Crystallizations of compact 4-manifolds minimizing combinatorially defined PL-invariants

https://arxiv.org/abs/2004.07894

joint work with M.R. Casali and C. Gagliardi

Crystallization theory: a representation theory for compact PL n-manifolds

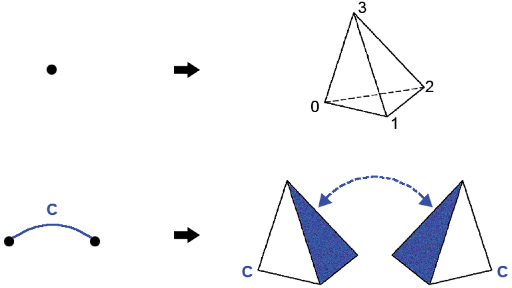
$(n+1)$-colored graph (Γ, γ)

- $\Gamma = (V(\Gamma), E(\Gamma))$ regular graph of degree $n+1$,
- $\gamma : E(\Gamma) \to \Delta_n = \{0, \ldots, n\}$ such that $\gamma(e) \neq \gamma(f)$ for each pair of adjacent edges $e, f \in E(\Gamma)$ (*edge-coloration*)

graph = multigraph (multiple edges allowed, loops forbidden)

The pseudocomplex $K(\Gamma)$

1) take a n-simplex $\sigma(x)$ for every vertex $x \in V(\Gamma)$, and label its vertices by Δ_n;
2) if $x, y \in V(\Gamma)$ are joined by a c-colored edge, identify the $(n-1)$-faces of $\sigma(x)$ and $\sigma(y)$ opposite to c-labelled vertices, so that equally labelled vertices coincide.



- $\star$ for each $c \in \Delta_n$, each connected component of $\Gamma_{\hat{c}} = \Gamma_{\Delta_d - \{c\}}$ ($\hat{c}$-residue of Γ) represents $Lk(v_c, K'(\Gamma)), \quad v_c \in V(K(\Gamma))$.
- $\star$ $|K(\Gamma)|$ is a closed n-manifold if and only if, for every $c \in \Delta_n$, each $\hat{c}$-residue of Γ represents $\mathbb{S}^{n-1}$.
- $\star$ $|K(\Gamma)|$ is a singular n-manifold if and only if, for every $c \in \Delta_n$, each $\hat{c}$-residue of Γ represents a closed connected PL $(n-1)$-manifold.

A singular (PL) n-manifold ($n > 1$) is a compact connected n-dimensional polyhedron admitting a simplicial triangulation where the links of vertices are closed connected PL $(n-1)$-manifolds.
Vertices whose links are not PL $(n-1)$-spheres are called singular.

Singular manifolds/bounded manifolds

{(closed) manifolds} $\subset$ {singular manifolds} $\subset$ {pseudomanifolds}

From a singular n-manifold N to a n-manifold with boundary $\check{N}$
(by deleting regular neighbourhoods of singular vertices)

From a n-manifold M with boundary to a singular n-manifold $\hat{M}$
(by capping off the boundary components with cones over them)

Existence Theorem (Pezzana, Casali - C. - Grasselli)

Any orientable (resp. non-orientable) compact PL n-manifold M^n can be represented by a bipartite (resp. non-bipartite) $(n+1)$-colored graph (Γ, γ) that can always be supposed to be contracted, i.e. for each color $c \in \Delta_n$, either $\Gamma_{\hat{c}}$ is connected or no connected component of $\Gamma_{\hat{c}}$ represents an $(n-1)$-sphere ($=$ either $K(\Gamma)$ has only one c-colored vertex or all c-colored vertices of $K(\Gamma)$ are singular).

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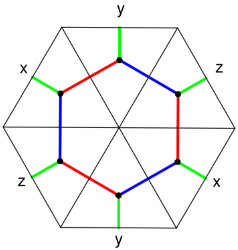
The gem-complexity of a compact PL n-manifold M^n is

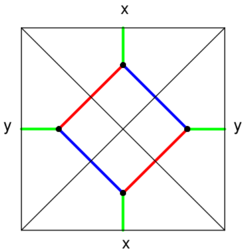
$$k(M^n) = p - 1$$

where $2p =$ minimum order of an $(n+1)$-colored graph representing M^n

Regular embeddings

A cellular embedding $\phi : |\Gamma| \to F$ of a $(n+1)$-colored graph (Γ, γ) into a (closed) surface F is called a regular embedding if there exists a cyclic permutation $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)$ of Δ_n s.t. each connected component of $F - \phi(|\Gamma|)$ is an open ball bounded by the image of an $\{\varepsilon_i, \varepsilon_{i+1}\}-$ colored cycle of Γ $(\forall i \in \mathbb{Z}_n)$.





The regular genus

Gagliardi, 1981

For each $(n+1)$-colored graph (Γ, γ) and for every cyclic permutation ε of Δ_n, there exists a *regular embedding* of Γ into a suitable surface F_ε. Moreover:

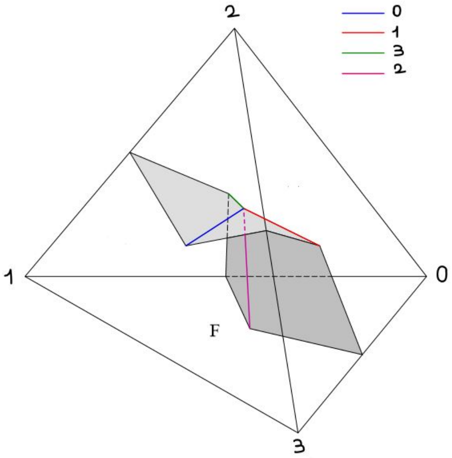
- F_ε is orientable if and only if Γ is bipartite;
- ε and ε^{-1} induce the same embedding.

Definition

The *regular genus* $\rho_\epsilon(\Gamma)$ of Γ with respect to ε is the classical genus (resp. half of the genus) of the orientable (resp. non-orientable) surface F_ε :

$$\sum_{i \in \mathbb{Z}_{d+1}} g_{\varepsilon_i \varepsilon_{i+1}} + (1-n)p = 2 - 2\rho_\varepsilon(\Gamma)$$

Heegaard splitting of M^3 : (H_0, H_1) s.t. $M^3 = H_0 \cup H_1, \; H_0 \cap H_1 = F$
$H_0 = N(K_{02}) \; H_1 = N(K_{13}) \qquad \mathcal{H}(M^3) = \mathcal{G}(M^3)$



Regular embeddings and Gurau degree

Regular genus of Γ

$$\rho(\Gamma) = \min\{\rho_\varepsilon(\Gamma) \ / \quad \varepsilon \text{ cyclic permutation of } \Delta_n\}$$

Gurau degree

Given a $(n+1)$-colored graph (Γ, γ), then

$$\omega_G(\Gamma) = \sum_{i=1}^{\frac{n!}{2}} \rho_{\varepsilon^{(i)}}(\Gamma)$$

where the $\varepsilon^{(i)}$'s are the cyclic permutations of Δ_n up to inverse.

PL invariants

Definition

Regular genus of a compact PL n-manifold M^n:

$$\mathcal{G}(M^n) = \min \{\rho(\Gamma) \mid (\Gamma, \gamma) \text{ represents } M^n\}$$

Gurau degree (G-degree) of a compact PL n-manifold M^n:

$$\mathcal{D}_G(M^n) = \min \{\omega_G(\Gamma) \mid (\Gamma, \gamma) \text{ represents } M^n\}$$

The minimum is always realized by a contracted graph.

Dimension 4: $\mathbf{TOP} \neq \mathbf{PL} \quad \mathbf{PL} = \mathbf{DIFF}$

Simple crystallizations of closed PL 4-manifolds

Dimension 4: TOP $\neq$ PL PL = DIFF

Definition (B. Basak - J. Spreer 2016)

A *simple crystallization* of a closed PL 4-manifold M^4 is a contracted 5-colored graph Γ, representing M^4, such that $g_{i,j,k} = 1, \ \forall i, j, k \in \Delta_4$

Equivalently, the 1-skeleton of $K(\Gamma)$ coincides with 1-skeleton of the standard 4-simplex

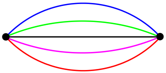
If M^4 admits simple crystallizations, then M^4 is simply-connected

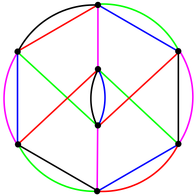
Basak - Spreer, 2016

- ⋆ all closed "standard" simply-connected PL 4-manifolds[a] have simple crystallizations
- ⋆ connected sums of simple crystallizations of closed PL 4-manifolds are simple crystallizations of the connected sum
- ⋆ examples of simple crystallizations of pairs of homeomorphic but not PL-homeomorphic closed 4-manifolds ($K3\#\overline{\mathbb{CP}^2}$ and $3\mathbb{CP}^2\#20\overline{\mathbb{CP}^2}$)

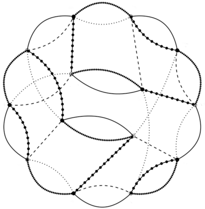
[a] i.e., $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $K3$

The (unique) simple crystallizations of $\mathbb{S}^4$ and $\mathbb{CP}^2$





A simple crystallization of $\mathbb{S}^2 \times \mathbb{S}^2$



Simple crystallizations and PL invariants

Casali - Cristofori - Gagliardi, 2016

A closed simply-connected PL 4-manifold M^4 admits simple crystallizations if and only if $k(M) = 3\beta_2(M)$.

If M^4 admits simple crystallizations, then $\mathcal{G}(M) = 2\beta_2(M)$.

Generalizations:

- ⋆ semi-simple: not simply-connected (Basak-Casali 2017), not closed (Casali-Cristofori 2019)
- ⋆ weak semi-simple: crystallizations for which formula $\mathcal{G}(M) = 2\beta_2(M)$ is a characterization (Basak 2018)
- ⋆ larger class where the invariants are not determined only by the TOP structure, but are easily computable (Casali-Cristofori 2020)
- ⋆ taking into consideration the G-degree (Casali-Cristofori-Gagliardi 2020)

Lower bounds

Notations: M^4 = a compact 4-manifold with empty or connected boundary;
$\widehat{M}^4$ = associated singular manifold
$rk(\pi_1(M^4)) = m \quad rk(\pi_1(\widehat{M}^4)) = m' \quad (0 \le m' \le m)$
χ = Euler characteristic

Casali-Cristofori-Gagliardi 2020

$$\mathcal{G}(M^4) \ge 2\chi(\widehat{M}^4) + 5m - 2(m - m') - 4$$

$$\mathcal{D}_G(M^4) \ge 12\left[2\chi(\widehat{M}^4) + 5m - 2(m - m') - 4\right]$$

$$k(M^4) \ge 3\chi(\widehat{M}^4) + 10m - 4(m - m') - 6$$

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Corollary

$$\mathcal{D}_G(M^4) = 6\Big[\chi(\widehat{M}^4) - 2 + k(M^4)\Big].$$

color 4 is the unique singular color

$$t_{j,k,l} = (g_{j,k,l} - 1) - m' \geq 0 \quad \forall j,k,l \in \Delta_3$$
$$t_{j,k,4} = (g_{j,k,4} - 1) - m \geq 0 \quad \forall j,k \in \Delta_3$$

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$t_{j,k,4} = (g_{j,k,4} - 1) - m \geq 0 \quad \forall j,k \in \Delta_3$

Let Γ be an order $2p$ crystallization of a compact 4-manifold M^4 with empty or connected boundary and $\varepsilon \in \mathcal{P}_4$:

$$\rho_\varepsilon(\Gamma) \;=\; 2\chi(\widehat{M}^4) + 5m - 2(m-m') - 4 + \sum_{i\in\mathbb{Z}_5} t_{\varepsilon_i,\varepsilon_{i+2},\varepsilon_{i+4}}$$

$$\omega_G(\Gamma) \;=\; 6\Big[4\chi(\widehat{M}^4) + 10m - 4(m-m') - 8 + \sum_{i,j,k\in\mathbb{Z}_5} t_{i,j,k}\Big]$$

$$p-1 \;=\; 3\chi(\widehat{M}^4) + 10m - 4(m-m') - 6 + \sum_{i,j,k\in\mathbb{Z}_5} t_{i,j,k}$$

$\mathcal{P}_4 =$ set of all cyclic permutations $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 = 4)$ of Δ_4.

Weak semi simple crystallizations

Definition (Basak 2017, Casali-Cristofori 2019)

Let Γ be a crystallization of a compact PL 4-manifold M^4 with empty or connected boundary and let color 4 be its (unique) possible singular color.

Γ is called semi-simple if

$g_{j,k,l} = 1 + m' \;\; \forall\, j,k,l \in \Delta_3$ and

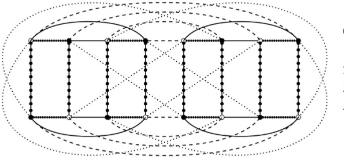
$g_{j,k,4} = 1 + m \;\; \forall\, j,k \in \Delta_3$.

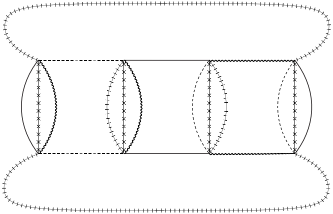
Γ is called weak semi-simple with respect to a permutation $\varepsilon \in \mathcal{P}_4$ if

$g_{\varepsilon_i,\varepsilon_{i+2},\varepsilon_{i+4}} = 1 + m \;\; \forall\, i \in \{0,2,4\}$ and

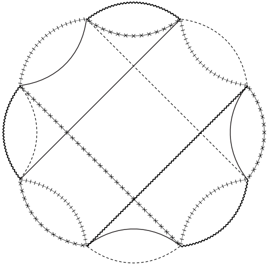
$g_{\varepsilon_i,\varepsilon_{i+2},\varepsilon_{i+4}} = 1 + m' \;\; \forall\, i \in \{1,3\}$.

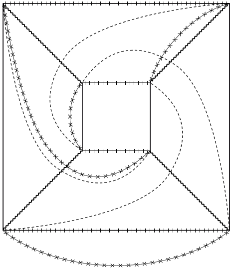
Semi-simple crystallizations of $\mathbb{RP}^4$ and $\mathbb{S}^1 \times \mathbb{D}^3$





Simple crystallizations of $\mathbb{S}^2 \times \mathbb{D}^2$ and ξ_2





Casali-Cristofori-Gagliardi 2020

$$\mathcal{G}(M^4) \geq 2\chi(\widehat{M}^4) + 5m - 2(m - m') - 4$$

equality holds if and only if M^4 admits a weak semi-simple crystallization.

$$\mathcal{D}_G(M^4) \geq 12\Big[2\chi(\widehat{M}^4) + 5m - 2(m - m') - 4\Big]$$

equality holds if and only if M^4 admits a semi-simple crystallization.

$$k(M^4) \geq 3\chi(\widehat{M}^4) + 10m - 4(m - m') - 6$$

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M^4	$\mathcal{G}(M^4)$	$\mathcal{D}_G(M^4)$	$k(M^4)$	
$\mathbb{S}^4$	0	0	0	simple
$\mathbb{CP}^2$	2	24	3	simple
$\mathbb{S}^2 \times \mathbb{S}^2$	4	48	6	simple
$\mathbb{S}^1 \times \mathbb{S}^3$ and $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^3$	1	12	4	semi-simple
$\mathbb{RP}^4$	3	36	7	semi-simple
$K3$	44	528	66	simple
ξ_2	2	24	3	simple
$\mathbb{S}^2 \times \mathbb{D}^2$	2	24	3	simple
$\mathbb{Y}^4_h$ and $\widetilde{\mathbb{Y}}^4_h$	h	$12h$	$3h$	semi-simple
ξ_c $(c \in \mathbb{Z}^+ - \{1,2\})$	2	$\leq 12c$	$\leq 2c-1$	weak simple

.

$\varepsilon_{\hat{i}} = (\varepsilon_0, \dots, \varepsilon_{i-1}, \varepsilon_{i+1}, \dots, \varepsilon_n)$

Casali-Cristofori-Gagliardi 2020

Γ is weak semi-simple with respect to the cyclic permutation $\varepsilon \in \mathcal{P}_4$ if and only if

$$\mathcal{G}(M^4) = \rho(\Gamma) = \rho_\varepsilon(\Gamma) = 2\chi(\widehat{M}^4) + 5m - 2(m - m') - 4$$

equivalently, if and if

$$\rho_{\varepsilon_{\hat{i}}}(\Gamma_{\hat{i}}) = \chi(\widehat{M}^4) + m + m' - 2 \ \ \forall i \in \Delta_3 \quad \text{and} \quad \rho_{\hat{4}}(\Gamma_{\hat{4}}) = \chi(\widehat{M}^4) + 2m - 2.$$

$\varepsilon_{\hat{i}} = (\varepsilon_0, \dots, \varepsilon_{i-1}, \varepsilon_{i+1}, \dots, \varepsilon_n)$

Casali-Cristofori-Gagliardi 2020

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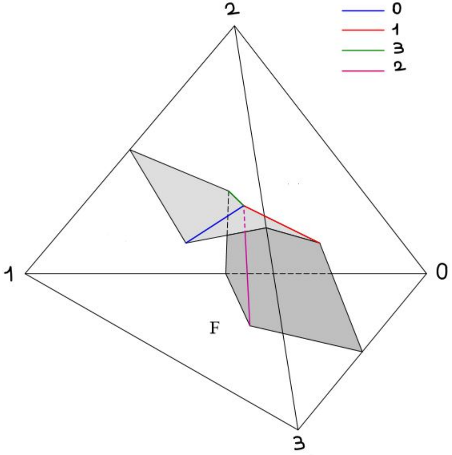
M^4 admits a semi-simple crystallization if and only of

$$k(M^4) = \frac{3\mathcal{G}(M^4) + 5m - 2(m - m')}{2}$$

Heegaard splitting of M^3 : (H_0, H_1) s.t. $M^3 = H_0 \cup H_1$, $H_0 \cap H_1 = F$
$H_0 = N(K_{02})$ $H_1 = N(K_{13})$ $\mathcal{H}(M^3) = \mathcal{G}(M^3)$



Definition (Gay-Kirby 2016)

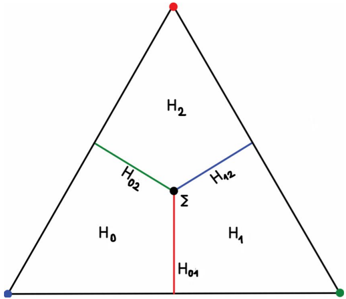
A trisection of a closed PL orientable 4-manifold M^4 is a triple $\mathcal{T} = (H_0, H_1, H_2)$ of 4-dimensional submanifolds of M^4, such that:

(a) $M^4 = H_0 \cup H_1 \cup H_2$ and H_0, H_1, H_2 have pairwise disjoint interiors

(b) H_0, H_1, H_2 are 4-dimensional handlebodies

(c) $H_{01} = H_0 \cap H_1$, $H_{02} = H_0 \cap H_2$ and $H_{12} = H_1 \cap H_2$ are 3-dimensional handlebodies;

(d) $\Sigma(\mathcal{T}) = H_0 \cap H_1 \cap H_2$ is a closed connected surface (which is called *central surface*).



- ⋆ Any closed orientable smooth 4-manifold admits a trisection (Gay-Kirby 2016)
- ⋆ There is an algorithm to obtain a trisection from any triangulation of a closed orientable PL 4-manifold (Bell-Hass-Rubinstein-Tillmann 2018)
- ⋆ Computation of the trisection genus of all closed simply-connected standard PL 4-manifolds (Spreer-Tillmann 2018)

Casali-Cristofori 2019

For each crystallization (Γ, γ) of a compact orientable PL 4-manifold with empty (resp. connected boundary) M^4 and for each $\varepsilon \in \mathcal{P}_4$, a triple $\mathcal{T}(\Gamma, \varepsilon) = (H_0, H_1, H_2)$ of submanifolds of M^4 is constructed, such that

(a') $M^4 = H_0 \cup H_1 \cup H_2$ and H_0, H_1, H_2 have pairwise disjoint interiors

(b') H_1, H_2 are 4-dimensional handlebodies; H_0 is a 4-disk (resp. is (PL) homeomorphic to $\partial M^4 \times [0,1]$)

(c') $H_{01} = H_0 \cap H_1$, $H_{02} = H_0 \cap H_2$ are 3-dimensional handlebodies

(d') the central surface $\Sigma(\mathcal{T}(\Gamma, \varepsilon)) = H_0 \cap H_1 \cap H_2$ is a closed connected surface of genus $\rho_{\hat{4}}(\Gamma_{\hat{4}})$

Casali-Cristofori 2019

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G-trisection genus of M^4:

$$g_{GT}(M^4) = min\{genus(\Sigma(\mathcal{T}(\Gamma, \varepsilon))) \mid \mathcal{T}(\Gamma, \varepsilon) \text{ is a B-trisection of } M^4\}$$

Casali-Cristofori 2019

If Γ is a weak semi-simple crystallization of M^4 with respect to $\varepsilon \in \mathcal{P}_4$, then

- $\Sigma(\mathcal{T}(\Gamma, \varepsilon))$ is of minimal genus among all central surfaces relative to crystallizations of M^4 and permutations of $\mathcal{P}_4$
- furthermore, if $\mathcal{T}(\Gamma, \varepsilon)$ is a B-trisection

$$g_{GT}(M^4) \;=\; \frac{1}{2}(\rho_\varepsilon(\Gamma) + m) \;=\; \beta_2(M^4) + \beta_1(M^4) + 2\left(m - \beta_1(M^4)\right)$$

In particular, if M^4 is simply-connected,

$$g_{GT}(M^4) \;=\; \frac{1}{2}\rho_\varepsilon(\Gamma) \;=\; \beta_2(M^4)$$

Additivity of the invariants and finiteness-to-one

Exotic structures and $\mathcal{D}_G$

$\mathcal{D}_G$ does not satisfy the additivity property, within the set of closed PL 4-manifolds.

Example: let N and N' be two of the infinitely many different PL manifolds homeomorphic to $\mathbb{CP}^2\#(\#_2(-\mathbb{CP}^2))$.

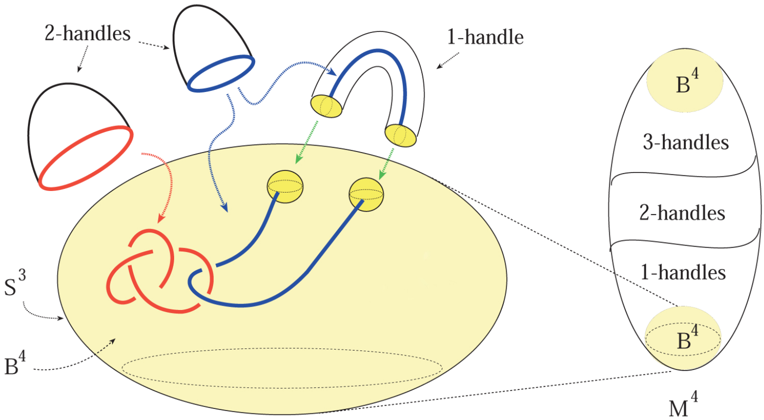
By Wall theorem: $\exists h \geq 0$ s.t. $N\#(\#_h\,\mathbb{S}^2 \times \mathbb{S}^2) \cong_{PL} N'\#(\#_h\,\mathbb{S}^2 \times \mathbb{S}^2)$

By Wall theorem and additivity:
$\mathcal{D}_G(N) = \mathcal{D}_G(N') \quad \Longrightarrow \quad$ impossible by finiteness property of $\mathcal{D}_G$.

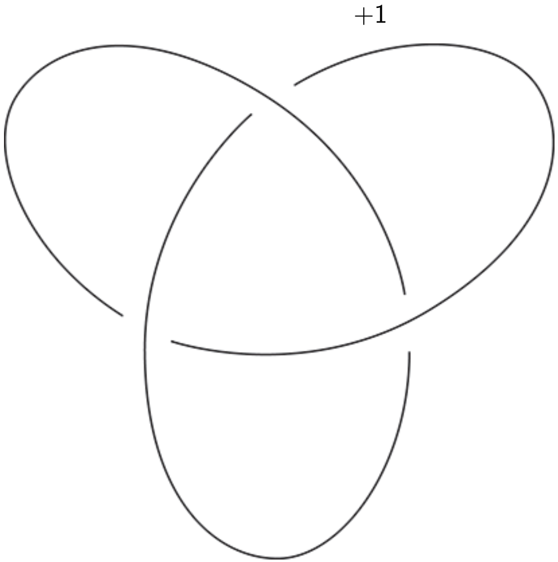
Additivity

- Graph-connected sums of weak semi-simple (resp. semi-simple) crystallizations are weak semi-simple (resp. semi-simple)
- If M_1 and M_2 admit weak semi-simple (resp. semi-simple) crystallizations, then, $M_1 \sharp M_2$ admits weak semi-simple (resp. semi-simple) crystallizations
- Gem-complexity and G-degree are additive within the class of compact PL 4-manifolds admitting semi-simple crystallizations
- Regular genus and G-trisection genus are additive within the class of compact PL 4-manifolds admitting weak semi-simple crystallizations

Work in progress



(Picture from S. Akbulut's book "4-manifolds")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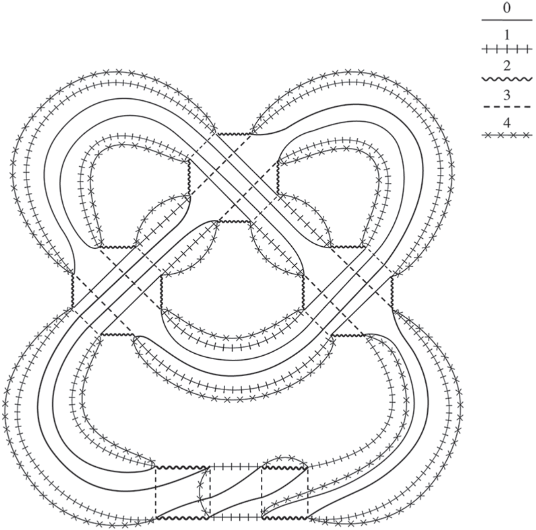
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THANK YOU