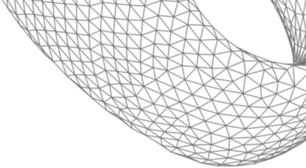

The full-Ward Takahashi identity for random tensors

Dimensons 3 and 4

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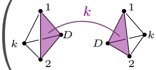
Institute Camille Jordan,
Lyon, 3rd. February

Motivation

- Random Geometry framework ("Quantum Gravity")

$$\mathcal{Z} = \sum\!\!\!\!\!\!\int_{\substack{\text{topologies}\\ \text{geometries}}} \mathcal{D}[g] \exp\left(-S_{\mathrm{EH}}[g]\right) \sim \sum\!\!\!\!\!\!\int_{(\,\cdot\,) \in \substack{\text{topologies}\\ \text{geometries}}} \mu\left(\,\cdot\,\right)$$



$$\leftarrow \int \mathcal{D}[\boldsymbol{\varphi}] \mathrm{e}^{-S(\boldsymbol{\varphi})}$$

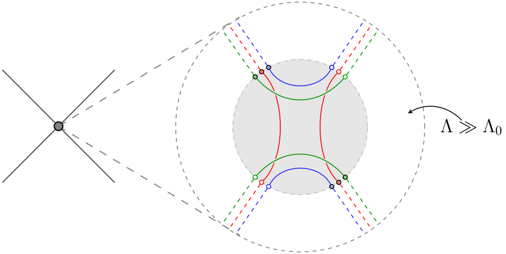
  - Gravity Quanta – 'fundamental blocks of space(time)', simplexes
  - Interactions – *how* to glue those quanta, i.e. gluing simplexes

- Random matrices do that successfully for 2D. Random tensor models is a higher-dimensional arena $S(\boldsymbol{\varphi})$, together with QFT-techniques, based on this idea

Outline

- topology and geometry of random tensor models ($\rightsquigarrow$ gravity)
  - boundaryless manifolds, vacuum graphs
  - manifolds with boundary and sundry graph-encoded topological operations
- Non-perturbative approach to quantum tensor fields (graph-theory is ubiquitous, though)

φ^4-intearction Λ_0



  - full Ward-Takahashi Identities: non-perturbative, systematic approach
  - Schwinger-Dyson equations: equations for the multiple-point functions

Random matrix theory: ensembles

- Nuclear physics: Wigner's model of heavy nuclei
- Stochastics: $E \subset M_N(\mathbb{K})$:

$$\mathcal{Z} = \int_E \mathrm{d}\mu$$

  Statistics of random eigenvalues; study limit $N \to \infty$; universality, μ-independence (tensor models too: book by R. Gurău)

- usually, for certain polynomial $P(x) = Nx^2/2 + N\,V(x)$,

$$\mathcal{Z} = \int_E \mathrm{d}M\,\mathrm{e}^{-\mathrm{Tr}\,P(M)} = \int_E \underbrace{\mathrm{d}M\,\mathrm{e}^{-\frac{N}{2}\mathrm{Tr}\,M^2}}_{\mathrm{d}\mu_0} {}^{-N\mathrm{Tr}\,V(M)} = \int_E \mathrm{d}\mu_0\ \mathrm{e}^{-N\mathrm{Tr}\,V(M)}$$

  Interesting deviations from this are, for instance:
  - Kontsevich's model with Airy function interaction $\mathrm{i}\mathrm{Tr}M^3$ and non-trivial Gaußian term, $\mathrm{Tr}(EM^2)$.
  - NCG-models for which the potential is *not* the trace of a polynomial $\mathrm{Tr}M^2 \cdot \mathrm{Tr}M^2$
  - From noncommutative-QFT: Grosse-Wulkenhaar model

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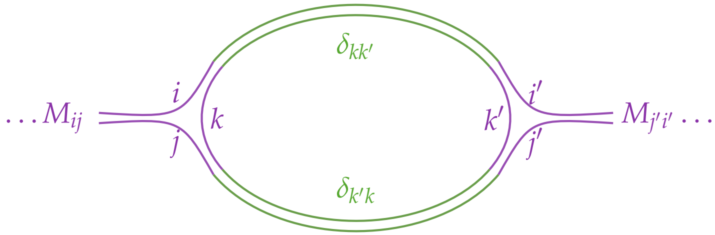
Power-counting in N: $\mathcal{Z} = \int_{E\subset M_N(\mathbb{R})} \mathrm{d}M \, \mathrm{e}^{-\frac{N}{2}\mathrm{Tr}(M^2)+N\lambda\mathrm{Tr}(M^3)}$



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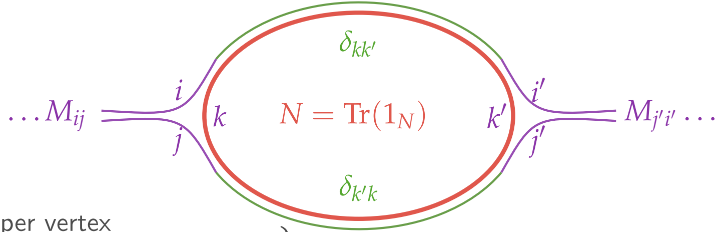
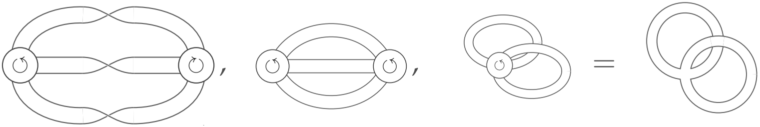


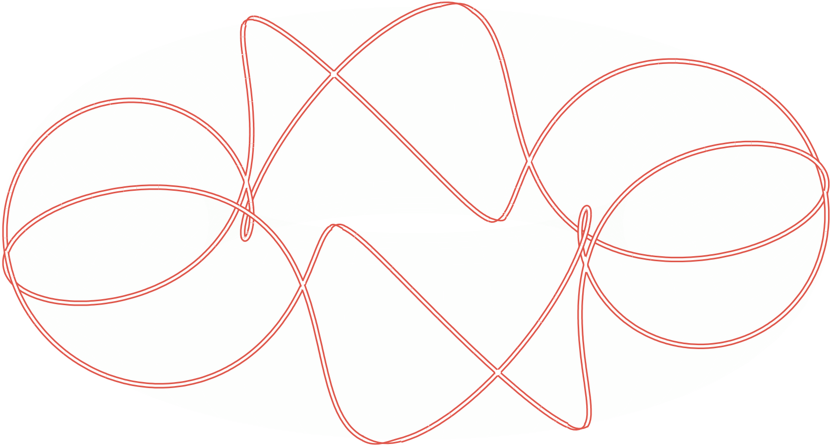
- N per vertex
- N^{-1} factor per propagator
- N per $\sum_{a,b} \delta_{ab}\delta_{ba}$

$$\left.\right\} \mathcal{A}(\mathcal{G}) \sim N^{V(\mathcal{G}) - E(\mathcal{G}) + F(\mathcal{G})} = N^{\chi(\Sigma^g(\mathcal{G}))}$$

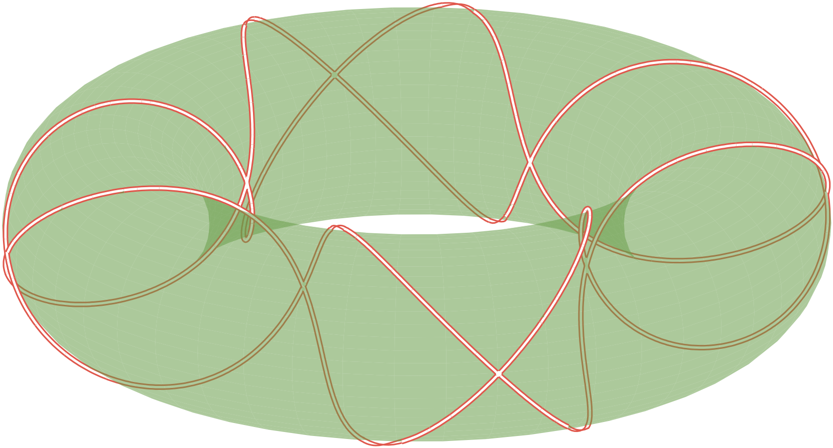
For generic models models, Feynman diagrams are ribbon graphs:



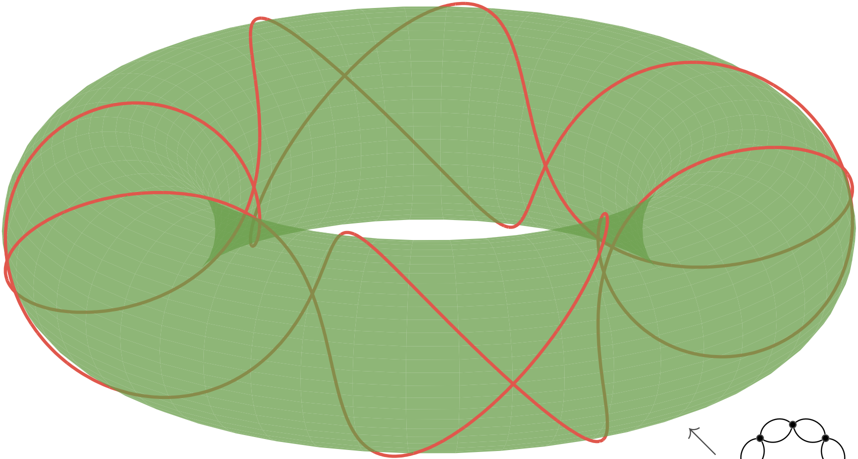
- An example of ribbon graph of a quartic model:

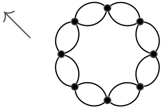


- Any ribbon graph uniquely determines a closed, orientable Riemannian surface, Σ^g:



- Usual scalar fields fail to generate "faces"





Coloured Tensor Models

- a quantum field theory for tensors $\varphi_{a_1 \ldots a_D}$ and $\overline{\varphi}_{a_1 \ldots a_D}$
- the indices transform under *different* representations of

$$G = \mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \ldots \times \mathrm{U}(N_D)$$

- for $g \in G$, $g = (U^{(1)}, \ldots, U^{(D)})$, $U^{(a)} \in \mathrm{U}(N_a)$,

$$\varphi_{a_1 a_2 \ldots a_D} \xrightarrow{g} (\varphi')_{a_1 a_2 \ldots a_D} = U^{(1)}_{a_1 b_1} U^{(2)}_{a_2 b_2} \ldots U^{(D)}_{a_D b_D} \varphi_{b_1 \ldots b_D}$$

- the complex conjugate tensor $\overline{\varphi}_{a_1 a_2 \ldots a_D}$ transforms as

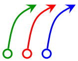
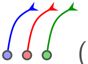
$$\overline{\varphi}_{a_1 a_2 \ldots a_D} \xrightarrow{g} (\overline{\varphi}')_{a_1 a_2 \ldots a_D} = \overline{U}^{(1)}_{a_1 b_1} \overline{U}^{(2)}_{a_2 b_2} \ldots \overline{U}^{(D)}_{a_D b_D} \overline{\varphi}_{b_1 b_2 \ldots b_D}$$

- G-invariants serve as *interaction vertices*

$$S[\varphi, \overline{\varphi}] = \sum_i \tau_i \mathrm{Tr}_{\mathcal{B}_i}(\varphi, \overline{\varphi}) = \mathrm{Tr}_{\mathcal{B}_2}(\overline{\varphi}, \varphi) + \sum_\alpha \lambda_\alpha \mathrm{Tr}_{\mathcal{B}_\alpha}(\overline{\varphi}, \varphi)$$

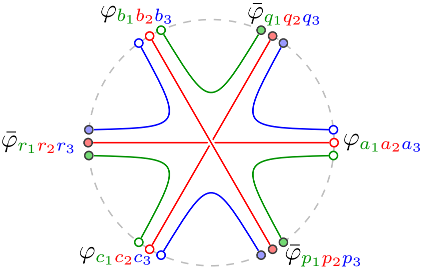
- For $D = 2$, rectangular matrices, $\varphi \in M_{N_1 \times N_2}(\mathbb{C})$ and $\varphi \longmapsto U^{(1)} \varphi (U^{(2)})^{\mathrm{t}}$ and invariants are $\mathrm{Tr}((\varphi\varphi^\dagger)^q)$, $q \in \mathbb{Z}_{\geq 1}$

- traces $\mathrm{Tr}_{\mathcal{B}}$ are indexed by vertex-bipartite regularly edge-D-coloured graphs $\mathcal{B}$ (“*D-coloured graphs*”).
- In $D = 3$-colours (with obvious generalization to higher ranks) we associate

$\varphi_{a_1 a_2 a_3} \mapsto$  and $\bar{\varphi}_{p_1 p_2 p_3} \mapsto$  (reversed order!)

- Contract: $\xrightarrow{1} = \delta_{a_1 p_1}$ $\quad \xrightarrow{2} = \delta_{a_2 p_2}$ $\quad \xrightarrow{3} = \delta_{a_3 p_3}$

Rank-3-Example: One way (out of 7) to obtain a sixth-order vertex is

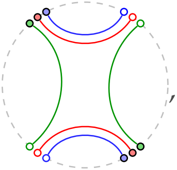
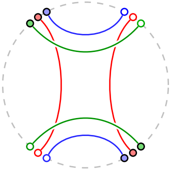
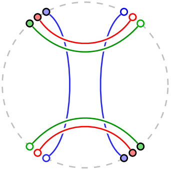


$$\mathrm{Tr}_{K_c(3,3)}(\varphi, \bar{\varphi}) = \sum_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{p},\mathbf{q},\mathbf{r}} \left(\bar{\varphi}_{r_1 r_2 r_3} \bar{\varphi}_{q_1 q_2 q_3} \bar{\varphi}_{p_1 p_2 p_3} \right) \cdot \left(\delta_{a_1 p_1} \delta_{a_2 r_2} \delta_{a_3 q_3} \delta_{b_1 q_1} \delta_{b_2 p_2} \delta_{b_3 r_3} \delta_{c_1 r_1} \delta_{c_2 q_2} \delta_{c_3 p_3} \right) \cdot \left(\varphi_{a_1 a_2 a_3} \varphi_{b_1 b_2 b_3} \varphi_{c_1 c_2 c_3} \right)$$

Feynman diagrams: Choose an action, for instance, the φ^4_3-theory,

$$S[\varphi,\bar{\varphi}] = \mathrm{Tr}_{\mathcal{B}_2}(\varphi,\overline{\varphi}) + \lambda(\mathrm{Tr}_{\mathcal{V}_1}(\varphi,\overline{\varphi}) + \mathrm{Tr}_{\mathcal{V}_2}(\varphi,\overline{\varphi}) + \mathrm{Tr}_{\mathcal{V}_3}(\varphi,\overline{\varphi}))$$

and

$\mathcal{V}_1 =$ , $\mathcal{V}_2 =$ , $\mathcal{V}_3 =$ ,

$$Z[J,\bar{J}] = \frac{\int \mathcal{D}[\varphi,\bar{\varphi}]\, \mathrm{e}^{\mathrm{Tr}_{\mathcal{B}_2}(\bar{J}\varphi)+\mathrm{Tr}_{\mathcal{B}_2}(\bar{\varphi}J)-N^2 S[\varphi,\overline{\varphi}]}}{\int \mathcal{D}[\varphi,\bar{\varphi}]\, \mathrm{e}^{-N^2 S[\varphi,\overline{\varphi}]}},$$ with $\mathrm{Tr}_{\mathcal{B}_2} \leftrightarrow$ 

$$\mathrm{d}\mu_C(\varphi,\overline{\varphi}) := \mathcal{D}[\varphi,\bar{\varphi}]\mathrm{e}^{-N^2 S_0[\varphi,\overline{\varphi}]} := \prod_a \frac{\mathrm{d}\varphi_a \mathrm{d}\overline{\varphi}_a}{2\pi\mathrm{i}} \mathrm{e}^{-N^2 \mathrm{Tr}_{\mathcal{B}_2}(\varphi,\overline{\varphi})}$$

- Write for Wick's contractions w.r.t. the Gaußian measure

$$\int \mathrm{d}\mu_C(\varphi,\overline{\varphi})\varphi_{\mathbf{a}}\overline{\varphi}_{\mathbf{p}} = C(\mathbf{a},\mathbf{p}) = \delta_{\mathbf{ap}} = \mathbf{a}$$ $\mathbf{p}$

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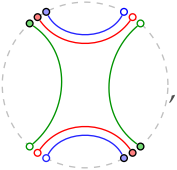
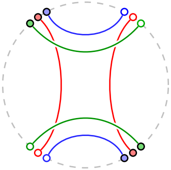
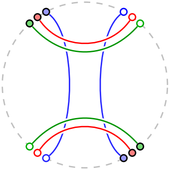
$$S[\varphi,\bar\varphi] = \mathrm{Tr}_{\mathcal{B}_2}(\varphi,\overline{\varphi}) + \lambda\big(\mathrm{Tr}_{\mathcal{V}_1}(\varphi,\overline{\varphi}) + \mathrm{Tr}_{\mathcal{V}_2}(\varphi,\overline{\varphi}) + \mathrm{Tr}_{\mathcal{V}_3}(\varphi,\overline{\varphi})\big)$$

and

$\mathcal{V}_1 =$ , $\mathcal{V}_2=$ , $\mathcal{V}_3 =$ ,

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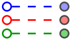
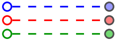
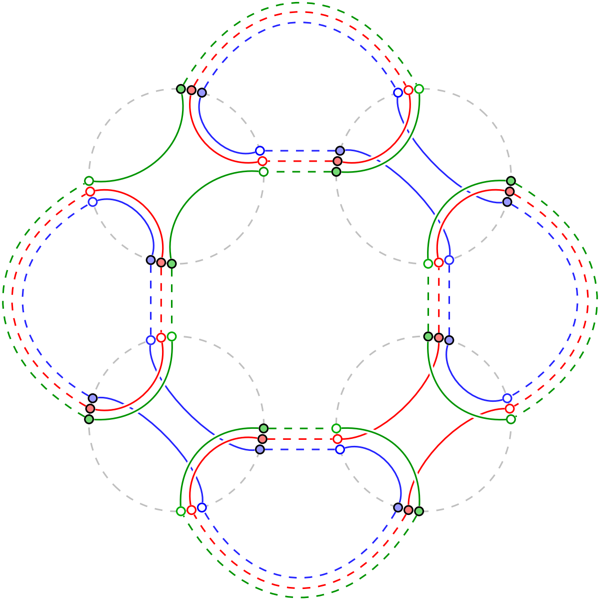
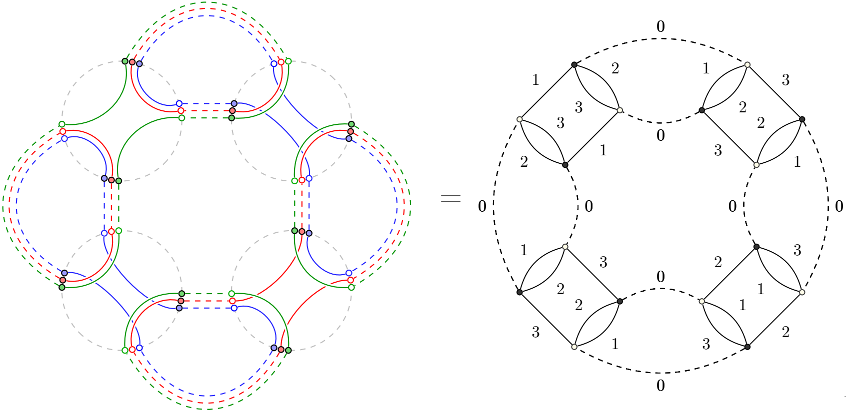
- Write  for Wick's contractions w.r.t. the Gaußian measure

$$\int \mathrm{d}\mu_C(\varphi,\overline{\varphi})\varphi_{\mathbf{a}}\overline{\varphi}_{\mathbf{p}} = C(\mathbf{a},\mathbf{p}) = \delta_{\mathbf{ap}} = \mathbf{a}$$  $\mathbf{p}$

- Example. An $\mathcal{O}(\lambda^4)$-contribution (vacuum sector)



$$\subset \int \mathrm{d}\mu_C(\mathcal{V}_1\mathcal{V}_2[\mathcal{V}_3]^2)$$

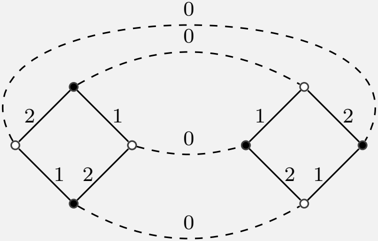


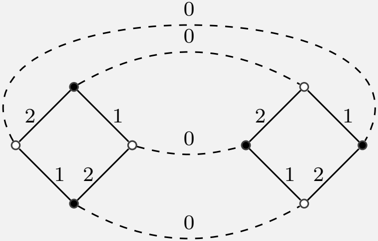
Vertex bipartite regularly edge–D-coloured graphs

- Feynman graphs of a model V, $\mathsf{Feyn}_D(V)$ are $(D+1)$-coloured. Crystallization theory or GEMs [Pezzana, '74] says all PL-manifolds of dimension D can be represented as $D+1$-coloured graphs, $\mathsf{Grph}_{\mathrm{c},D+1}$.
- $\mathcal{G}^{(p)} = \mathcal{G}$'s p-bubbles = "conn. subgraphs with edges in p colours"
$$\mathcal{G}^{(0)} = \mathcal{V}(\mathcal{G}), \quad \mathcal{G}^{(1)} = \mathcal{E}(\mathcal{G}), \quad \mathcal{G}^{(2)} = \mathcal{F}(\mathcal{G}), \ldots$$

Rank-2 models

For an interaction $\lambda \mathrm{Tr}((\varphi\bar{\varphi})^2)$, different connected $\mathcal{O}(\lambda^2)$-graphs are





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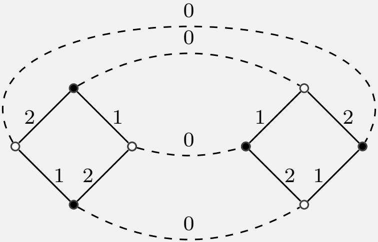
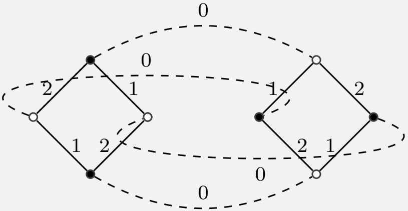
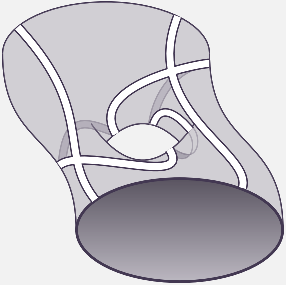




Can be drawn on a sphere

Can be drawn on $\mathbb{T}^2$

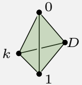
(capped twice)



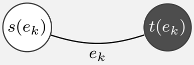
The complex $\Delta(\mathcal{G})$

- for each vertex $v \in \mathcal{G}^{(0)}$, add a D-simplex σ_v to $\Delta(\mathcal{G})$ with colour-labelled vertices $\{0,1,\ldots,D\}$

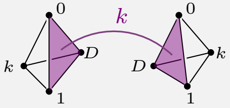
$$\varphi, \bar\varphi \longmapsto v \longmapsto$$



- for each edge $e_k \in \mathcal{G}_k^{(1)}$ of arbitrary colour k, one identifies the two $(D-1)$-simplices $\sigma_{s(e_k)}$ and $\sigma_{t(e_k)}$ that do not contain the colour k.



$\longmapsto$



,

edges come from either $\varphi_{a_1\ldots a_k \ldots a_D}\delta_{a_k p_k}\bar\varphi_{p_1\ldots p_k\ldots p_D}$ $(k \neq 0)$ or $\varphi_{\mathbf{a}}\bar\varphi_{\mathbf{p}}$ $(k=0)$.

[Gurău, '09] and [Bonzom, Gurău, Riello, Rivasseau, '11];

$$\mathcal{A}(\mathcal{G}) = \lambda^{V(\mathcal{G})/2} N^{\underbrace{F(\mathcal{G}) - \frac{D(D-1)}{4}V(\mathcal{G})}_{=: D - \frac{2}{(D-1)!}\omega(\mathcal{G})}} = \exp(-S_{\mathrm{Regge}}[N,D,\lambda])$$

$\rightsquigarrow$ generalizes g; not topol. invariant

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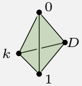

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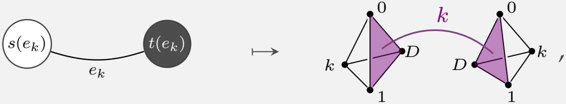

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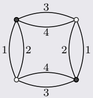
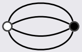
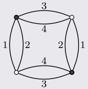
Invariants of the associated complex.

- Graph-homology $H^{\mathrm{bbl}}_\star$. Chain groups are generated by p-bubbles ($p \leq D$)

$H^{\mathrm{bbl}}_\star\Big($  $\Big) = H_\star(\mathbb{S}^3) = H^{\mathrm{bbl}}_\star($  $)$ but $\omega\big($  $\big) \neq \omega($  $)$

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- Fundamental group. Gagliardi's algorithm (here 4-coloured crystallizations).
  - Choose two colours i, j. To each face $\mathcal{G}^{i,j}_\alpha \rightsquigarrow x_\alpha$ (generator).
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    $\rightsquigarrow R(\mathcal{G}^{k,l}_\gamma) =$ alt'd product of the generators touching the vertices of $\mathcal{G}^{k,l}_\gamma$
  - Presentation: $\pi_1(\mathcal{G}) = \langle x_1, \dots x_F | x_F, \{R(\mathcal{G}^{i,k}_\gamma)\}\rangle$. $\pi_1|\Delta(\mathcal{G})| \cong \pi_1(\mathcal{G})$

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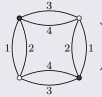
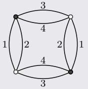
$H_\star^{\text{bbl}}\Big($  $\Big) = H_\star(\mathbb{S}^3) = H_\star^{\text{bbl}}\Big($  $\Big)$ but $\omega\big($  $\big) \neq \omega\big($  $\big)$

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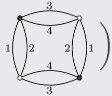
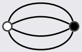
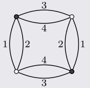
  $H^{\text{bbl}}_\star\Big($  $\Big) = H_\star(\mathbb{S}^3) = H^{\text{bbl}}_\star\big($  $\big)$ but $\omega\big($  $\big) \neq \omega\big($  $\big)$

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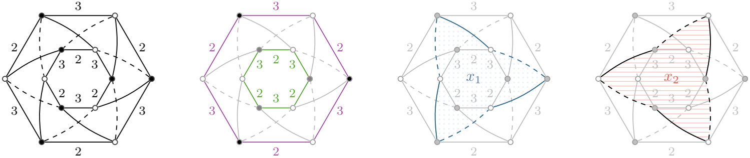
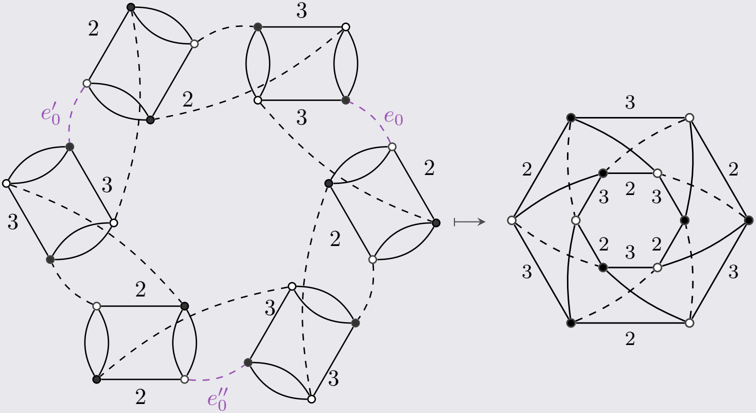
$$i = 0, j = 1, \quad R(\Gamma^{23}_1) = x_1^{+1} x_2^{-1} x_1^{+1} x_2^{-1} x_1^{+1} x_2^{-1}$$

$\Gamma =$ 

$$\pi_1(\Gamma) \cong \langle x_1, x_2 \,|\, x_2, R(\Gamma^{23}_1)\rangle = \langle x_1 \,|\, x_1^3\rangle \cong \mathbb{Z}_3 \cong \pi_1(L_{3;*})\,.$$

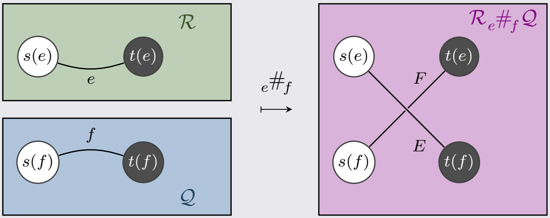
Dipole reduction (1-dipole example)



- Given a model V, how to go backwards? Algorithm?
- probably $|\Delta(\mathsf{Grph}_{c,D+1})| \stackrel{?}{=} |\Delta(\mathsf{Feyn}_D(V))|$, for V sufficiently simple, e.g. $\varphi^4_{D,\mathrm{m}}$.

"Connected sum" is additive with respect to ω

e, f of the same colour:

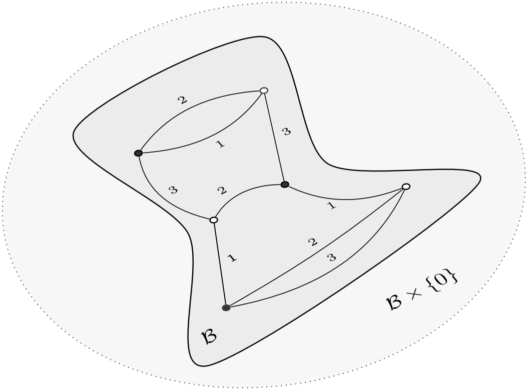


- $|(\mathcal{R}_e\#_f\mathcal{Q})^{(2)}| = |\mathcal{R}^{(2)}| + |\mathcal{Q}^{(2)}| - D$
- $$|\mathcal{R}^{(2)}| = \frac{1}{2}\binom{D}{2}\mathcal{R}^{(0)} + D - \frac{2\omega(\mathcal{R})}{(D-1)!}$$
  $$|\mathcal{Q}^{(2)}| = \frac{1}{2}\binom{D}{2}\mathcal{Q}^{(0)} + D - \frac{2\omega(\mathcal{Q})}{(D-1)!}$$
- $\omega(\mathcal{R}_e\#_f\mathcal{Q}) = \omega(\mathcal{R}) + \omega(\mathcal{Q})$
- P. Cristofori: dipole moves lead to the known # of the crystallization-theory.

- for coloured-0 edges e and f, ${}_e\#_f$ restricts to a binary operation on $\mathrm{Feyn}_D(V)$

The cone $C : \mathsf{Grph}_{c,D} \longrightarrow \mathsf{Grph}_{c,D+1}$

C adds one colour. In the QFT-context, it is the 0-colour.



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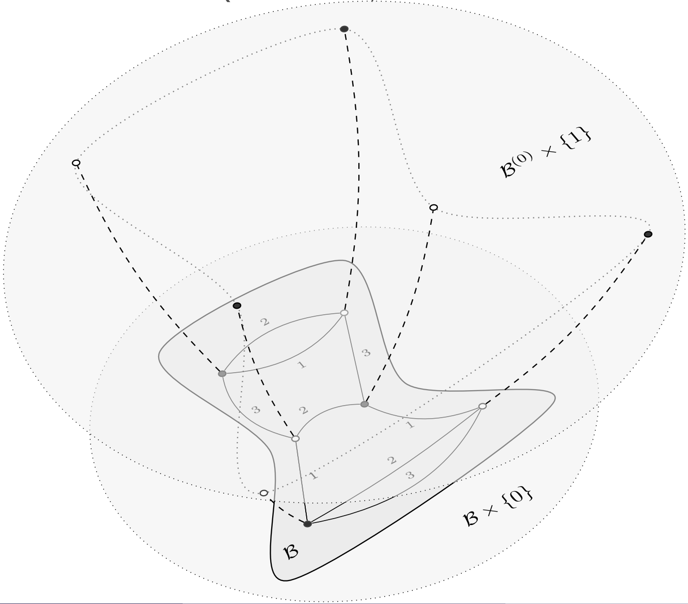
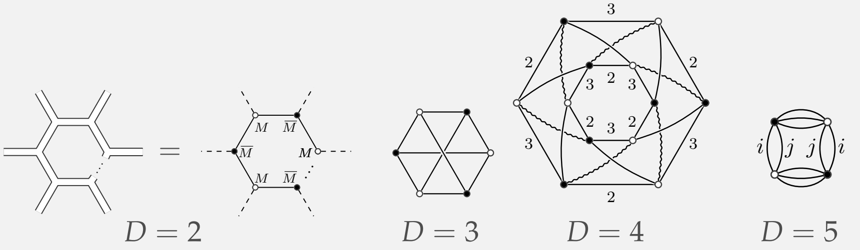


Graphs of $\mathrm{CTM}_D\ V(\Phi)$	closed connected	closed disconn.	open conn.	open disc.
D-coloured graphs	1. Observables (2. Boundaries) $\mathrm{Grph}_{c,D}$	Boundary gr. $\mathrm{im}\partial \subset \amalg \mathrm{Grph}_{c,D}$	$\emptyset$	$\emptyset$
$(D+1)$-col. graphs	Vacuum gr. $\mathrm{Feyn}_D^0(V)$	$\emptyset$	$\mathrm{Feyn}_D(V) \subset \cup_k \mathrm{Grph}_{c,D+1}^{(2k)}$	$\emptyset$

$\mathrm{Grph}_{c,D}$ $(D = 2,\ldots,5)$-examples



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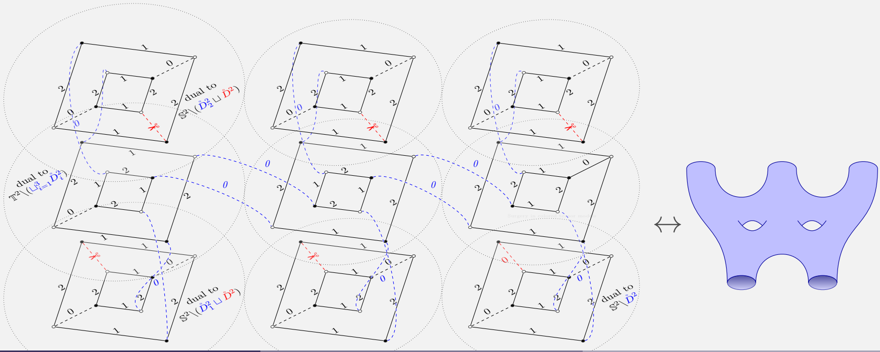
$\mathrm{Feyn}_2((\varphi\bar{\varphi})^2) \subset \cup_k \mathrm{Grph}_{c,2+1}^{(2k)}$ $\quad$ ($2k$ = number of external legs)



GRAPHS OF $\mathrm{CTM}_D\ V(\Phi)$	closed connected	closed disconn.	open conn.	open disc.
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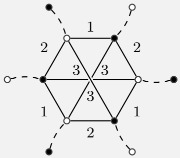
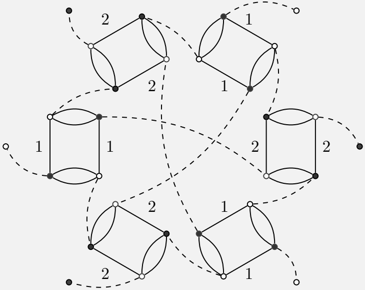
$\mathsf{Feyn}_3(\varphi^4) \subset \mathsf{Grph}_{c,3+1}^{(6)}$



$\notin \mathsf{Feyn}_3(\varphi^4)$



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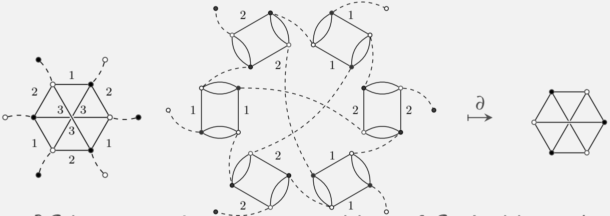
Boundary sector $\mathrm{im}\,\partial \subset \amalg\mathrm{Grph}_{c,D}$

$(D = 3)$-example.



$\partial\mathcal{G}$ has as vertices the external legs of $\mathcal{G}$. And has a i-coloured edge between them for each $0i$-bicoloured edge in $\mathcal{G}$

Ward-Takahashi Identity

- motivated by the WTI for matrix models by [DGMR];
- WTI fully exploited by [GW]
- for T_a^α a hermitian generator of the a-th summand of Lie$(\mathrm{U}(N)^D)$,

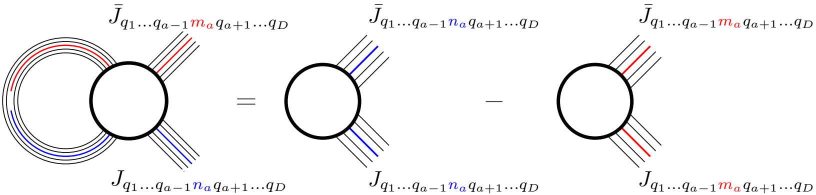
$$\frac{\delta \log Z[J,\bar{J}]}{\delta (T_a^\alpha)_{m_a n_a}} = 0.$$

- this implies

$$\sum_{p_i \in \mathbb{Z}} E(m_a,n_a) \frac{\delta^2 Z[J,\bar{J}]}{\delta J_{p_1 \ldots p_{a-1} m_a p_{a+1} \ldots p_D} \bar{J}_{p_1 \ldots p_{a-1} n_a p_{a+1} \ldots p_D}}$$

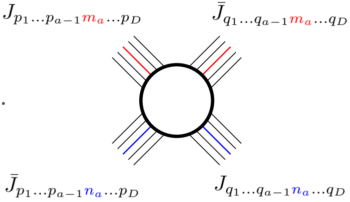
$$= \sum_{p_i \in \mathbb{Z}} \left\{ \bar{J}_{p_1 \ldots p_{a-1} m_a p_{a+1} \ldots p_D} \frac{\delta}{\delta \bar{J}_{p_1 \ldots p_{a-1} n_a \ldots p_D}} - J_{p_1 \ldots p_{a-1} n_a p_{a+1} \ldots p_D} \frac{\delta}{\delta J_{p_1 \ldots p_{a-1} m_a \ldots p_D}} \right\} Z[J,\bar{J}] \, .$$

with $E(m_a,n_a) = E_{p_1 \ldots p_{a-1} m_a p_{a+1} \ldots p_D} - E_{p_1 \ldots p_{a-1} n_a p_{a+1} \ldots p_D}$.



where the LHS is

$$\sum_{p_i \in \mathbb{Z}} \left(E_{p_1 \ldots p_{a-1} m_a p_{a+1} \ldots p_D} - E_{p_1 \ldots p_{a-1} n_a p_{a+1} \ldots p_D} \right) \cdot$$



Aims: understand the combinatorics of the correlation functions recover the (m_a, n_a)-symmetric part.

Expansion of the free energy

- $\partial\,\mathsf{Feyn}_D(V)$ is the boundary sector of the model V

$$W[J,\bar{J}] = \sum_{k=1}^{\infty} \sum_{\substack{\mathcal{B}\in\partial\mathsf{Feyn}_D(V(\varphi,\bar\varphi))\\ 2k=\#(\mathcal{B}^{(0)})}} \frac{1}{|\mathrm{Aut_c}(\mathcal{B})|}\; G^{(2k)}_{\mathcal{B}} \star \mathbb{J}(\mathcal{B})\,.$$

- Coloured automorphisms of $\mathcal{B}$

$$(\mathbb{J}(\mathcal{B}))(\underbrace{\mathbf{a}^1,\ldots,\mathbf{a}^k}_{(\mathbb{Z}^D)^k}) = J_{\mathbf{a}^1}\cdots J_{\mathbf{a}^k}\bar{J}_{\mathbf{p}^1}\cdots\bar{J}_{\mathbf{p}^k}$$

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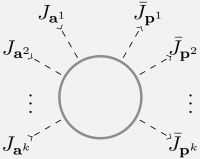
- 

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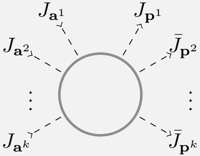
- 

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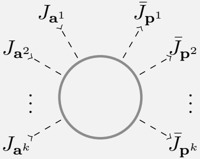
- 

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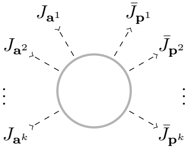
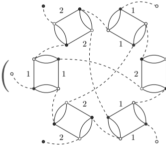
Green's functions

- $\mathcal{G} = $ 

$\rightsquigarrow\ \mathcal{B} = \partial\mathcal{G}\ \rightsquigarrow\ \mathbb{J}(\mathcal{B})\{\mathbf{a}^i\} = \prod_{i=1}^{k} J_{\mathbf{a}^i}\bar{J}_{\mathbf{p}^i}$

- One can derive a functional $X[J,\bar{J}]$ with respect to a graph. For instance:

$\partial\Big($  $\Big) =$

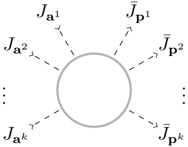
$$\frac{\partial X[J,\bar{J}]}{\partial\ \mathbf{a}\,\mathbf{b}\,\mathbf{c}} = \frac{\partial^6 X[J,\bar{J}]}{\partial J_{\mathbf{a}}\,\partial J_{\mathbf{b}}\,\partial J_{\mathbf{c}}\,\partial \bar{J}_{a_1c_2b_3}\partial \bar{J}_{b_1a_2c_3}\partial \bar{J}_{c_1b_2a_3}}$$

- So: $\dfrac{\partial}{\partial\ \mathbf{a}\,\mathbf{b}\,\mathbf{c}}\Big(\ \mathbf{e}\,\mathbf{f}\,\mathbf{g}\ \Big) = \delta^{\mathbf{e}}_{\mathbf{a}}\delta^{\mathbf{f}}_{\mathbf{b}}\delta^{\mathbf{g}}_{\mathbf{c}} + \delta^{\mathbf{g}}_{\mathbf{a}}\delta^{\mathbf{e}}_{\mathbf{b}}\delta^{\mathbf{f}}_{\mathbf{c}} + \delta^{\mathbf{f}}_{\mathbf{a}}\delta^{\mathbf{g}}_{\mathbf{b}}\delta^{\mathbf{e}}_{\mathbf{c}} \leftrightarrow \mathrm{Aut_c}(\ ) \simeq \mathbb{Z}_3$

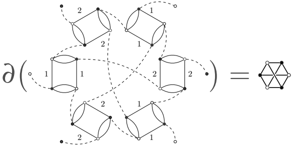
Lemma

$$G^{(2k)}_{\mathcal{B}}(\mathbf{a}^1,\ldots,\mathbf{a}^k) = \frac{\partial^{2k} W[J,\bar{J}]}{\partial\mathbb{J}(\mathcal{B})(\mathbf{a}^1,\ldots,\mathbf{a}^k)}\bigg|_{J=\bar{J}=0}$$ *are all non-trivial*, $\mathcal{B} \in \mathrm{im}\partial$

Green's functions

- $\mathcal{G} = \vdots$  $\vdots \;\rightsquigarrow\; \mathcal{B} = \partial\mathcal{G} \;\rightsquigarrow\; \mathbb{J}(\mathcal{B})\{\mathbf{a}^i\} = \prod_{i=1}^{k} J_{\mathbf{a}^i}\bar{J}_{\mathbf{p}^i}$

- One can derive a functional $X[J,\bar{J}]$ with respect to a graph. For instance:



$$\frac{\partial X[J,\bar{J}]}{\partial \; {}_{\mathbf{b}}\overset{\mathbf{a}}{\cdot}{}_{\mathbf{c}}} = \frac{\partial^6 X[J,\bar{J}]}{\partial J_{\mathbf{a}}\,\partial J_{\mathbf{b}}\,\partial J_{\mathbf{c}}\,\partial \bar{J}_{a_1c_2b_3}\partial \bar{J}_{b_1a_2c_3}\partial \bar{J}_{c_1b_2a_3}}$$

- So: $\dfrac{\partial}{\partial \; {}_{\mathbf{b}}\overset{\mathbf{a}}{\cdot}{}_{\mathbf{c}}}\Big({}_{\mathbf{f}}\overset{\mathbf{e}}{\cdot}{}_{\mathbf{g}} \Big) = \delta^{\mathbf{e}}_{\mathbf{a}}\delta^{\mathbf{f}}_{\mathbf{b}}\delta^{\mathbf{g}}_{\mathbf{c}} + \delta^{\mathbf{g}}_{\mathbf{a}}\delta^{\mathbf{e}}_{\mathbf{b}}\delta^{\mathbf{f}}_{\mathbf{c}} + \delta^{\mathbf{f}}_{\mathbf{a}}\delta^{\mathbf{g}}_{\mathbf{b}}\delta^{\mathbf{e}}_{\mathbf{c}} \leftrightarrow \mathrm{Aut_c}(\;) \simeq \mathbb{Z}_3$

Lemma

$$G^{(2k)}_{\mathcal{B}}(\mathbf{a}^1,\ldots,\mathbf{a}^k) = \frac{\partial^{2k} W[J,\bar{J}]}{\partial\mathbb{J}(\mathcal{B})(\mathbf{a}^1,\ldots,\mathbf{a}^k)}\bigg|_{J=\bar{J}=0}$$ *are all non-trivial*, $\mathcal{B} \in \mathrm{im}\partial$

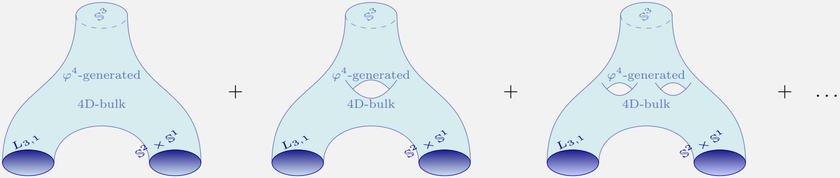
Boundary graphs and bordisms

- The correlation functions $G^{2k}_{\mathcal{B}}$ support a $1/N$-expansion in sectors of the same value of ω:

$$G^{(2k)}_{\mathcal{B}} = \sum_{\omega} G^{(2k,\omega)}_{\mathcal{B}}, \qquad \omega \in ((D-1)!/2)\cdot \mathbb{Z}_{\geq 0} \ \ (D \geq 3)$$

  as matrix models do in the genus..

- since ∂ represents the 'boundary of a simplicial complex', one can give a bordism-interpretation to the Green's functions
- for instance, if $|\Delta(\mathcal{B})| = \mathbb{S}^3 \sqcup \mathbb{S}^2 \times \mathbb{S}^1 \sqcup L_{3,1}$

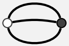


Lemma

The boundary sector of rank-D quartic melonic models is all of $\amalg\mathrm{Grph}_{c,D}$.

- For $D = 3$, all quartic vertices are melonic. The lowest order boundary connected graphs are:

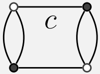
$\mathcal{M}$ — $k = 1$



ω: 0

$\mathrm{Aut}_c(\mathcal{M}) = \{*\}$ — #: 1

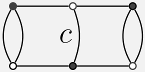
$\mathcal{V}_c$ — $k = 2$



ω: 0

$\mathrm{Aut}_c(\mathcal{V}_c) = \mathbb{Z}_2$ — #: 3

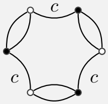
$\mathcal{E}_c$ — $k = 3$



ω: 0

$\mathrm{Aut}_c(\mathcal{E}_c) = \{*\}$ — #: 3

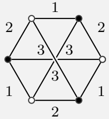
$\mathcal{Q}_c$ — $k = 3$



ω: 0

$\mathrm{Aut}_c(\mathcal{Q}_c) = \mathbb{Z}_3$ — #: 3

$K_c(3,3)$ — $k = 3$

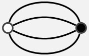


ω: 1

$\mathrm{Aut}_c(K_c(3,3)) = \mathbb{Z}_3$ — #: 1

- $D = 4$-connected boundary graphs

$\mathcal{M}$ $k = 1$



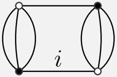
ω 0

$\mathrm{Aut}_c(\mathcal{M}) = \{*\}$

\# 1

No counting needed

$\mathcal{V}_i$ $k = 2$



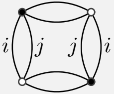
ω 0

$\mathrm{Aut}_c(\mathcal{V}_i) = \mathbb{Z}_2$

\# 4

For any colour $i \in \{1, 2, 3, 4\}$

$\mathcal{N}_{ij}$ $k = 2$



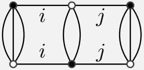
ω 1

$\mathrm{Aut}_c(\mathcal{N}_{ij}) = \mathbb{Z}_2$

\# 3

Since $\mathcal{N}_{ij} = \mathcal{N}_{ji}$ one imposes $i < j$

$\mathcal{E}_{ij}$ $k = 3$



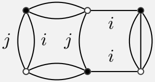
ω 0

$\mathrm{Aut}_c(\mathcal{E}_{ij}) = \{*\}$

\# 6

$\mathcal{E}_{ij} = \mathcal{E}_{ji}$ $i < j$
$i, j \in \{1, 2, 3, 4\}$

$\mathcal{Q}_{ij}$ $k = 3$



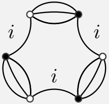
ω 1

$\mathrm{Aut}_c(\mathcal{Q}_{ij}) = \{*\}$

\# 12

$\mathcal{Q}_{ij} \neq \mathcal{Q}_{ji}$
arbirary colours i, j

$\mathcal{C}_i$ $k = 3$

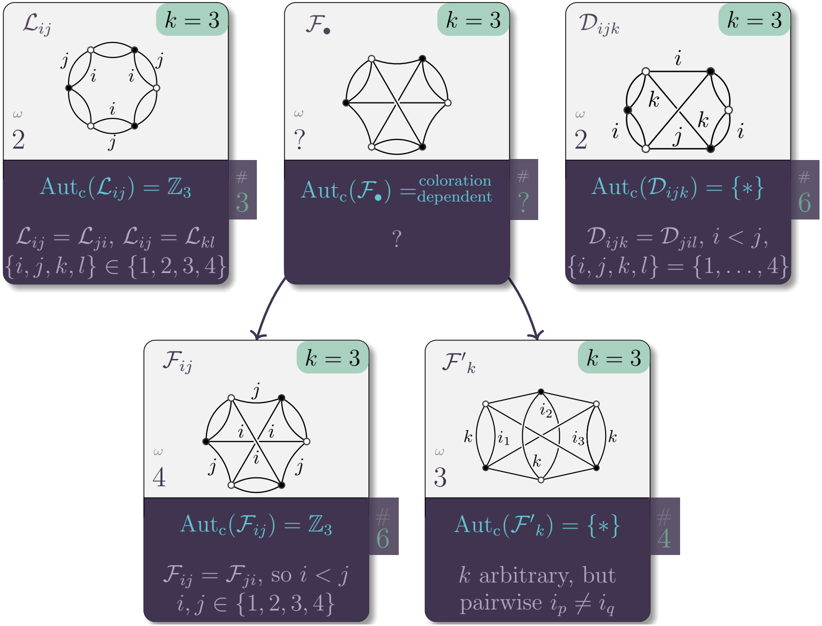


ω 0

$\mathrm{Aut}_c(\mathcal{C}_i) = \mathbb{Z}_3$

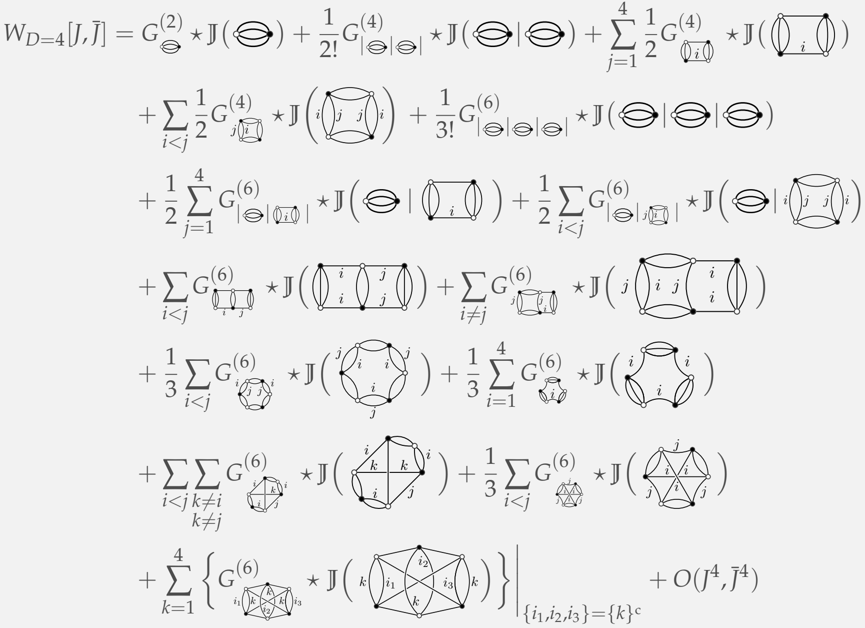
\# 4

arbirary colour i



$\mathcal{L}_{ij}$
$k=3$
ω 2
$\mathrm{Aut_c}(\mathcal{L}_{ij}) = \mathbb{Z}_3$
3
$\mathcal{L}_{ij} = \mathcal{L}_{ji}$, $\mathcal{L}_{ij} = \mathcal{L}_{kl}$
$\{i,j,k,l\} \in \{1,2,3,4\}$
$\mathcal{F}_\bullet$
$k=3$
ω ?
$\mathrm{Aut_c}(\mathcal{F}_\bullet) =$ coloration dependent
?
?
$\mathcal{D}_{ijk}$
$k=3$
ω 2
$\mathrm{Aut_c}(\mathcal{D}_{ijk}) = \{*\}$
6
$\mathcal{D}_{ijk} = \mathcal{D}_{jil}$, $i<j$,
$\{i,j,k,l\} = \{1,\dots,4\}$
$\mathcal{F}_{ij}$
$k=3$
ω 4
$\mathrm{Aut_c}(\mathcal{F}_{ij}) = \mathbb{Z}_3$
6
$\mathcal{F}_{ij} = \mathcal{F}_{ji}$, so $i<j$
$i,j \in \{1,2,3,4\}$
$\mathcal{F}'_k$
$k=3$
ω 3
$\mathrm{Aut_c}(\mathcal{F}'_k) = \{*\}$
4
k arbitrary, but
pairwise $i_p \neq i_q$

$W_{D=4}[J,\bar{J}]$



Theorem (Full Ward-Takahashi Identity for arbitrary tensor models)

If the kinetic form E in $\mathrm{Tr}_2(\bar\varphi, E\varphi)$ of a rank-D tensor model is such that

$$E_{p_1\dots p_{a-1}m_a p_{a+1}\dots p_D} - E_{p_1\dots p_{a-1}n_a p_{a+1}\dots p_D} = E(m_a, n_a) \quad \text{for each } a = 1,\dots,D$$

then its partition function $Z[J,\bar J]$, as a consequence of unitary invariance of the measure ($\delta Z[J,\bar J]/\delta(T^a)_{m_a n_a} = 0$, T^a a generator of $\mathfrak{u}(N)$), satisfies

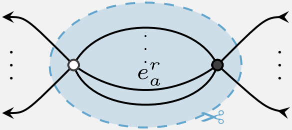
$$\sum_{p_i\in\mathbb{Z}} \frac{\delta^2 Z[J,\bar J]}{\delta J_{p_1\dots p_{a-1}m_a p_{a+1}\dots p_D}\delta \bar J_{p_1\dots p_{a-1}n_a p_{a+1}\dots p_D}} - \left(\delta_{m_a n_a} Y^{(a)}_{m_a}[J,\bar J]\right)\cdot Z[J,\bar J]$$
$$= \sum_{p_i\in\mathbb{Z}} \frac{1}{E(m_a,n_a)}\left(\bar J_{p_1\dots m_a\dots p_D}\frac{\delta}{\delta \bar J_{p_1\dots n_a\dots p_D}} - J_{p_1\dots n_a\dots p_D}\frac{\delta}{\delta J_{p_1\dots m_a\dots p_D}}\right) Z[J,\bar J]$$

where

$$Y^{(a)}_{m_a}[J,\bar J] := \sum_{k=1}^{\infty}{\sum_{\mathcal{B}}}' \frac{1}{|\mathrm{Aut}_{\mathrm{c}}(\mathcal{B})|}\langle\langle G^{(2k)}_{\mathcal{B}}, \mathcal{B}\rangle\rangle_{m_a}$$
$$= \sum_{k=1}^{\infty}{\sum_{\mathcal{B}}}' \frac{1}{|\mathrm{Aut}_{\mathrm{c}}(\mathcal{B})|}\sum_{r=1}^{k}\left(\Delta^{\mathcal{B}}_{m_a,r} G^{(2k)}_{\mathcal{B}}\right)\star \mathbb{J}(\mathcal{B}\ominus e^r_a)\,.$$

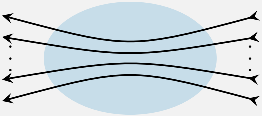
Defining $\mathcal{B} \longmapsto \mathcal{B} \ominus e_a^r$ and $\Delta^{\mathcal{B}}_{m_a,r} : (\mathbb{C})^{\mathbb{Z}^{k\cdot D}} \longrightarrow (\mathbb{C})^{\mathbb{Z}^{(k-1)\cdot D}}$ $(r = 1, \ldots, k)$

Locally:



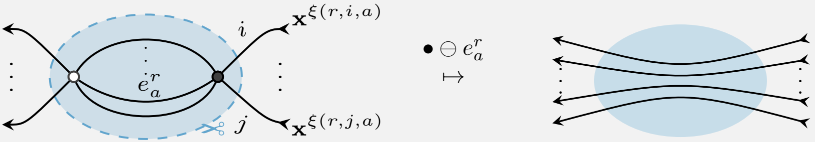
$\bullet \ominus e_a^r$

$\mapsto$



Defining $\mathcal{B} \mapsto \mathcal{B} \ominus e_a^r$ and $\Delta^{\mathcal{B}}_{m_a,r} : (\mathbb{C})^{\mathbb{Z}^{k\cdot D}} \to (\mathbb{C})^{\mathbb{Z}^{(k-1)\cdot D}}$ $(r = 1, \dots, k)$

Locally:



Let

$\mathbf{w} = (m_a, \{q_h\}_{h\in I(e_r^a)}, \{x_g^{\xi(r,g,a)}\}_{g\in A(e_r^a)})$ (colour-ordered);

- q_h is a dummy variable for each colour-h removed edge other than e_a^r
- $x_g^{\xi(r,g,a)}$ colour-g entry of $\mathbf{x}^{\xi(r,g,a)}$

Set for $F : (\mathbb{Z}^D)^k \to \mathbb{C}$,

$$(\Delta^{\mathcal{B}}_{m_a,r}F)(\mathbf{x}^1, \dots, \widehat{\mathbf{x}^r}, \dots, \mathbf{x}^k) = \sum_{\{q_h\}} F(\mathbf{x}^1, \dots, \mathbf{x}^{r-1}, \mathbf{w}(m_a, \mathbf{x}, \mathbf{q}), \dots, \mathbf{x}^k) \text{ and}$$

$$\langle\!\langle G^{(2k)}_{\mathcal{B}}, \mathcal{B}\rangle\!\rangle_{m_a} := \sum_{r=1}^{k} \left(\Delta^{\mathcal{B}}_{m_a,r} G^{(2k)}_{\mathcal{B}}\right) \star \mathbb{J}(\mathcal{B} \ominus e_a^r)$$

Examples of $\langle\langle G^{(2k)}_{\mathcal{B}}, \mathcal{B}\rangle\rangle_{m_a}$

- for instance, for $D = 3$, $a = 2$

$$\langle\langle G^{(2)}, \rangle\rangle_{m_2} = \Delta_{m_2,1} G^{(2)} \star \mathbb{J}(\varnothing) = \sum_{q_1,q_3\in\mathbb{Z}} G^{(2)}(q_1, m_2, q_3)\,.$$

- In $D = 4$, for $\mathcal{F}'_c =$, one has

$$\mathcal{F}'_c \ominus e^1_a = \mathcal{F}'_c \ominus e^3_a = \qquad , \quad \mathcal{F}'_c \ominus e^2_a =$$

$$\begin{aligned}\langle\langle G^{(6)}_{\mathcal{F}'_c}, \mathcal{F}'_c\rangle\rangle_{m_a} = \ & \Delta_{m_a,1} G^{(6)}_{\mathcal{F}'_c} \star \mathbb{J}(\qquad) \\ & + \Delta_{m_a,2} G^{(6)}_{\mathcal{F}'_c} \star \mathbb{J}(\qquad) \\ & + \Delta_{m_a,3} G^{(6)}_{\mathcal{F}'_c} \star \mathbb{J}(\qquad)\end{aligned}$$

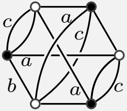
$$\Delta_{m_a,2} G^{(6)}_{\mathcal{F}'_c}(\mathbf{y}, \mathbf{z})$$

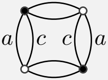
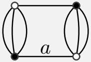
$$= \sum_{q_c} G^{(6)}_{\mathcal{F}'_c}(\mathbf{y}, (m_a, y_b, q_c, z_d), \mathbf{z})$$

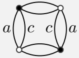
Examples of $\langle\!\langle G^{(2k)}_{\mathcal{B}}, \mathcal{B} \rangle\!\rangle_{m_a}$

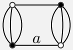
- for instance, for $D = 3$, $a = 2$

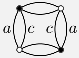
$\langle\!\langle G^{(2)}$  ,  $\rangle\!\rangle_{m_2} = \Delta_{m_2,1} G^{(2)}$  $\star \mathbb{J}(\emptyset) = \sum_{q_1,q_3 \in \mathbb{Z}} G^{(2)}$  (q_1, m_2, q_3).

- In $D = 4$, for $\mathcal{F}'_c =$  , one has

$\mathcal{F}'_c \ominus e^1_a = \mathcal{F}'_c \ominus e^3_a =$  , $\mathcal{F}'_c \ominus e^2_a =$ 

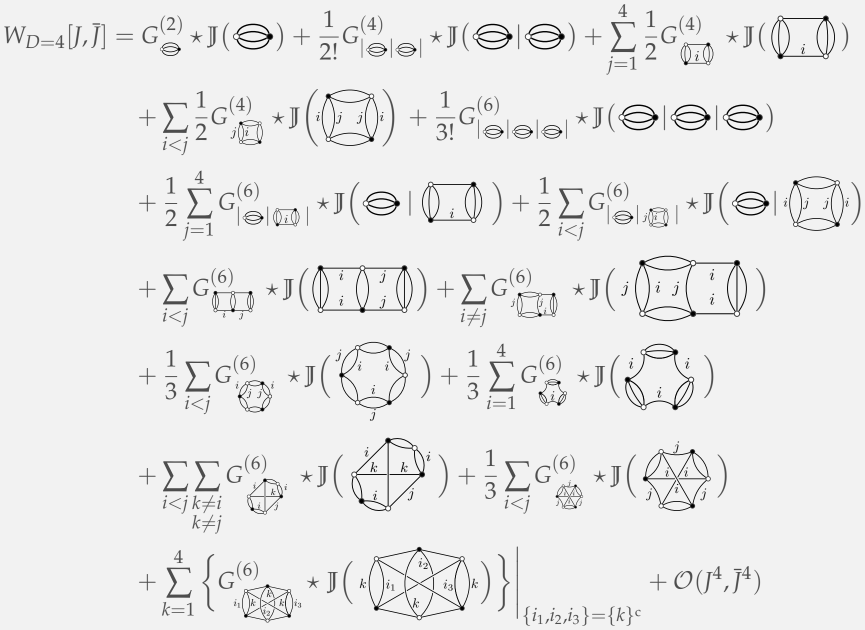
$\langle\!\langle G^{(6)}_{\mathcal{F}'_c}, \mathcal{F}'_c \rangle\!\rangle_{m_a} = \Delta_{m_a,1} G^{(6)}_{\mathcal{F}'_c} \star \mathbb{J}\big($  $\big)$

$+ \Delta_{m_a,2} G^{(6)}_{\mathcal{F}'_c} \star \mathbb{J}\big($  $\big)$

$+ \Delta_{m_a,3} G^{(6)}_{\mathcal{F}'_c} \star \mathbb{J}\big($  $\big)$

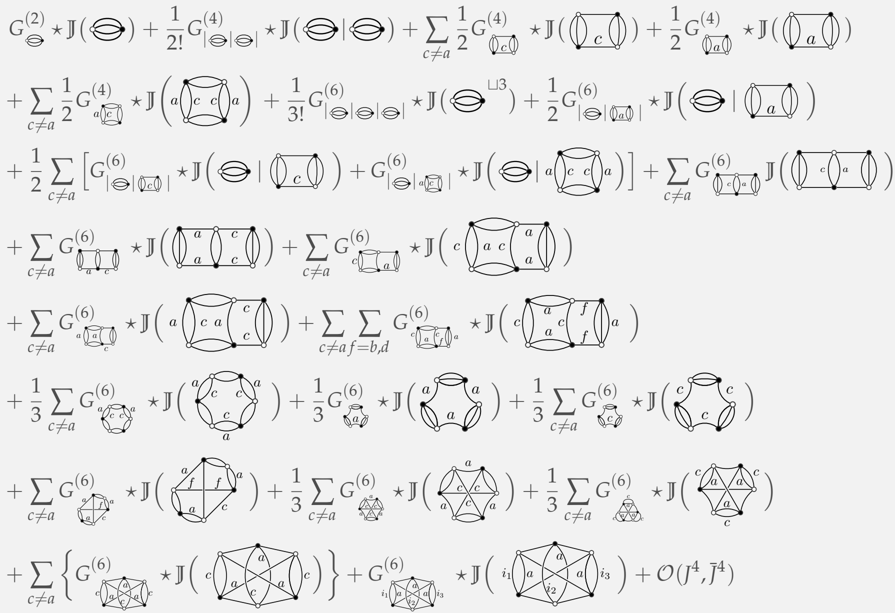
$$\Delta_{m_a,2} G^{(6)}_{\mathcal{F}'_c}(\mathbf{y}, \mathbf{z}) = \sum_{q_c} G^{(6)}_{\mathcal{F}'_c}(\mathbf{y}, (m_a, y_b, q_c, z_d), \mathbf{z})$$

$W_{D=4}[J,\bar{J}]$



$W_{D=4}[J,\bar{J}] = G^{(2)} \star \mathbb{J}(\;) + \frac{1}{2!} G^{(4)} \star \mathbb{J}(\;|\;) + \sum_{j=1}^{4} \frac{1}{2} G^{(4)} \star \mathbb{J}(\;)$
$+ \sum_{i<j} \frac{1}{2} G^{(4)} \star \mathbb{J}(\;) + \frac{1}{3!} G^{(6)} \star \mathbb{J}(\;|\;|\;)$
$+ \frac{1}{2} \sum_{j=1}^{4} G^{(6)} \star \mathbb{J}(\;|\;) + \frac{1}{2} \sum_{i<j} G^{(6)} \star \mathbb{J}(\;|\;)$
$+ \sum_{i<j} G^{(6)} \star \mathbb{J}(\;) + \sum_{i \neq j} G^{(6)} \star \mathbb{J}(\;)$
$+ \frac{1}{3} \sum_{i<j} G^{(6)} \star \mathbb{J}(\;) + \frac{1}{3} \sum_{i=1}^{4} G^{(6)} \star \mathbb{J}(\;)$
$+ \sum_{i<j} \sum_{\substack{k \neq i \\ k \neq j}} G^{(6)} \star \mathbb{J}(\;) + \frac{1}{3} \sum_{i<j} G^{(6)} \star \mathbb{J}(\;)$
$+ \sum_{k=1}^{4} \left\{ G^{(6)} \star \mathbb{J}(\;) \right\} \Big|_{\{i_1,i_2,i_3\}=\{k\}^c} + \mathcal{O}(J^4, \bar{J}^4)$

Singling out a in each graph, $W_{D=4}[J,\bar{J}]$ reads:

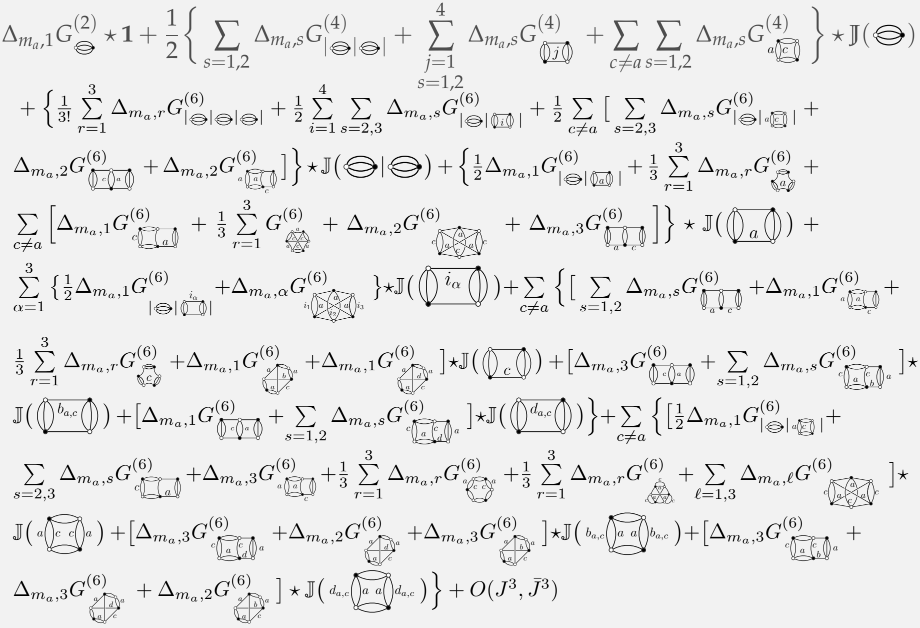


Expansion of $Y^{(a)}_{m_a}[J,\bar{J}]$ for $D=4$

$$\Delta_{m_a,1} G^{(2)} \star \mathbf{1} + \frac{1}{2}\Big\{ \sum_{s=1,2} \Delta_{m_a,s} G^{(4)}_{|\;|\;|} + \sum_{\substack{j=1\\ s=1,2}}^{4} \Delta_{m_a,s} G^{(4)}_{j} + \sum_{c\neq a}\sum_{s=1,2} \Delta_{m_a,s} G^{(4)}_{a\,c} \Big\} \star \mathbb{J}(\;) + \mathcal{O}(J^2,\bar{J}^2)$$

(One needs only $\nearrow$ in order to derive the equation for the 2-pt function;
for the 4-pt functions one also needs...)

Expansion of $Y^{(a)}_{m_a}[J,\bar{J}]$ for $D=4$: $\{a,b,c,d\}=\{1,2,3,4\}$, $b<d$ and $(i_1,i_2,i_3)=\{a\}^c$



$(D=4)$ Equation for the 2-point function

For $E_{\mathbf{x}} = m^2 + |\mathbf{x}|^2$ a Laplacian on $\mathbb{T}^4$

$$
\begin{aligned}
G^{(2)}(\mathbf{x}) = \frac{1}{E_{\mathbf{x}}} + \frac{(-\lambda)}{E_{\mathbf{x}}} \Bigg\{ \sum_{a=1}^{4} \Bigg[2 \cdot G^{(2)}(\mathbf{x}) \\
\cdot \Big(\sum_{q_{i_1(a)}} \sum_{q_{i_2(a)}} \sum_{q_{i_3(a)}} G^{(2)}(x_a, q_{i_1(a)}, q_{i_2(a)}, q_{i_3(a)}) \Big) + \sum_{q_{i_1(a)}} \sum_{q_{i_2(a)}} \sum_{q_{i_3(a)}} \Big(\\
G^{(4)}_{|\,|\,|}(x_a, q_{i_1(a)}, q_{i_2(a)}, q_{i_3(a)}; \mathbf{x}) + G^{(4)}_{|\,|\,|}(\mathbf{x}; x_a, q_{i_1(a)}, q_{i_2(a)}, q_{i_3(a)}) \Big) \\
+ \sum_{c \neq a} \sum_{q_{b(a,c)}} \sum_{q_{d(a,c)}} \Big(G^{(4)}_{c}(x_a, x_c, q_b, q_d; \mathbf{x}) + G^{(4)}_{c}(\mathbf{x}; x_a, x_c, q_b, q_d) \Big) \\
+ \sum_{c \neq a} \sum_{q_c} \Big(G^{(4)}_{a\,c}(x_a, x_b, q_c, x_d; \mathbf{x}) + G^{(4)}_{a\,c}(\mathbf{x}; x_a, x_b, q_c, x_d) \Big) + 2 G^{(4)}_{a}(\mathbf{x}; \mathbf{x}) \\
+ \sum_{y_a} \frac{2}{|x_a|^2 - |y_a|^2} \Big(G^{(2)}(\mathbf{x}) - G^{(2)}(y_a, x_{i_1(a)}, x_{i_2(a)}, x_{i_3(a)}) \Big) \Bigg] \Bigg\}
\end{aligned}
$$

Conclusions & outlook

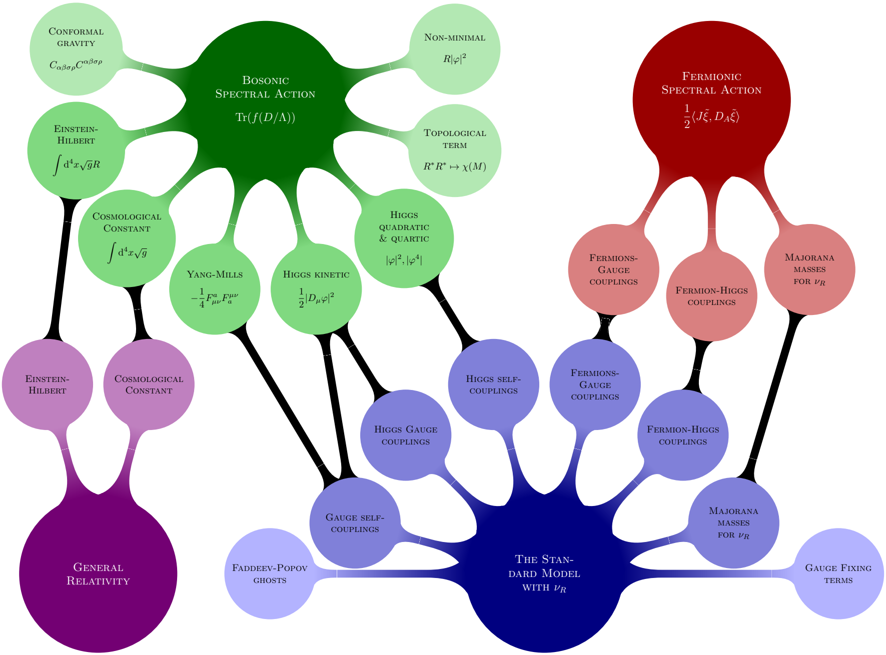
- (Coloured) tensor field theories [Ben Geloun, Bonzom, Carrozza, Gurău, Krajewski, Oriti, Ousmane-Samary, Rivasseau, Ryan, Tanasa, Vignes-Tourneret,...] provide a framework for $3 \leq D$-dimensional random geometry
  - A new Ward-Takahashi identity (bare parameters) based on [Disertori-Gurău-Magnen-Rivasseau] and [Grosse-Wulkenhaar] has been found
    - ★ non-perturbative
    - ★ universal: same for each interaction vertices
    - ★ full (information has been recovered)
    - ★ provides a method to systematically obtain exact equations for correlation functions
  - Eq. for the 2-point function was obtained. Correlation function there can be expanded in Gurău-degree's sectors ($\rightsquigarrow$ decouple?)
  - A bordism interpretation of the correlation functions was given.
- Apply this techniques for SYK-like ([Sachdev-Ye-Kitaev]) models [Witten]
- *Outlook.* The tensor model graphs are not canonically simplicial complexes. Aiming at gauge theories on a random (quantum) spacetime, representations of graphs directly in categories of (almost-commutative) spectral triples should be possible.

Non-Classical geometry of classical fields

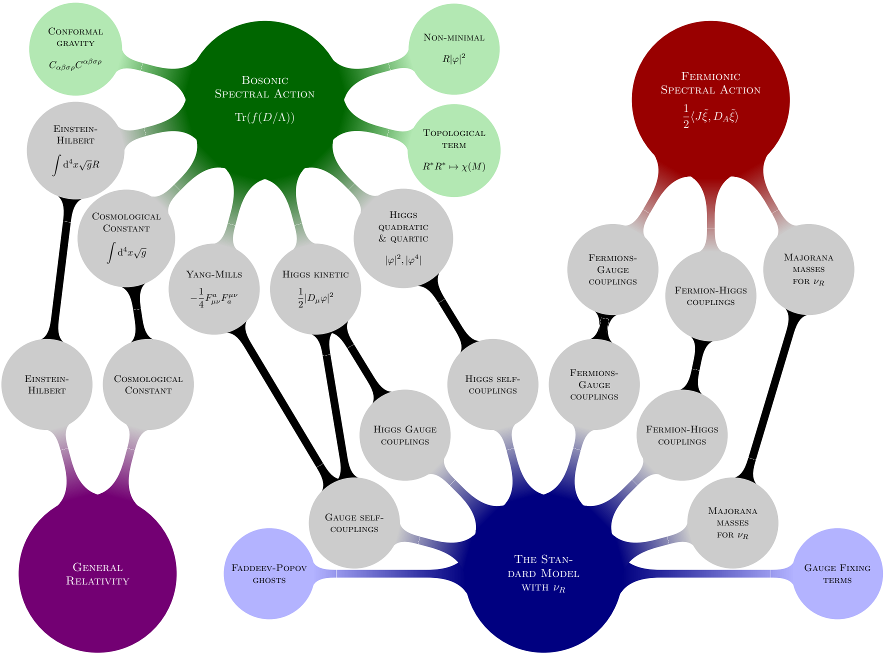
Brute-force summary of NCG-approach to SM: The 'geometry' determines the form D_F, thus of interactions

$$
D_F = \left(\begin{array}{cccccccccccccccccc}
0 & 0 & \Upsilon_\nu^* & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_R^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Upsilon_e^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Upsilon_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Upsilon_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^* & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^* & \otimes 1_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Upsilon_u & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Upsilon_d & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Upsilon_R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_\nu^T & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_e^T & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^T & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^T & \otimes 1_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_u & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_d & 0 & 0 &
\end{array}\right)
$$

"all forces are gravity in disguise"



Conformal gravity
$C_{\alpha\beta\sigma\rho}C^{\alpha\beta\sigma\rho}$
Bosonic Spectral Action
$\mathrm{Tr}(f(D/\Lambda))$
Non-minimal
$R|\varphi|^2$
Fermionic Spectral Action
$\frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$
Einstein-Hilbert
$\int d^4x\sqrt{g}R$
Topological term
$R^*R^* \mapsto \chi(M)$
Cosmological Constant
$\int d^4x\sqrt{g}$
Higgs quadratic & quartic
$|\varphi|^2, |\varphi^4|$
Yang-Mills
$-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a$
Higgs kinetic
$\frac{1}{2}|D_\mu\varphi|^2$
Fermions-Gauge couplings
Fermion-Higgs couplings
Majorana masses for ν_R
Einstein-Hilbert
Cosmological Constant
Higgs self-couplings
Fermions-Gauge couplings
Higgs Gauge couplings
Fermion-Higgs couplings
Gauge self-couplings
Majorana masses for ν_R
General Relativity
Faddeev-Popov ghosts
The Standard Model with ν_R
Gauge Fixing terms



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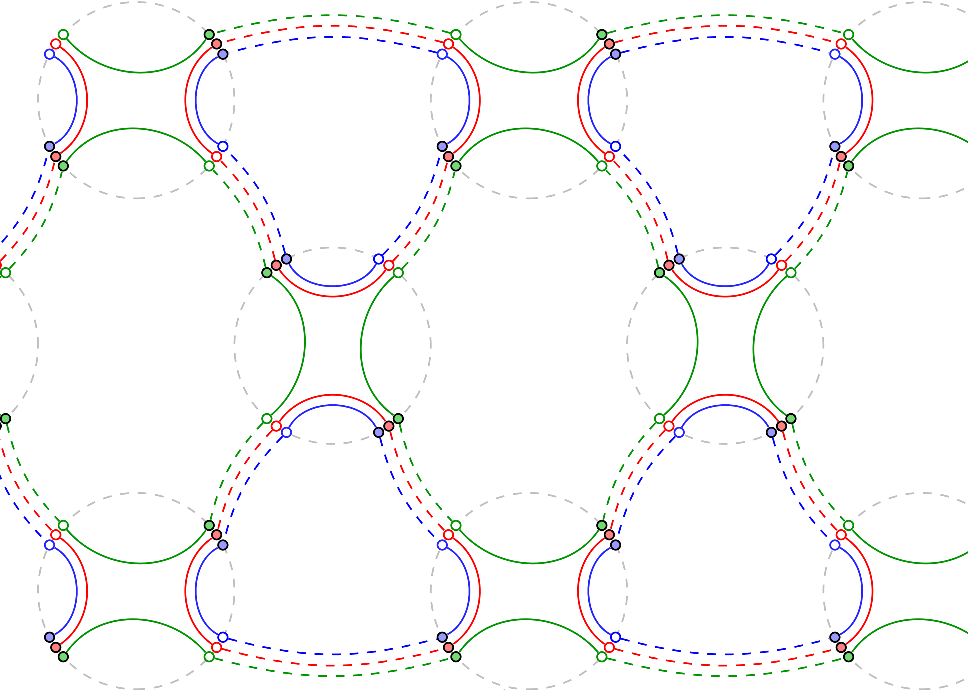

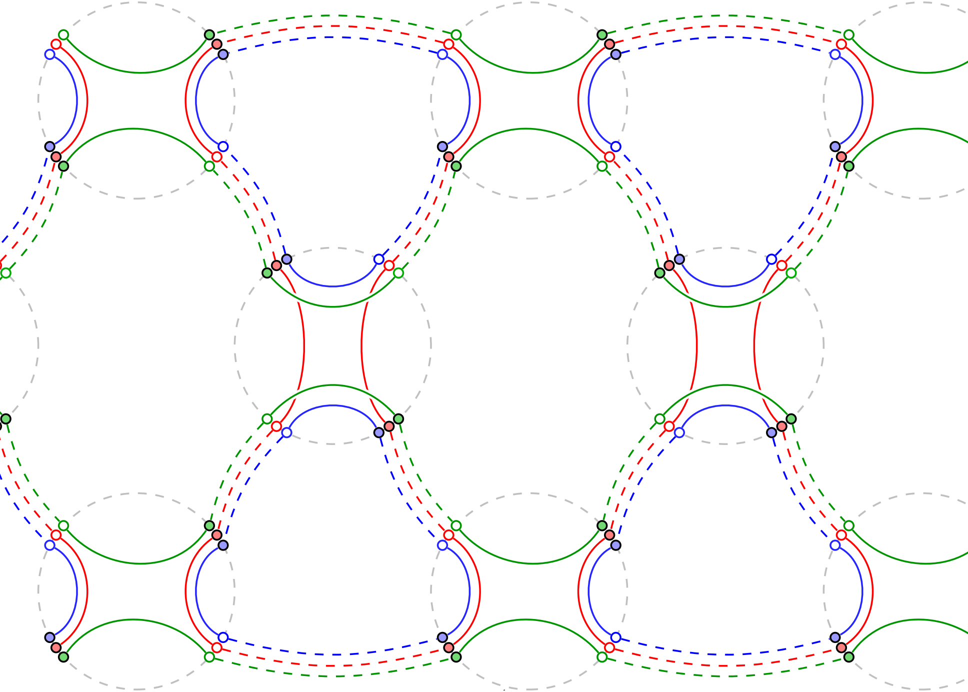

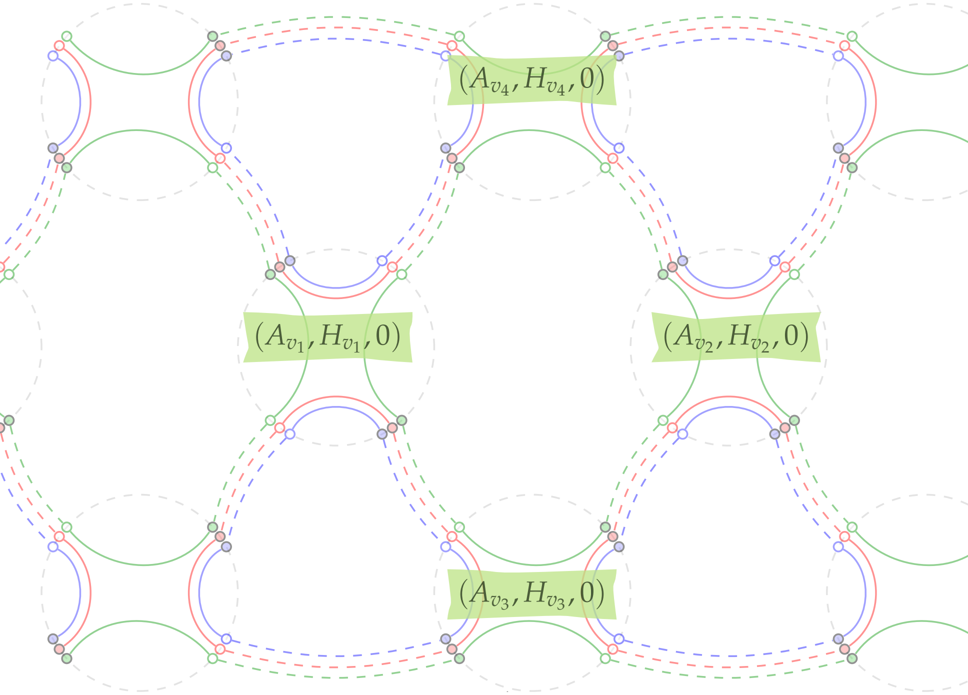

$(A_{v_4}, H_{v_4}, 0)$
$(A_{v_1}, H_{v_1}, 0)$
$(A_{v_2}, H_{v_2}, 0)$
$(A_{v_3}, H_{v_3}, 0)$

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Thank you for your attention!