The tensor journal club goes vir tu al!

The Wigner semicircle law for tensors

Răzvan Gurău

The resolvent for matrices

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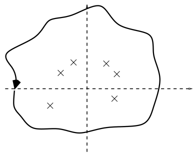
The singular locus of the resolvent is the *spectrum of T*

Why is the resolvent useful?

Spectral theorem:

$$f(T) = \int_\gamma \frac{dw}{2\pi\imath} f(w) \frac{1}{w - T}$$

γ simple curve encircling the spectrum.



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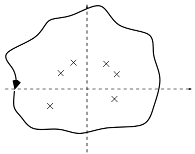

Single trace invariants

$$\frac{1}{N}\mathrm{Tr}\big[f(T)\big] = \int_\gamma \frac{dw}{2\pi\imath} f(w)\, \underbrace{\frac{1}{N}\mathrm{Tr}\left[\frac{1}{w - T}\right]}_{\omega(w;T)\ \text{function}}$$

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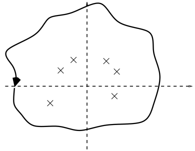

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Finite dimension – $\omega(w;T)$ has poles of order 1 at the eigenvalues of T with residues the multiplicities.

The Wigner semicircle law

Take T a real symmetric random matrix, distributed on GOE

$$d\nu(T) = \Big(\prod_{a\leq b} dT_{ab}\Big)\ \exp\Big\{ -\frac{N}{4}\sum_{a,b} T_{ab}^2 \Big\}$$

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Spectral representation: cut at $[-2, 2]$ and

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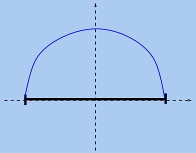

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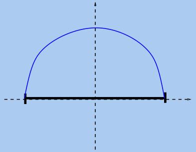

$$\lim_{N\to\infty} \int d\nu(T)\, \frac{1}{N}\mathrm{Tr}[f(T)] = \int d\lambda\, f(\lambda)\, \rho(\lambda)$$

The Gaussian 2−spin model

For $\phi_a \in \mathbb{R}^N$ Gaussian field:

$$\mathcal{Z}(w;T) = \int [d\phi] \;\exp\Big\{ \underbrace{-\frac{1}{2}\phi_a\Big(\delta_{ab} - \frac{T_{ab}}{w}\Big)\phi_b}_{S(\phi)} \Big\} = \frac{1}{\sqrt{\det\left(1 - \frac{T}{w}\right)}}$$

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The resolvent is the two point function of the Gaussian 2–spin model

$$\omega(w;T) = \frac{w^{-1}}{\mathcal{Z}(w;T)} \int [d\phi] \ \frac{\phi^2}{N} \ e^{-S(\phi)} \qquad \omega(w;T) = \frac{1}{w} - \frac{2}{N}\frac{d}{dw}\ln\mathcal{Z}(w;T)$$

Tensors and Eigenvalues

T real symmetric tensor of order p has N^p real components

$$T_{a_1 \ldots a_p} = T_{a_{\sigma(1)} \ldots a_{\sigma(p)}} \ , \qquad T_{b_1 \ldots b_p} = \sum_{a_1 \ldots a_p} O_{a_1 b_1} \ldots O_{a_p b_p} T_{a_1 \ldots a_p}$$

$$\left(Tx^p = \sum_{a_1 \ldots a_p} T_{a_1 \ldots a_p} x_{a_1} \ldots x_{a_p}, \ Tx^{p-1} = \sum_{a_2 \ldots a_p} T_{a_1 \ldots a_p} x_{a_2} \ldots x_{a_p} \right)$$

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Eigenvectors x_i – critical points of $\{ T x^p | \, x^2 = 1 \}$. Eigenvalues $\lambda_i = T x_i^p$

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Real symmetric tensors have:

- at least two real eigenvalues (a continuous function on a compact set with no boundary attains its extrema which are critical points)
- at most $[(p-1)^N - 1]/(p-2)$ complex eigenvalues

Trace invariants

Invariants in correspondence with p valent graphs $\mathcal{B}$:

$$\mathrm{Tr}_{\mathcal{B}}(T) = \sum_{\{a\}} \prod_{v \text{ vertices}} T_{a_1^v \dots a_p^v} \prod_{(v,w) \text{ edges}} \delta_{a_i^v a_i^w}$$

Many graphs with n vertices...

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Balanced trace invariant I_n
– sum connected rooted
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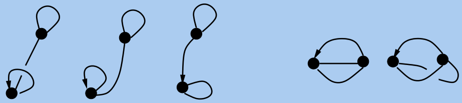

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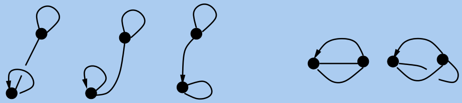

$p = 2$ balanced invariant $I_n = \mathrm{Tr}(T^n)$

$p \geq 3$ more complicated

$$I_2(T) = 3 \sum_{a,b} T_{aab} T_{aab} + 2 \sum_{a,b,c} T_{abc} T_{abc}$$

The p–spin model, $p \geq 3$

Partition function of the Gaussian p–spin model:

$$\mathcal{Z}(w,T) = \int [d\phi] e^{-S(\phi)} \; , \qquad S(\phi) = \frac{1}{2}\phi^2 - \frac{1}{p\;w} \underbrace{\sum_{\{a\}} T_{a_1 \dots a_p} \phi_{a_1} \dots \phi_{a_p}}_{T\phi^p}$$

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Directional Borel Leroy sum of its perturbative series $w = |w|e^{\mathrm{i}\,\psi}$:

$$\mathcal{Z}_{\pm}(w;T) = \int_{e^{\mathrm{i}\,\theta_{\pm}}\mathbb{R}^N} [d\phi]\; e^{-S(\phi)} \,, \qquad \theta_{\pm} = \frac{1}{p}\left(\psi \mp \frac{\pi}{2}\right)$$

Cuts along $\mathbb{R}$, dominated by the largest and smallest real eigenvalues of T

$$\mathcal{Z}_{+}(y;T) - \mathcal{Z}_{-}(y;T) \sim \begin{cases} e^{-\frac{p-2}{2p}\left(\frac{y}{\lambda_{\max}}\right)^{\frac{2}{p-2}}} + \dots \,, & y > 0 \\ e^{-\frac{p-2}{2p}\left(\frac{y}{\lambda_{\min}}\right)^{\frac{2}{p-2}}} + \dots \,, & y < 0 \end{cases} .$$

The tensor resolvent

The tensor resolvent – two point function of the p–spin model

$$\omega(w;T) = \frac{w^{-1}}{\mathcal{Z}(w;T)} \int [d\phi] \, \frac{\phi^2}{N} e^{-S(\phi)} \,, \quad \omega(w;T) = \frac{1}{w} - \frac{p}{N} \frac{d}{dw} \ln \mathcal{Z}(w;T)$$

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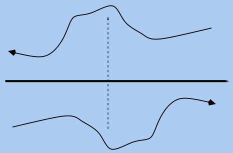
Spectral theorem for balanced invariants:

$$I(T) = \sum a_n \frac{I_n(T)}{N} \;, \quad h_I(w) = \sum_n a_n w^n$$

$$I(T) = \int_\gamma \frac{dw}{2\pi\imath}\, h_I(w)\, \omega(w;T)$$



The generalized Wigner law

Take T a real symmetric random tensor, distributed on GOE

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Spectral representation: cut at $[-p^{p/2}/(p-1)^{(p-1)/2}, p^{p/2}/(p-1)^{(p-1)/2}]$ and

$$\omega(w) = \int_{cut} d\lambda\, \frac{\rho(\lambda)}{w-\lambda} \ , \qquad \rho(\lambda) = |\lambda| P_p(\lambda^2)$$

$$P_p(x) = \sum_{k=1}^{p-1} \Lambda_{k,p}\, x^{\frac{k-p}{p}}\, {}_{p-1}F_{p-2}\left(\left\{ 1 - \frac{1+j}{p-1} + \frac{k}{p} \right\}_{j=1}^{p-1}, \left\{ 1 + \frac{k-j}{p} \right\}_{\substack{j=1 \\ j\neq k}}^{p-1}; \frac{(p-1)^{p-1}}{p^p} x \right) ,$$

$$\Lambda_{k,p} = \frac{1}{(p-1)^{3/2}} \sqrt{\frac{p}{2\pi}} \left(\frac{(p-1)^{p-1}}{p^p} \right)^{\frac{k}{p}} \frac{\prod_{j=1,\dots p-1}^{j\neq k} \Gamma\left(\frac{j-k}{p}\right)}{\prod_{j=1}^{p-1} \Gamma\left(\frac{j+1}{p-1} - \frac{k}{p} \right)} \ .$$

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$$\lim_{N\to\infty} \int d\nu(T)\, I(T) = \int d\lambda\, h_I(\lambda)\, \rho(\lambda)$$

THE $p = 3$ CASE

$$\omega(w) = \frac{\imath}{3^{1/2}}\left[\left(\sqrt{1-\frac{3^3/2^2}{w^2}} - \imath\frac{3^{3/2}/2}{w}\right)^{1/3} - \left(\sqrt{1-\frac{3^3/2^2}{w^2}} + \imath\frac{3^{3/2}/2}{w}\right)^{1/3}\right],$$

$$\rho(\lambda) = \frac{1}{2\pi|\lambda|^{1/3}}\left(\frac{3^3}{2^2}\right)^{1/6}\left[\left(1+\sqrt{1-\frac{\lambda^2}{3^3/2^2}}\right)^{1/3} - \left(1-\sqrt{1-\frac{\lambda^2}{3^3/2^2}}\right)^{1/3}\right],$$

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Annealed is

$$\int [dT][d\phi]\ \exp\left\{ -\frac{N^{p-1}}{2p}\sum T^2_{a_1\dots a_p} - \frac{1}{2}\phi^2 + \frac{1}{p\,w}T\phi^p \right\}$$

integrate out T

Saddle point

$$\omega(w) = \frac{1}{w} - \frac{p}{N}\frac{d}{dw}\ln\int d\nu(T)\mathcal{Z}(w;T)$$

$$\int d\nu(T)\mathcal{Z}(w;T) \sim \int d\rho\, e^{Nf(\rho)} \qquad f(\rho) = \ln\rho - \frac{1}{2}\rho^2 + \frac{1}{2pw}\rho^{2p}$$

Resolvent is $\omega(w) = w^{-1}\rho_0^2$ with ρ_0 dominant saddle point.

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Saddle point equations

$$f'(\rho) = \frac{1}{\rho}\big(1 - \rho^2 + \frac{1}{w^2}\rho^{2p}\big) \;\;\Rightarrow\;\; \rho_0^2 = T_p(w^{-2})$$

Signal + Gaussian noise:

$$A_{a_1 \ldots a_p} = \frac{b}{N^{\frac{p}{2}-1}} \, v_{a_1} \ldots v_{a_p} + T_{a_1 \ldots a_p} \; ,$$

with v a fixed vector in $\mathbb{R}^N$ with $v^2 = 1$ and T GOE.

The spiked model and the detection threshold

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$$b_t = \frac{(p-1)^{p/2}}{(p-2)^{(p-2)/2}}$$

at b_t the largest non removable singularity of the resolvent jumps from $p^{p/2}/(p-1)^{(p-1)/2}$ to $p^{p/2}$.

Saddle point revisited

$$\omega(w) = \frac{1}{w} - \frac{p}{N}\,\frac{d}{dw}\ln\int d\theta d\rho\; e^{Nf(\theta,\rho)}$$

$$f(\theta,\rho) = \ln(\sin\theta) + \ln\rho - \frac{1}{2}\rho^2 + \frac{b}{wp}\,\rho^p\cos^p\theta + \frac{1}{2pw^2}\rho^{2p}\ .$$

Resolvent is $\omega(w) = \frac{1}{w}\rho_\star^2$ with $\star$ the dominant saddle point

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Saddle point equations:

$$\partial_\theta f = \frac{\cos\theta}{\sin\theta} - \frac{b}{w}\,\rho^p\cos^{p-1}\theta\sin\theta\;,\quad \partial_\rho f = \frac{1}{\rho}\left(1 - \rho^2 + \frac{b}{w}\,\rho^p\cos^p\theta + \frac{1}{w^2}\,\rho^{2p}\right)\;,$$

Two solutions $\theta_0 = \pi/2$, $\rho_0^2 = T_p(w^{-2})$ and (θ_1, ρ_1^2) with $\theta_1 \neq \pi/2$.

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At b_t the saddle point (θ_1, ρ_1^2) becomes dominant at the largest non removable singularity of $\omega(w)$.

Problem 1: spiked melons?

Quenched is

$$\int [dT]\ \exp\left\{ -\frac{N^{p-1}}{2p} \sum T^2_{a_1 \ldots a_p} \right\} \ln\left(\int [d\phi]\ \exp\left\{ -\frac{1}{2}\phi^2 + \frac{1}{p\,w} T\phi^p \right\} \right)$$

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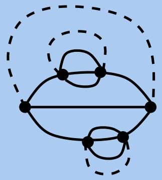
Feynman expand in $\phi \to$ maps
Logarithm $\to$ connected maps.

$\int [dT], N \to \infty \to$ melons, each brings 1.

$\omega(w) \to$ generating function of rooted melons.



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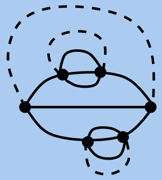

- build the graphical representation for the spiked model.
- what is b_t in this language?

Problem 2: best rank 1 approximation

The real eigenpair with maximal eigenvalue yields the best rank one approximation of T

$$\inf_{\alpha,x}\left\{ ||T - \alpha x^{\otimes p}||_F \,\middle|\, x^2 = 1 \right\}$$

Substituting $T \to \lambda v^{\otimes p}$ in any invariant $\mathcal{B}_n$ with n vertices $\mathrm{Tr}_{\mathcal{B}_n}(T) = \lambda^n$

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For $p=2$

$$\mathrm{Tr}(T^n)\sim\lambda^n+\text{small}$$

Is for $p\geq 3$

$$\mathrm{Tr}_{\mathcal{B}_n}(T)\sim\lambda^n+\text{small}\quad ?$$

Problem 3: reconstruct the signal

Identify the signal v in the spiked model

$$A_{a_1 \ldots a_p} = \frac{b}{N^{\frac{p}{2}-1}} \, v_{a_1} \ldots v_{a_p} + T_{a_1 \ldots a_p} \ ,$$

for $b > b_t$