Homotopical algebraic quantum field theory

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Seminar Talk @ Séminaire de Physique Mathématique,
Institut Camille Jordan, Lyon, 17 November 2017.

Based on works with different subsets of

Collaborators := $\{$C. Becker, M. Benini, U. Schreiber, R. J. Szabo, L. Woike$\}$

Outline

1. Explain why

   AQFT is insufficient to describe gauge theories

2. Present ideas/observations indicating that the key to resolve this problem is

   homotopical AQFT := homotopical algebra + AQFT

3. Discuss our results and inform you about the state-of-the-art of our development of homotopical AQFT

AQFT vs Gauge Theory

AQFT on Lorentzian manifolds

◇ **Basic idea** (Locally Covariant QFT) [Brunetti,Fedenhagen,Verch; ...]

$$\mathsf{Loc} \xrightarrow{\quad \text{functor } \mathfrak{A} \quad} \mathsf{Alg}$$

category of spacetimes — category of algebras

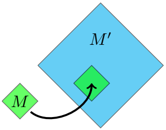
⇝ "Coherent assignment of observable algebras to spacetimes"

– $\mathfrak{A}(M)$ = observables we can measure in M

– $\mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(M')$ = embedding of observables along $f : M \to M'$

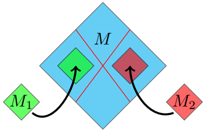
◇ **BFV axioms** (motivated from physics)

Isotony:



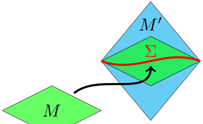
$\mathfrak{A}(M) \xrightarrow{\text{mono}} \mathfrak{A}(M')$

Causality:



$[\mathfrak{A}(M_1), \mathfrak{A}(M_2)] = \{0\}$

Time-slice:



$\mathfrak{A}(M) \xrightarrow{\text{iso}} \mathfrak{A}(M')$

Local-to-global property

For every spacetime M, the global algebra $\mathfrak{A}(M)$ can be "recovered" from the algebras $\mathfrak{A}(U)$ corresponding to suitable sub-spacetimes $U \subseteq M$.

◇ Different ways to formalize this property:

1. **Cosheaf property:** $\mathfrak{A} : \mathsf{Loc} \to \mathsf{Alg}$ is cosheaf (w.r.t. suitable topology)

   ↯ only true for extremely special covers $\Rightarrow$ too strong condition

2. **Additivity:** $\mathfrak{A}(M) \cong \bigvee_\alpha \mathfrak{A}(U_\alpha)$ for suitable covers $\{U_\alpha \subseteq M\}$ [Fewster; ...]

   ✓ true in examples ↯ need to know $\mathfrak{A}(M)$

3. **Universality:** $\mathfrak{A}(M)$ is isomorphic to *Fredenhagen's universal algebra* corresponding to $\{U \subseteq M \ : \text{ open, causally compatible and } U \cong \mathbb{R}^m\}$

   ✓ $\mathfrak{A}$ determined by restriction $\mathfrak{A}_{©} : \mathsf{Loc}_{©} \to \mathsf{Alg}$ via left Kan extension

   ✓ true in examples [Lang]

$$
\begin{tikzcd}
\mathsf{Loc}_{©} \arrow[rr, "\mathfrak{A}_{©}"] \arrow[dr, "\text{inclusion}"'] & \arrow[d, phantom, "\Downarrow"] & \mathsf{Alg} \\
& \mathsf{Loc} \arrow[ur, "\mathfrak{A}"'] &
\end{tikzcd}
$$

Does $U(1)$-Yang-Mills theory fit into AQFT?

- Differential cohomology groups = gauge orbit spaces

$$\widehat{H}^2(M) \cong \frac{\{\text{ principal } U(1)\text{-bundles } P \to M \text{ with connection } A \text{ }\}}{\{\text{ gauge transformations }\}}$$

- Solution spaces of $U(1)$-Yang-Mills theory

$$\mathcal{F}(M) := \big\{ \, h \in \widehat{H}^2(M) \; : \; \delta \, \mathrm{curv}(h) = 0 \, \big\}$$

are Abelian Fréchet-Lie groups with natural presymplectic structure ω_M

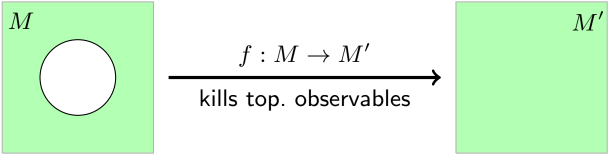
Theorem [Becker,AS,Szabo:1406.1514]

Weyl quantization of smooth Pontryagin dual of $(\mathcal{F}(M), \omega_M)$ defines functor $\mathfrak{A} : \mathsf{Loc} \to \mathsf{Alg}$ which satisfies causality and time-slice, but violates isotony and local-to-global properties.

NB: Similar results for S-duality invariant theory [Becker,Benini,AS,Szabo:1511.00316] and also for less complete approaches based on A-fields or F-fields [Sanders,Dappiaggi,Hack; Fewster,Lang; . . .]

What is the source of these problems?

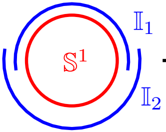
1. Isotony fails because gauge theories carry topological charges



top. charges in hole $H^2(M;\mathbb{Z})$ and $H^{m-2}(M;\mathbb{Z})$

no top. charges

2. Local-to-global property fails because we took gauge invariant observables

$\widehat{H}^2(\mathbb{S}^1) \cong U(1)$ ⟵  ⟶ $\widehat{H}^2(\mathbb{I}_{1/2}) \cong 0$

1. Violation of isotony is a physical feature, hence we have to accept that!
2. Violation of local-to-global property is an artifact of our description by gauge invariant observables, hence we can improve that!

Groupoids vs Gauge Orbit Spaces

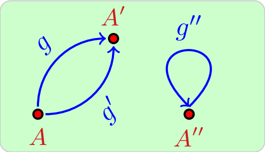
Groupoids of gauge fields

◇ Let's consider for the moment gauge theory on $M \cong \mathbb{R}^m$

1. Gauge fields $A \in \Omega^1(M, \mathfrak{g})$
2. Gauge transformations $g \in C^\infty(M, G)$ acting as $A \triangleleft g = g^{-1}\, A\, g + g^{-1}\mathrm{d}g$

◇ **Groupoid** of gauge fields on M

$$\mathcal{G}(M) := \Omega^1(M, \mathfrak{g})//C^\infty(M, G) =$$



Two groupoids are "the same" not only when isomorphic, but also when **weakly equivalent** $\rightsquigarrow$ model category/homotopical algebra (à la Quillen)

◇ Non-redundant information encoded in the groupoid $\mathcal{G}(M)$

1. Gauge orbit space $\pi_0(\mathcal{G}(M)) = \Omega^1(M, \mathfrak{g})/C^\infty(M, G)$
2. Automorphism groups $\pi_1(\mathcal{G}(M), A) = \{g \in C^\infty(M, G) \,:\, A \triangleleft g = A\}$

! Gauge invariant observables ignore the π_1's, hence are incomplete!

Groupoids and local-to-global properties

- Groupoids of gauge fields satisfy very strong local-to-global property

Homotopy sheaf property

For all manifolds M and all open covers $\{U_\alpha \subseteq M\}$, the canonical map

$$\mathcal{G}(M) \xrightarrow{\ \sim\ } \mathrm{holim}\Big(\prod_\alpha \mathcal{G}(U_\alpha) \rightrightarrows \prod_{\alpha\beta} \mathcal{G}(U_{\alpha\beta}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \prod_{\alpha\beta\gamma} \mathcal{G}(U_{\alpha\beta\gamma}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots \Big)$$

is a weak equivalence in Grpd.

- Precise formulation of the familiar "gluing up to gauge transformation"

$$\Big\{ \big(\{A_\alpha\},\{g_{\alpha\beta}\}\big) \,:\, A_\beta|_{U_{\alpha\beta}} = A_\alpha|_{U_{\alpha\beta}} \triangleleft g_{\alpha\beta}\ ,\ \ g_{\alpha\beta}\, g_{\beta\gamma} = g_{\alpha\gamma} \text{ on } U_{\alpha\beta\gamma} \Big\}$$

$$\Big\Updownarrow \ \text{1:1}$$

$$\Big\{ \text{ gauge fields on } M \ \Big\}$$

- **Crucial Point:** Taking into account the groupoids of gauge fields, rather than only the gauge orbit spaces, there are very strong homotopical local-to-global properties for classical gauge theories!

Cosimplicial observable algebras

What are “function algebras” on groupoids?

- QFT needs quantized ‘algebras’ of functions on the ‘spaces’ of fields
  - ✓ Space of fields $\mathcal{F}(M)$ is set (+ smooth structure)
    $\rightsquigarrow \quad \mathcal{O}(M) = C^\infty(\mathcal{F}(M))$ has the structure of an algebra
  - **?** Space of fields $\mathcal{G}(M)$ is groupoid (+ smooth structure)
    $\rightsquigarrow \quad \mathcal{O}(M) =$ “$C^\infty(\mathcal{G}(M))$” $=$ **?** has which algebraic structure?
- Nerve construction $N : \mathsf{Grpd} \to \mathsf{sSet}$ assigns the simplicial set

$$N(\mathcal{G}(M)) = \Big(\, \Omega^1(M,\mathfrak{g}) \; \leftleftarrows \; \Omega^1(M,\mathfrak{g}) \times C^\infty(M,G) \; \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} \; \cdots \, \Big)$$

- Taking level-wise smooth functions gives **cosimplicial algebra**

$$\mathcal{O}(M) = \Big(\, C^\infty\big(\Omega^1(M,\mathfrak{g})\big) \; \rightrightarrows \; C^\infty\big(\Omega^1(M,\mathfrak{g}) \times C^\infty(M,G)\big) \; \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \; \cdots \, \Big)$$

NB: These constructions can be made mathematically precise!
For algebraic geometry, see e.g. [Toën: Champs affines].

Relation to the BRST formalism and ghost fields

◇ Dual Dold-Kan correspondence gives equivalence $\mathsf{cAlg} \rightleftarrows \mathsf{dgAlg}^{\geq 0}$

⇒ Equivalent description of $\mathcal{O}(M)$ in terms of **differential graded algebra**

$$\mathcal{O}_{\mathrm{dg}}(M) = \Big(\, C^\infty\big(\Omega^1(M,\mathfrak{g})\big) \stackrel{\mathrm{d}}{\longrightarrow} C^\infty\big(\Omega^1(M,\mathfrak{g}) \times C^\infty(M,G)\big) \; \stackrel{\mathrm{d}}{\longrightarrow} \; \cdots \Big)$$

◇ Considering only infinitesimal gauge transformations $C^\infty(M,\mathfrak{g})$

$$\mathcal{O}_{\mathrm{dg}}(M) \xrightarrow{\text{van Est map}} \underbrace{C^\infty\big(\Omega^1(M,\mathfrak{g})\big)}_{\text{gauge field observables}} \otimes \underbrace{\wedge^\bullet C^\infty(M,\mathfrak{g})^*}_{\text{ghost field observables}}$$

The cosimplicial algebra $\mathcal{O}(M)$ (or equivalently our dg-algebra $\mathcal{O}_{\mathrm{dg}}(M)$) describes non-infinitesimal analogs $C^\infty(M,G)$ of ghost fields $C^\infty(M,\mathfrak{g})$

⇒ BRST formalism for **finite** gauge transformations

Working definition for homotopical AQFT

Working definition (intentionally imprecise)

A **homotopical AQFT** is a (weak) functor $\mathfrak{A} : \mathsf{Loc} \to \mathsf{dgAlg}$ to the model category of noncommutative dg-algebras, which satisfies the following axioms:

1. *Causality:* Given causally disjoint $M_1 \xrightarrow{f_1} M \xleftarrow{f_2} M_2$, there exist a (coherent) cochain homotopy λ_{f_1,f_2} such that

$$[\,\cdot\,,\,\cdot\,]_{\mathfrak{A}(M)} \circ \big(\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)\big) = \lambda_{f_1,f_2} \circ \mathrm{d} + \mathrm{d} \circ \lambda_{f_1,f_2}$$

2. *Time-slice:* Given Cauchy morphism $f : M \to M'$, there exists a (coherent) homotopy inverse $\mathfrak{A}(f)^{-1}$ of $\mathfrak{A}(f)$.

3. *Universality:* $\mathfrak{A} : \mathsf{Loc} \to \mathsf{dgAlg}$ is the homotopy left Kan extension of its restriction $\mathfrak{A}_{©} : \mathsf{Loc}_{©} \to \mathsf{dgAlg}$.

Rem: 'Coherent' in e.g. 1.) means that the homotopies for different commutations of more than 2 observables (e.g. $a\,b\,c \to a\,c\,b \to c\,a\,b$ vs $a\,b\,c \to c\,a\,b$) coincide up to specified higher homotopies.

Precise definition requires **operads** [Benini,AS,Woike:1709.08657 & work in progress]

$$\text{homotopical AQFT} \;:=\; \text{AQFT}_\infty\text{-algebra} + \text{operadic universality}$$

Local-to-global property in Abelian gauge theory

Universal global gauge theory observables

- For $G = U(1)$ and $M \cong \mathbb{R}^m$, $\mathcal{G}(M)$ can be described by chain complex

$$\mathcal{G}_{\mathrm{chain}}(M) = \Big(\; \Omega^1(M) \xleftarrow{\frac{1}{2\pi i} \; \mathrm{d}\log} C^\infty(M, U(1)) \; \Big)$$

- Smooth Pontryagin dual cochain complex of observables

$$\mathcal{O}_{\copyright}(M) := \Big(\; \Omega^{m-1}_{\mathrm{c}}(M) \xrightarrow{\quad \mathrm{d} \quad} \Omega^m_{\mathrm{c};\mathbb{Z}}(M) \; \Big)$$

- Homotopy left Kan extension of $\mathcal{O}_{\copyright} : \mathsf{Loc}_{\copyright} \to \mathsf{Ch}^{\geq 0}$

$$\mathcal{O}(M) := \mathrm{hocolim}\Big(\mathcal{O}_{\copyright} : \mathsf{Loc}_{\copyright} \downarrow M \longrightarrow \mathsf{Ch}^{\geq 0} \Big)$$

Theorem [Benini,AS,Szabo:1503.08839]

1. For $M \cong \mathbb{R}^m$, $\mathcal{O}_{\copyright}(M)$ and $\mathcal{O}(M)$ are naturally weakly equivalent.
2. For every M, $\mathcal{O}(M)$ weakly equivalent to dual Deligne complex on M.

- **Crucial Point:** Our homotopical version of "Fredenhagen's universal algebra" produces the correct global observables in Abelian gauge theory, in contrast to the non-homotopical version [Dappiaggi,Lang; Fewster,Lang]!

Toy-models of homotopical AQFT

AQFT on structured spacetimes

◇ **Basic idea** [Benini,AS:1610.06071]

1. Consider AQFT $\mathfrak{A} : \mathsf{Str} \to \mathsf{Alg}$ on category of spacetimes with extra geometric structures, i.e. category fibered in groupoids $\pi : \mathsf{Str} \to \mathsf{Loc}$.
   ($\pi^{-1}(M)$ is groupoid of structures over M, e.g. spin structures, gauge fields)

2. Regard $\mathfrak{A}$ as a trivial homotopical AQFT $\mathfrak{A} : \mathsf{Str} \to \mathsf{dgAlg}$ via embedding $\mathsf{Alg} \to \mathsf{dgAlg}$ of algebras into dg-algebras.

3. Perform homotopy right Kan extension

$$\begin{tikzcd} \mathsf{Str} \arrow[rr, "\mathfrak{A}"] \arrow[dr, "\pi"'] & \arrow[d, phantom, "\Uparrow"] & \mathsf{dgAlg} \\ & \mathsf{Loc} \arrow[ur, "\mathsf{hoU}_\pi \mathfrak{A}"'] & \end{tikzcd}$$

   to induce a nontrivial homotopical AQFT $\mathsf{hoU}_\pi\mathfrak{A}$ on Loc.

◇ **Physical interpretation:** Homotopy right Kan extension turns the background fields described by $\pi^{-1}(M)$ into observables in $\mathsf{hoU}_\pi\mathfrak{A}(M)$.

Properties of $\mathsf{hoU}_\pi\mathfrak{A}$

◇ Explicit description of degree 0 of $\mathsf{hoU}_\pi\mathfrak{A}(M)$

$$\mathsf{hoU}_\pi\mathfrak{A}(M)^0 = \prod_{S\in\pi^{-1}(M)} \mathfrak{A}(S) \ni \Big(a : \pi^{-1}(M) \ni S \longmapsto a(S) \in \mathfrak{A}(S)\Big)$$

◇ **Physical interpretation:** Combination of classical gauge field observables and quantum matter field observables!

Theorem [Benini,AS:1610.06071]

Assume that $\pi : \mathsf{Str} \to \mathsf{Loc}$ is strongly Cauchy flabby. Then the homotopy right Kan extension $\mathsf{hoU}_\pi\mathfrak{A} : \mathsf{Loc} \to \mathsf{dgAlg}$ satisfies the causality and time-slice axioms of homotopical AQFT. (Coherences just established in low orders.)

✓ First toy-models satisfying the new homotopical AQFT axioms!

(Proving universality is hard: $\mathrm{hocolim}$'s in dgAlg are beyond our current technology.)

Stack of non-Abelian Yang-Mills fields

Yang-Mills stack

- **Motivation:** Prior to deformation quantization, we have to understand the geometry of the groupoid of Yang-Mills solutions and the Cauchy problem

$\rightsquigarrow$ Stacks $\cong$ presheaves of groupoids X on Cart satisfying descent [Hollander]

- **Basic idea:** Smooth structure on X is encoded by specifying groupoid $X(\mathbb{R}^k)$ of *all* smooth maps $\mathbb{R}^k \to X$ for *all* $\mathbb{R}^k$ in Cart (functor of points)
- $\exists$ abstract model categorical construction of the stack of non-Abelian Yang-Mills solutions $G\mathbf{Sol}(M)$ [Benini,AS,Schreiber:1704.01378]
- Explicit description of $G\mathbf{Sol}(M)$ up to weak equivalence

$$G\mathbf{Sol}(M)(\mathbb{R}^k) = \begin{cases} \mathrm{obj}: & \text{smoothly } \mathbb{R}^k\text{-parametrized Yang-Mills solutions } (\mathbf{A},\mathbf{P}) \\ \mathrm{mor}: & \text{smoothly } \mathbb{R}^k\text{-parametrized gauge transformations} \\ & \mathbf{h} : (\mathbf{A},\mathbf{P}) \to (\mathbf{A}',\mathbf{P}') \end{cases}$$

- For $M \cong \mathbb{R}^m$ even simpler in terms of vertical geometry on $M \times \mathbb{R}^k \to \mathbb{R}^k$

$$(\mathbf{A},\mathbf{P}) = A \in \Omega^{1,0}(M \times \mathbb{R}^k, \mathfrak{g}) \qquad \text{s.t.} \qquad \delta_A^{\mathrm{vert}} F^{\mathrm{vert}}(A) = 0$$

Stacky Cauchy problem

- $\exists$ map of stacks $\mathrm{data}_\Sigma : G\mathbf{Sol}(M) \to G\mathbf{Data}(\Sigma)$ assigning to Yang-Mills solutions their initial data on Cauchy surface $\Sigma \subseteq M$

Def: The stacky Cauchy problem is well-posed if data_Σ is a weak equivalence.

Theorem [Benini,AS,Schreiber:1704.01378]

The stacky Yang-Mills Cauchy problem is well-posed if and only if the following hold true, for all $\mathbb{R}^k$ in Cart:

1. For all $(\mathbf{A}^\Sigma, \mathbf{E}, \mathbf{P}^\Sigma)$ in $G\mathbf{Data}(\Sigma)(\mathbb{R}^k)$, there exists $(\mathbf{A}, \mathbf{P})$ in $G\mathbf{Sol}(M)(\mathbb{R}^k)$ and iso $\mathbf{h}^\Sigma : \mathrm{data}_\Sigma(\mathbf{A}, \mathbf{P}) \to (\mathbf{A}^\Sigma, \mathbf{E}, \mathbf{P}^\Sigma)$ in $G\mathbf{Data}(\Sigma)(\mathbb{R}^k)$.
2. For any other iso $\mathbf{h}'^\Sigma : \mathrm{data}_\Sigma(\mathbf{A}', \mathbf{P}') \to (\mathbf{A}^\Sigma, \mathbf{E}, \mathbf{P}^\Sigma)$ in $G\mathbf{Data}(\Sigma)(\mathbb{R}^k)$, there exists unique iso $\mathbf{h} : (\mathbf{A}, \mathbf{P}) \to (\mathbf{A}', \mathbf{P}')$ in $G\mathbf{Sol}(M)(\mathbb{R}^k)$, such that $\mathbf{h}'^\Sigma \circ \mathrm{data}_\Sigma(\mathbf{h}) = \mathbf{h}^\Sigma$.

! Note that this is stronger than Cauchy problem for gauge equivalence classes!

! Interesting smoothly $\mathbb{R}^k$-parametrized Cauchy problems! To the best of my knowledge, positive results only known for $\mathbb{R}^0$ [Chrusciel,Shatah; Choquet-Bruhat].

Summary and Outlook

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- Quantum gauge theories are **NOT** contained in the AQFT framework
- To capture crucial homotopical features of classical gauge theories, one needs "higher algebras" to formalize quantum gauge theories

  $\Rightarrow$ **Homotopical AQFT**

- Already very promising results:
  - ✓ Local-to-global property of observables [Benini,AS,Szabo:1503.08839]
  - ✓ Toy-models of homotopical AQFT [Benini,AS:1610.06071]
  - ✓ Yang-Mills stack and stacky Cauchy problem [Benini,AS,Schreiber:1704.01378]
  - ✓ Operadic approach to AQFT [Benini,AS,Woike:1709.08657 & work in progress]
- Open problems/Work in progress:
  1. Construct proper examples of dynamical and quantized gauge theories
  2. What's the physics behind "higher algebras"?

Thanks for your attention.