PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

Multiple scaling limits of $\mathrm{U}(N)^2 \times \mathrm{O}(D)$ multi-matrix models

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Virtual Tensor Journal Club
Planet Earth – April 29th 2020

Work in collaboration with Dario Benedetti, Reiko Toriumi and Guillaume Valette.
arXiv:2003.02100

Motivations

- Matrix models $\rightarrow$ random surfaces.
  - large N expansion indexed by the genus g;
  - planar sector $\rightarrow$ Brownian sphere / Liouville Quantum Gravity at criticality;
  - higher-genus contributions included in a double-scaling.
- Tensor models $\rightarrow$ higher-dimensions.
  - large N expansion indexed by the Gurau degree ω;
  - $\omega = 0$ sector $\rightarrow$ melon diagrams $\rightarrow$ continuous random tree / branched polymers at criticality;
  - $\omega > 0$ contributions included in a double-scaling.

Hybrid situation of **large D multi-matrix models**: [Ferrari '17; Ferrari, Rivasseau, Valette '17;...]

- large collection of D $N \times N$ matrices, viewed as a $D \times N \times N$ tensor;
- generate decorated surfaces;
- large N $\rightarrow$ genus expansion;
- large D $\rightarrow$ planar sector dominated by melon diagrams $\rightarrow$ branched polymers.

How do higher genus contributions behave? Is it possible to escape the universality class of branched polymers in a suitable multiple-scaling limit?

Outline

1. The model and its double-scaling limit
2. Recursive characterization of all contributing graphs
3. Triple-scaling limit of the connected generating function: more random trees
4. Triple-scaling limit of the 2PI generating function: random planar maps
5. Summary and discussion

1. The model and its double-scaling limit

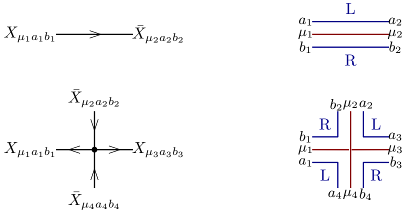
Model

[Ferrari '17]

$$\mathcal{F}(\lambda) = \log \int [dX] \, e^{-S[X, X^\dagger]}$$

$$S[X, X^\dagger] = N D \left(\mathrm{Tr} \left[X_\mu^\dagger X_\mu \right] - \frac{\lambda}{2} D \, \mathrm{Tr} \left[X_\mu^\dagger X_\nu X_\mu^\dagger X_\nu \right] \right)$$

- X_μ $N \times N$ complex matrix, $\mu \in \{1, \ldots, D\}$;
- For all $O \in \mathrm{O}(D)$, and $U_L, U_R \in \mathrm{U}(N)$: $X_\mu \to X'_\mu = O_{\mu\mu'} U_{(L)} X_{\mu'} U^\dagger_{(R)}$
- Propagator and vertex:



- Feynman graphs $\to$ ribbon graphs / quadrangulations decorated by $\mathrm{O}(D)$-loops.

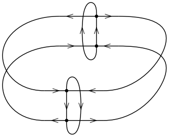
- Double asymptotic expansion at large N and large D:

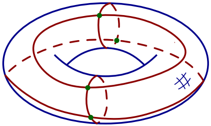
$$\mathcal{F}(\lambda) = \sum_{g \in \mathbb{N}} N^{2-2g} \sum_{\ell \in \mathbb{N}} D^{1+g-\frac{\ell}{2}} \mathcal{F}_{g,\ell}(\lambda)$$

where the *grade* $\ell \in \mathbb{N}$ is defined by

$$\frac{\ell}{2} = 1 + g + \frac{v}{2} - \varphi$$

Example. $g = 1$, $v = 4$, $\varphi = 4 \Rightarrow \ell = 0$.





- Relation to familiar combinatorial quantities in tensor models:

$$\omega - g = \frac{\ell}{2} = g_L + g_R$$

Double-scaling

$$\mathcal{F}(\lambda) = \sum_{g\in\mathbb{N}} N^{2-2g} \sum_{\ell\in\mathbb{N}} D^{1+g-\frac{\ell}{2}} \mathcal{F}_{g,\ell}(\lambda) = D^2 \sum_{g\in\mathbb{N}} \left(\frac{N}{\sqrt{D}}\right)^{2-2g} \sum_{\ell\in\mathbb{N}} D^{-\frac{\ell}{2}} \mathcal{F}_{g,\ell}(\lambda)$$

Introduce a new double-scaling parameter $M := \frac{N}{\sqrt{D}}$:

$$\mathcal{F}^{(0)}(M,\lambda) := \lim_{\substack{N,D\to\infty \\ M<\infty}} \frac{1}{D^2} \mathcal{F}(\lambda) = \sum_{g\geq 0} M^{2-2g} \mathcal{F}_{g,0}(\lambda)$$

1. Characterization of $\ell = 0$ graphs?
2. Critical behaviour of $\mathcal{F}_{g,0}(\lambda)$?
3. Triple-scaling by tuning simultaneously $\lambda \to \lambda_c$ and $M \to \infty$?

For convenience, we will focus on the two-point function:

$$\mathcal{G}^{(0)}(\lambda) \equiv \frac{N}{D} \left\langle \mathrm{Tr}\left[X_\mu^\dagger X_\mu\right] \right\rangle = \sum_{g\in\mathbb{N}} \mathcal{G}_g(\lambda) M^{2-2g}$$

with $\mathcal{G}_g(\lambda) = \#\{$rooted $\ell = 0$ graphs of genus $g\}$.

2. Recursive characterization of $\ell = 0$ graphs

Methodology

Same as in enumeration of Feynman graphs of fixed degree ω.

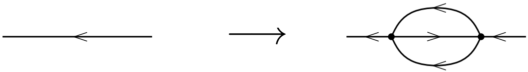
[Gurau, Schaeffer '13; Fusy, Tanasa '15]

- Organize the Feynman diagrams into equivalent classes, modulo insertion/deletion of infinite families of subgraphs that do not change ℓ.
- This defines the notion of scheme S_G of a Feynman graph G.
- Classify the schemes with $\ell = 0$, order by order in the genus g.

The two relevant families of subgraphs in our context are: two-point melon subgraphs and four-point ladder diagrams.

Interesting feature: in contrast to the construction of graphs of fixed degree ω, the construction of $\ell = 0$ graphs of fixed genus g will be entirely recursive.

Melons



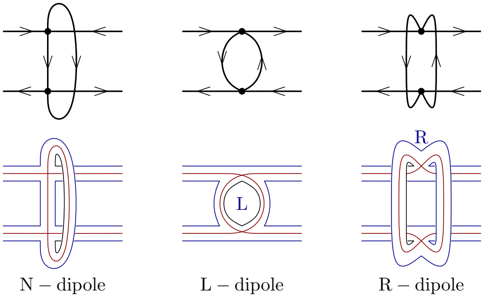
- If $g(G) = 0 = \ell(G)$, then $\omega = g + \frac{\ell}{2} = 0$ and therefore G is melonic.
- Moreover, melonic insertions/deletions preserve both ℓ and g $\rightarrow$ consider graphs up to melonic insertions.
- Generating function of two-point melonic graphs: [Bonzom, Gurau, Riello, Rivasseau '11...]

$$T(\lambda) = 1 + \lambda^2 \, T(\lambda)^4$$

Develops a square-root singularity at the critical value $\lambda_c = (3^3/4^4)^{1/2}$:

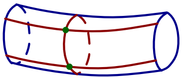
$$T(\lambda) \underset{\lambda \to \lambda_c^-}{\approx} \frac{1}{3}\left(4 - \sqrt{\frac{8}{3}}\sqrt{1 - \frac{\lambda^2}{\lambda_c^2}}\right)$$

Dipoles

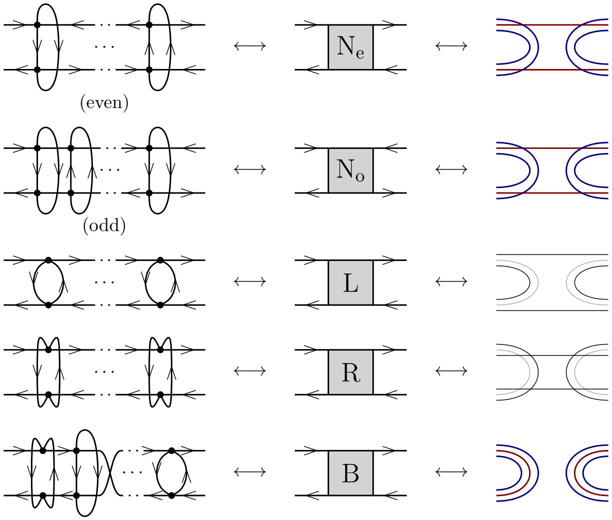


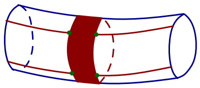
Topological visualisation

N − dipole



Ladders and ladder-vertices





ℓ and g are invariant under addition/removal of rungs in a ladder of a given type.

Schemes

Definition

The *scheme* S_G of a (rooted, connected) graph G is the graph obtained by:

- replacing any melonic 2-point function by a propagator;
- replacing any maximal ladder by the ladder-vertex of the corresponding type.

Proposition

If $S_{G_1} = S_{G_2}$*, then:*

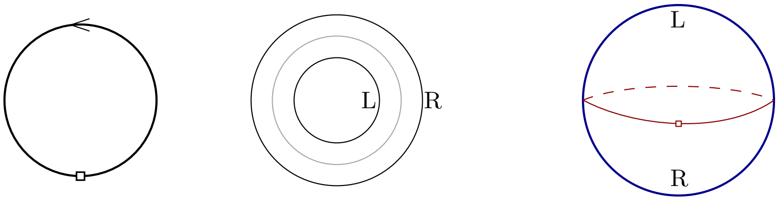
$$g(G_1) = g(G_2) \quad \text{and} \quad \ell(G_1) = \ell(G_2).$$

Theorem

Any $\ell = 0$ *scheme of genus* g *can be reconstructed from* $\ell = 0$ *schemes of genus* $g' < g$.

$\Rightarrow$ straightforward (though quickly impractical) algorithm to generate all $\ell = 0$ schemes of fixed genus. To be contrasted with classification of schemes of fixed degree ω.

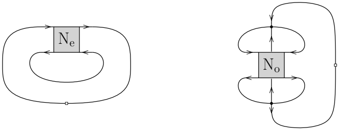
$\ell = 0$ scheme of genus 0



Represents all melonic two-point functions.

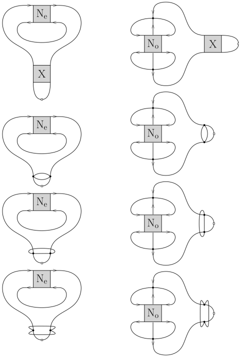
$\ell = 0$ schemes of genus 1

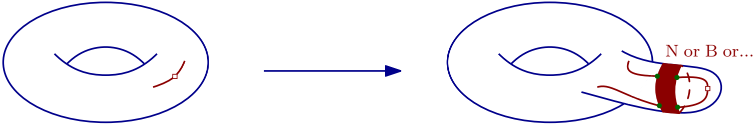
$\exists$ two 2PI schemes



S_1 S_2

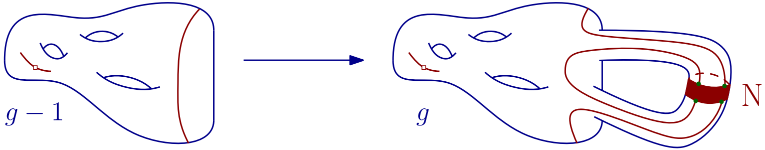
$\exists$ sixteen 2PR schemes



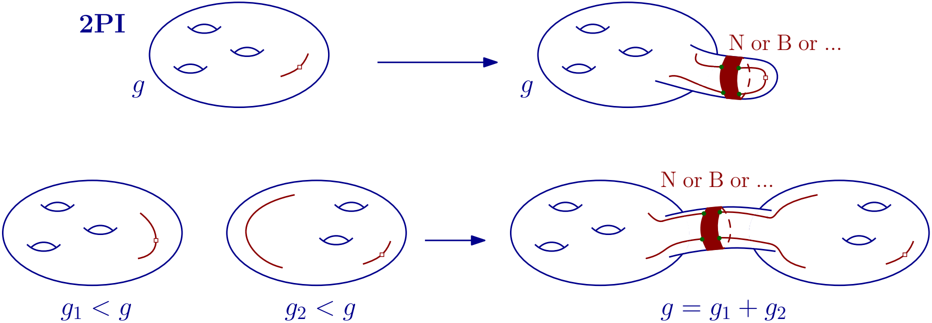


$\ell = 0$ schemes of genus $g \geq 2$

Generating 2PI schemes



Generating 2PR schemes



3. Triple-scaling limit of the connected generating function

Connected generating function

$$\mathcal{G}^{(0)}(\lambda) = \sum_{g \in \mathbb{N}} \mathcal{G}_g(\lambda) M^{2-2g}$$

with $\mathcal{G}_g(\lambda) = \#\{$rooted $\ell = 0$ graphs of genus $g\}$.

- Decompose $\mathcal{G}_g$ into a sum over a finite number of schemes:

$$\mathcal{G}_g(\lambda) = \sum_{\ell=0 \text{ scheme } S} \hat{\mathcal{G}}_S(\lambda)$$

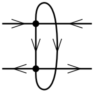
- Still too hard to resum all schemes of genus g.
- Focus on the subclass of schemes that contribute to the dominant singularity of $\mathcal{G}_g$

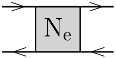
  $\rightarrow$ *dominant schemes*.

- Singularities can only arise from the resummation of melon and ladder diagrams.

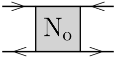
Dominant schemes

 $\longleftrightarrow \quad T(\lambda)$

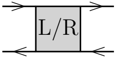
 $\longleftrightarrow \quad \lambda^2 T(\lambda)^4 =: u$



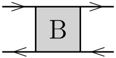
$\longleftrightarrow \quad \frac{u^2}{1-u^2}$



$\longleftrightarrow \quad \frac{u^3}{1-u^2}$



$\longleftrightarrow \quad \frac{u^2}{1-u^2}$



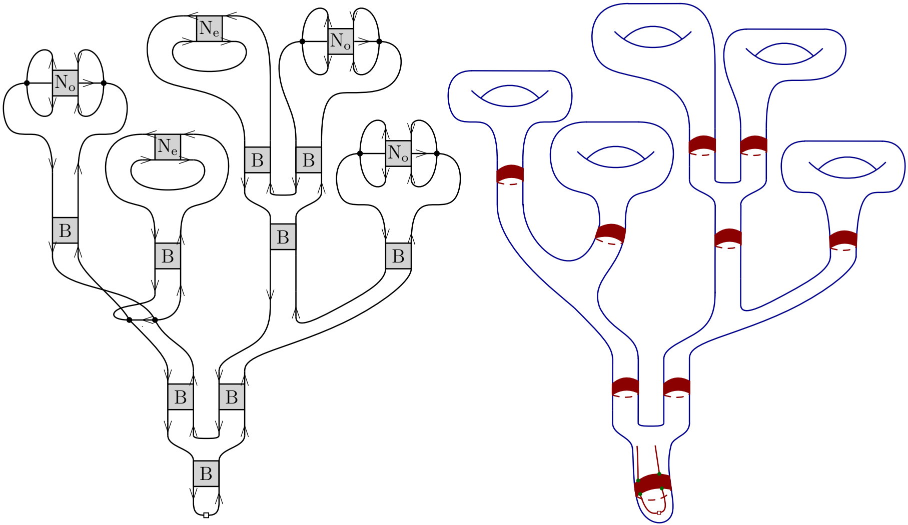
$\longleftrightarrow \quad \frac{6u^2}{(1-3u)(1-u)}$

Dominant singularity: $\lambda_c = (3^3/4^4)^1/2 \Leftrightarrow u_c = 1/3$

The dominant schemes are those that maximize the number of B-vertices at fixed g.

Theorem

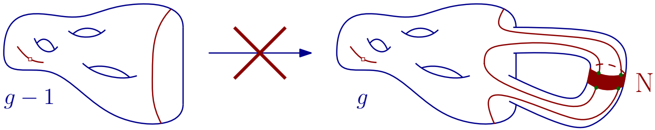
Dominant schemes of genus g have $2g-1$ B-vertices, and are in one-to-one correspondence with decorated plane binary trees.



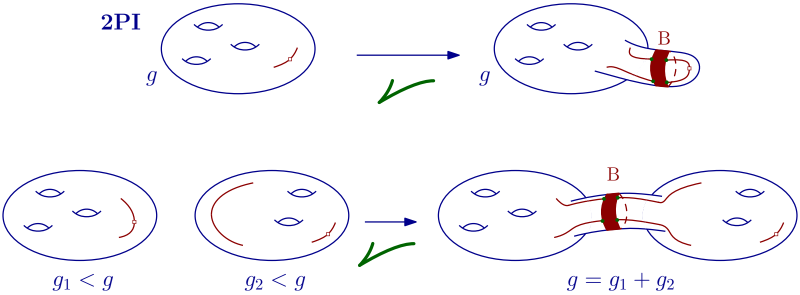
Dominant scheme of genus $g = 5$: there are $2g - 1 = 9$ B-vertices.

Idea of proof:

2PI



2PR



Show that those B-vertices must be glued along specific six-point functions. □

Resummation of dominant schemes

- The trees can be enumerated exactly: for any $g \geq 1$

$$\mathcal{G}_g(\lambda) \underset{\lambda \to \lambda_c^-}{\sim} \sum_{\text{dominant scheme } S} \hat{\mathcal{G}}_S(\lambda) \sim \frac{2}{3}\sqrt{\frac{8}{3}}\mathcal{T}_g \left(\frac{5}{48}\right)^g \left(\sqrt{1-\frac{\lambda^2}{\lambda_c^2}}\right)^{1-2g}$$

where $\mathcal{T}_g = \frac{1}{2g-1}\binom{2g-1}{g-1}$.

- The critical exponent is linear in the genus
  $\Rightarrow$ triple-scaling limit, with parameter: $\kappa^{-1} := M\left(1-\frac{\lambda^2}{\lambda_c^2}\right)^{1/2}$

$$\mathcal{D}(\kappa) := \frac{\kappa}{M}\left(\mathcal{G}^{(0)}(\lambda) - M^2 T(\lambda)\right) \sim \frac{2}{3}\sqrt{\frac{8}{3}}\sum_{g\geq 1}\mathcal{T}_g\left(\frac{5}{48}\right)^g \kappa^{2g}$$

$$= \left(\frac{2}{3}\right)^{\frac{3}{2}}\left(1-\sqrt{1-\frac{5}{12}\kappa^2}\right)$$

- Near $\kappa_c = \sqrt{\frac{12}{5}}$, large random trees (representing surfaces with large g) dominate:

$$\langle g \rangle = \frac{1}{2}\kappa\partial_\kappa \ln \mathcal{D}(\kappa) \simeq \frac{1}{2\sqrt{1-\kappa^2/\kappa_c^2}}$$

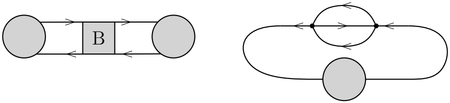
4. Triple-scaling limit of the 2PI generating function

Main idea

To avoid tree-like structures, one needs to:

- avoid the $\lambda = \lambda_c$ singularity of the melonic generating function;
- avoid the $u = 1/3$ singularity of the B-ladder generating function.

Melon diagrams and B-ladders have a common feature: they generate two-edge cuts / 2PR components.



$\Rightarrow$ restricting the sum to 2PI graphs kills these contributions, while still allowing N-ladders to proliferate.

$\Rightarrow$ this restriction allows to reach the $u = 1$ singularity of N-ladders, and tune them to criticality.

2PI generating function

- Define theory with modified covariance:

$$S[X,X^\dagger;m] = ND\left((1-m)\mathrm{Tr}\left[X_\mu^\dagger X_\mu\right] - \frac{\lambda}{2}\sqrt{D}\,\mathrm{Tr}\left[X_\mu^\dagger X_\nu X_\mu^\dagger X_\nu\right]\right)$$

$$\left\langle \mathrm{Tr}\left[X_\mu^\dagger X_\mu\right]\right\rangle_m = \frac{\int[dX]\, e^{-S[X,X^\dagger;m]}\mathrm{Tr}\left[X_\mu^\dagger X_\mu\right]}{\int[dX]\, e^{-S[X,X^\dagger;m]}}$$

- Define $m(\lambda)$ as the solution of:

$$\left\langle \mathrm{Tr}\left[X_\mu^\dagger X_\mu\right]\right\rangle_{m(\lambda)} = N\,, \qquad \text{with} \qquad m(0)=0.$$

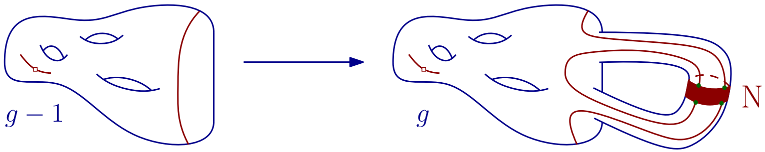
- Claim: $m(\lambda)$ is the generating function of rooted 2PI Feynman diagrams.

- In the double-scaling limit:

$$\mathcal{G}_{\mathrm{2PI}}^{(0)}(\lambda) = \lim_{\substack{N,D\to\infty \\ M<\infty}} \frac{N^2}{D} m(\lambda) = \sum_{g\in\mathbb{N}} \mathcal{G}_g^{\mathrm{2PI}}(\lambda) M^{2-2g}$$

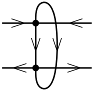
with $\mathcal{G}_g^{\mathrm{2PI}}(\lambda) = \#\{$rooted and 2PI $\ell=0$ graphs of genus $g\}$.

$\exists$ only one way of increasing the genus of a $\ell = 0$ 2PI graph:

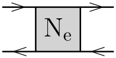


2PI-dominant schemes

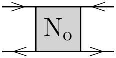
$\longleftrightarrow \quad 1$



$\longleftrightarrow \quad \lambda^2$



$\longleftrightarrow \quad \frac{\lambda^4}{1-\lambda^4}$



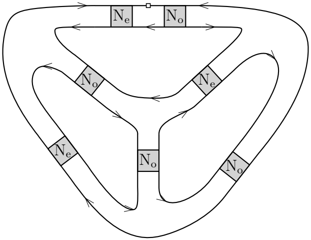
$\longleftrightarrow \quad \frac{\lambda^6}{1-\lambda^4}$

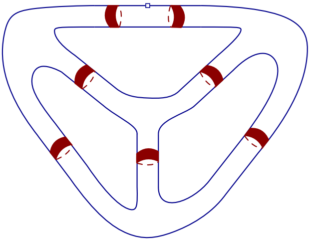
Dominant singularity: $\lambda_* = 1$

The 2PI-dominant schemes are those that maximize the number of N-vertices at fixed g.

Theorem

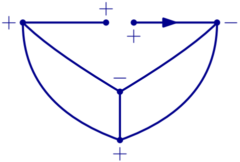
2PI-dominant schemes of genus g have $3g - 2$ N-vertices, and are in one-to-one correspondence with decorated rooted, cubic and bridgeless (i.e. 1PI) planar maps.

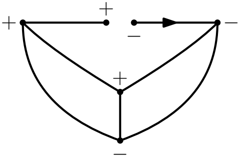




2PI-dominant scheme of genus $g = 3$: there are $3g - 2 = 7$ N-vertices. The N-ladders encode non-separating cuffs in a pants decomposition of the manifold.

Claim. 2PI-dominant schemes are in one-to-one correspondence with Ising states on rooted, cubic and bridgeless planar maps:





Resummation of 2PI-dominant schemes

- The combinatorial mapping to Ising yields: for any $g \geq 1$

$$\mathcal{G}^{(0)}_{\mathrm{2PI}}(\lambda) \underset{\lambda \to \lambda_*^-}{\sim} M^{2/3} \left(Z_{++}(\mathcal{C}_{\mathrm{N_o}}(\lambda^2) M^{-2/3}, \lambda^{-2}) + \lambda^2 Z_{+-}(\mathcal{C}_{\mathrm{N_o}}(\lambda^2) M^{-2/3}, \lambda^{-2}) \right)$$

  where

$$Z_{++}(t,x) = \sum_{T \in \mathcal{T}_{++}} t^{\epsilon(T)} x^{m(T)}$$

  is the grand-canonical partition function for the Ising model on random, cubic and bridgeless planar maps with boundary condition $(++)$. $\qquad (x = \mathrm{e}^{2\beta}\,, t = z\mathrm{e}^{-2\beta})$

- Such Ising partition functions are explicitly solvable in general:
  - by matrix-integral methods (effective two-matrix model); [Kazakov '86; Boulatov, Kazakov '87]
  - by bijective methods (Tutte equations with 2 catalytic variables). [Bernardi, Bousquet-Mélou '11]

- Here, we are only interested in the high-temperature limit:

$$\lambda \to \lambda_* = 1 \qquad \Leftrightarrow \qquad \beta \to 0 \qquad \Leftrightarrow \qquad x \to 1$$

$\Rightarrow$ the evaluation reduces to an enumeration problem, solvable by a one-matrix model or a Tutte equation with one catalytic variable.

- One finds:

$$\tilde{\mathcal{D}}(\kappa) := (1-\lambda)\mathcal{G}^{(0)}_{\rm 2PI}(\lambda) \underset{\lambda\to\lambda_*^-}{\sim} \frac{1}{2}\sum_{n\in\mathbb{N}} \left(\frac{\kappa^2}{16}\right)^n \mathcal{M}_n$$

where the triple-scaling parameter $\kappa^{-1} = M\,(1-\lambda)^{3/2}$ is kept fixed, and
$\mathcal{M}_n = \#\{$rooted, bridgeless and planar cubic maps with $2n$ vertices$\}$ (OEIS A000309)

- Well-known enumeration: [Tutte '62]

$$\mathcal{M}_n = \frac{2^n(3n)!}{(n+1)!(2n+1)!} \sim \frac{1}{4}\sqrt{\frac{3}{\pi}}\left(\frac{27}{2}\right)^n n^{-5/2}$$

- Singularity at $\kappa_c = \frac{8}{3\sqrt{6}}$:

$$\tilde{\mathcal{D}}(\kappa) \underset{\kappa\to\kappa_c^-}{\sim} \frac{1}{2\sqrt{3}}\left(1-\frac{\kappa^2}{\kappa_c^2}\right)^{3/2}$$

- Near $\kappa_c = \sqrt{\frac{12}{5}}$, large random planar maps (representing surfaces with large g) dominate:

$$\langle g\rangle = \langle n+1\rangle < \infty\,, \qquad \langle g^2\rangle \underset{\kappa\to\kappa_c^-}{\sim} K\left(1-\frac{\kappa^2}{\kappa_c^2}\right)^{-1/2}$$

5. Summary and discussion

Summary

- Multi-matrix model generating random surfaces decorated by loops.
- The large N parameter controls the genus of the surfaces. The large D parameter controls the loops.
- Double-scaling $M = N/\sqrt{D}$ $\rightarrow$ retains non-trivial contributions at arbitrary genus, on top of the melonic genus 0 sector.
- Result 1: combinatorial characterization of all graphs contributing to the double-scaling limit.
- Result 2: the connected partition function admits a triple-scaling limit dominated by surfaces of large genus proliferating like random trees.
- Result 2: the 2PI partition function admits a triple-scaling limit dominated by surfaces of large genus proliferating like random planar maps.

Outlook

- Arbitrary n-point functions: loop equations and topological recursion?
- SYK-like models and QFT: tractable higher-genus corrections to the melonic behaviour?
- Connection to recent works on Euclidean wormholes and the Page curve in 2d quantum gravity?
  [Saad, Shenker, Stanford, Witten, Penington, Almheiri, Engelhardt, Maxfield, Marolf,...]