = \tanh y_e = e^{-2\tilde{y}_e}, \quad \tilde{Y}_e = \tanh\tilde{y}_e = e^{-2y_e}$

$$Y = \mathcal{D}(\tilde{Y}) = \frac{(1-\tilde{Y})}{(1+\tilde{Y})}$$

Duality transform is involution, relating the graph and its dual

$$\tilde{Y} = \mathcal{D}(Y) = \frac{(1-Y)}{(1+Y)}$$

Its fixed point is critical Ising coupling for square lattice : $Y_c = -(1\pm\sqrt{2})$

The Tetrahedron & the 6j Symbol

Can have more fun on tetrahedron with high T/low T duality

Apply to 6j generating function :

$$4^3 \sum_{\{j_e\}} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \prod_v \Delta_v(j_e) \prod_e (-1)^{2k_e} T(2j_e+1, 2k_e+1) = \begin{Bmatrix} k_4 & k_5 & k_6 \\ k_1 & k_2 & k_3 \end{Bmatrix} \prod_{v^*} \Delta_{v^*}(k_e)$$

with transform coefficients given by power series (figurate numbers) :

$$Y \frac{(1-Y)^{2j}}{(1+Y)^{2(j+1)}} = \sum_{k \in \mathbb{N}/2} (-1)^{2k} T(2j+1, 2k+1)\, Y^{2k+1}$$

Could be related to self-duality of squared q-deformed 6j symbol ...

The Tetrahedron & the 6j Symbol

Lessons from the tetrahedron :

- Geometric characterization of Fisher zeroes for the Ising model : graph is planar but not flat, critical couplings defined by 3d embedding (dihedral angles)

- Low T / High T Ising duality gives relation between graph and dual graph for spin networks : non-perturbative relations for spinfoams? another path towards criticality ?