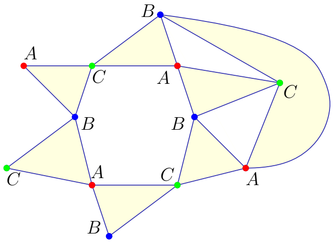
KP hierarchy for constellations



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LIPN – Combinatorics, Algorithms and Interactions – Paris University 13
IRIF – Université de Paris (2019-2020)

Sept. 21, 2020

virtual Tensor Journal Club

Abstract

The Hermitian 1-matrix model has been known for 30 years to satisfy the KP hierarchy. It is an infinite set of partial differential equations, here with respect to the coupling constants of the model. At least three proofs of this statement are known. Other systems in two-dimensional enumerative geometry such as, famously, intersection numbers of the moduli space of Riemann surfaces, satisfy the KP hierarchy. In this talk, I will introduce the KP hierarchy from the point of view of the Sato Grassmannian as an orbit of an action of $GL(\infty)$. This will then give us the opportunity to present the proof of Kazarian and Zograf that the generating function of bipartite maps satisfies the KP hierarchy. I will then focus on constellations, which are generalizations of bipartite maps, and are also the dual to the jackets of the bipartite edge-colored graphs of unitary-invariant tensor models. Constellations are known to also satisfy the KP hierarchy but only a proof via algebraic combinatorics is known. I will report some progress towards understanding their integrable properties using Tutte/Schwinger-Dyson–like equations. Only this last part is an original contribution.

What is KdV/KP/Toda hierarchy?

- Is it really in matrix model and combinatorial maps?
- All genus set of equations?
- Relation to loop equations?
- No teaching last year $\to$ opportunity to become a PhD once again!

Objective: Prove bipartite maps satisfy KP following Kazarian-Zograf

- Finite dim. Grassmannian
- Infinite dim. Sato Grassmannian and $\hat{GL}(\infty)$
- Boson-Fermion correspondence
- Kazarian-Zograf's "elementary" proof of KP for bipartite maps/dessins d'enfants
- Constellations are generalizations of bip maps
  Recent progress by G. Chapuy and M. Dolega
- Little personal contribution: W.i.p. with S. Dartois

Introduction

- Well-known for 30 years that matrix models satisfy KP hierarchy
- Related interest in Witten conjecture proved by Kontsevich
  Generating function of intersection numbers satisfies KP

Ubiquitous in combinatorial/enumerative geometry

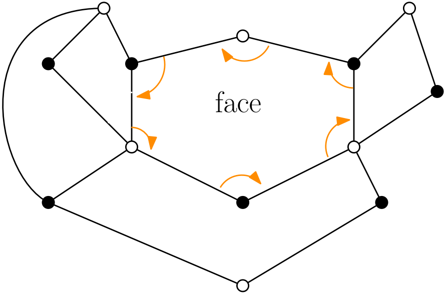
- Combinatorial maps and **constellations**
- Enumeration of **factorizations of permutations**

$$\sigma_1 \sigma_2 \cdots \sigma_d = 1$$

  with fixed cycle type and/or fixed number of cycles
- **Hurwitz numbers**
  Enumerations of branched covers of the sphere
- Double Hurwitz numbers [Okounkov]

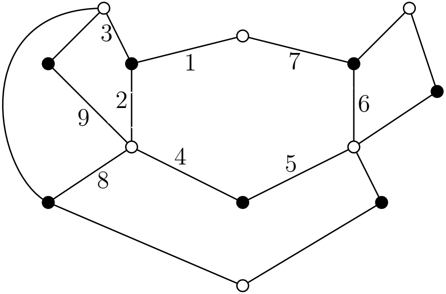
Example: bipartite maps

- Embedded graph without crossings = cyclic order around vertices
  $\Rightarrow$ Vertices, edges and faces



Example: bipartite maps

- Embedded graph without crossings = cyclic order around vertices
  $\Rightarrow$ Vertices, edges and faces
- Label edges



Example: bipartite maps

- Embedded graph without crossings = cyclic order around vertices
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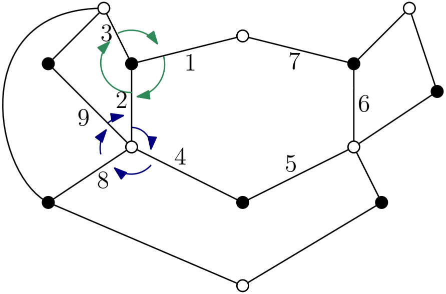
- Label edges
- Represent cyclic orders around vertices as perm. cycles

$$\sigma_\bullet = (123)\cdots \qquad \sigma_\circ = (2489)\cdots$$

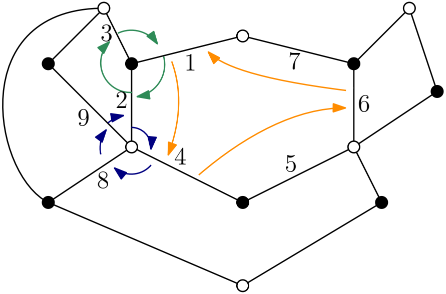


Example: bipartite maps

- Embedded graph without crossings = cyclic order around vertices
  $\Rightarrow$ Vertices, edges and faces
- Label edges
- Represent cyclic orders around vertices as perm. cycles

$$\sigma_\bullet = (123)\cdots \qquad \sigma_\circ = (2489)\cdots$$

- Faces as cycles of $\phi = \sigma_\circ \sigma_\bullet = (146)\cdots$



Generating function of labeled bipartite maps

Dictionary

	Bip. Maps	Weights	Matrix model
# Edges	n	t	$e^{-\frac{1}{t}\operatorname{tr} MM^\dagger}$
# White v.	$V_\circ$	N_1	Rectangular
# Black v.	$V_\bullet$	N_2	$M \in \mathbb{M}(N_1, N_2)$
# Faces of d°i	F_i	p_i	$e^{\frac{p_i}{i}\operatorname{tr}(MM^\dagger)^i}$

$$\tau(\vec{p}, t, N_1, N_2) = \sum \frac{t^n}{n!} N_1^{V_\circ} N_2^{V_\bullet} \prod_{i\geq 1} p_i^{F_i}$$

$$= \int dMdM^\dagger \; e^{-\frac{1}{t}\operatorname{tr} MM^\dagger + \sum_{i\geq 1} \frac{p_i}{i}\operatorname{tr}(MM^\dagger)^i}$$

What does it mean that it is a KP **tau-function**?

KP hierarchy

- Infinite number of PDEs
- KP equation for $u = 2\frac{\partial^2}{\partial p_1^2} \ln \tau$

$$12\frac{\partial^2 u}{\partial p_2^2} + \frac{\partial}{\partial p_1}\left(-12\frac{\partial u}{\partial p_3} + 6u\frac{\partial u}{\partial p_1} + \frac{\partial^3 u}{\partial p_1^3}\right) = 0$$

- Infinite number of eqs involving the other p_is

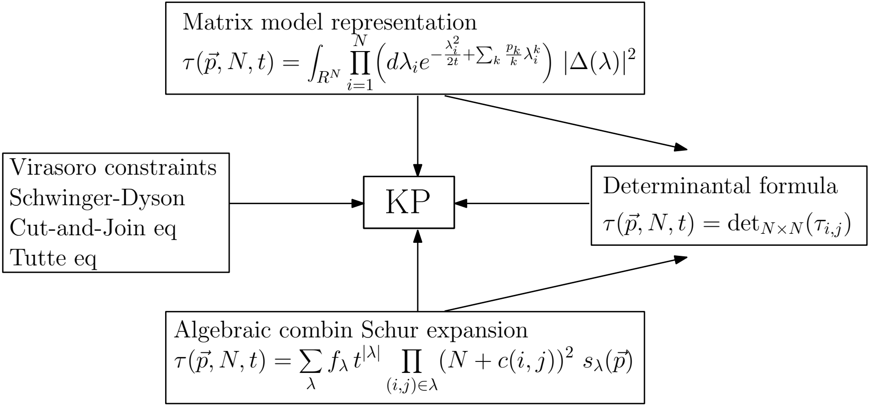
What can you do with that?

- For triangulations: reduces to ODE, solvable at large N
- Most efficient recursions for evaluating number of maps [Carrell-Chapuy, Louf]
- Used to study local limits of maps of high genus [Budzinski-Louf *Inv.*]
- Still many questions
  - Bijective versions of the hierarchy?
  - What to do with higher order eqs?

(Some) proofs of KP for maps

Let's get fancy

- **Thm** – Image of Plücker relations for Sato Grasmannian via boson-fermion correspondence
- **Thm** – Orbit of a solution under transitive action of $\widehat{GL}(\infty)$



Finite-dim Grassmannians

- Set of **k-dim. linear subspaces** in $\mathbb{C}^n$

$$\mathrm{Gr}(k,n) = \{\mathrm{span}(v_1,\dots,v_k), \quad v_1,\dots,v_k \text{ linearly indpt}\}$$

- **Action of $GL(n)$** on $\mathbb{C}^n \to$ action of $GL(n)$ on $\mathrm{Gr}(k,n)$

$$f \in GL(n) \quad \Rightarrow f\,\mathrm{span}(v_1,\dots,v_k) = \mathrm{span}(f(v_1),\dots,f(v_k))$$

- **Transitive** action

$$f(e_1) = v_1, \quad \dots, \quad f(e_k) = v_k$$

and complete the basis of $\mathbb{C}^n$

- Grassmannian as an **orbit** under $GL(n)$

$$|0\rangle = \mathrm{span}(e_1,\dots,e_k) \qquad \mathrm{Gr}(k,n) = GL(n)|0\rangle$$

Fact 1

Plücker embedding

- **Wedge/Exterior** product: antisymmetrized tensor product

$$v_1 \wedge v_2 = \frac{1}{2}\big(v_1 \otimes v_2 - v_2 \otimes v_1\big)$$

- k-th exterior power

$$\Lambda^k \mathbb{C}^n = \mathrm{span}\big(v_1 \wedge v_2 \wedge \cdots \wedge v_k\big)$$

- $\phi : \mathrm{Gr}(k, n) \to \mathbb{P}\Lambda^k \mathbb{C}^n$

$$\mathrm{span}\big(v_1, \ldots, v_k\big) \mapsto [v_1 \wedge \cdots \wedge v_k]$$

  ϕ is **injective**, but **not surjective**
- Point in $\mathrm{Gr}(k, n)$ = factorized wedge product

$$e_1 \wedge e_2 + e_3 \wedge e_2 = (e_1 + e_3) \wedge e_2 \in \phi(\mathrm{Gr}(k, n))$$
$$e_1 \wedge e_2 + e_3 \wedge e_4 \notin \phi(\mathrm{Gr}(k, n))$$

Plücker relations

Theorem – Fact 2

$x \in \mathbb{P}\Lambda^k\mathbb{C}^n$. Then

$$x \in \phi(\mathrm{Gr}(k,n)) \quad \Leftrightarrow \quad \text{Quadratic relations}$$

Philosophy: Look in and out of the subspace

- Say $x = [v_1 \wedge \cdots \wedge v_k] \in \phi(\mathrm{Gr}(k,n))$
- **Wedging** with vectors of the inside

$$v_i \wedge x = 0$$

- **Contractions** with $\perp$ vectors
  Denote $(w_1, \ldots, w_{n-k})$ orthogonal supplement to $(v_1, \ldots, v_k)$

$$\iota_{w_i^*} x = 0$$

  where $\iota_{w^*} v_1 \wedge v_2 \wedge \cdots = w^*(v_1) v_2 \wedge \cdots - w^*(v_2) v_1 \wedge \cdots + \cdots$
- That is sufficient

Semi-infinite wedge and Sato Grassmannian

- In infinite dim. $\mathbb{C}^\infty = \text{span}(e_i)_{i\in\mathbb{Z}}$, half-infinite wedge product

$$v_0 \wedge v_1 \wedge v_2 \wedge \cdots, \qquad \forall i \geq 0 \quad v_i \in \mathbb{C}^\infty$$

- Freeze the degrees of freedom at $-\infty$

$$v_0 \wedge v_1 \wedge v_2 \wedge \cdots \wedge e_{-k} \wedge e_{-k-1} \wedge e_{-k-2} \wedge \cdots$$

  Ex: $e_{31} \wedge e_{17} \wedge e_{-4} \wedge e_{-5} \wedge e_{-6} \wedge \cdots$

- **Semi-infinite wedge**: all linear superpositions of such vectors

$$\Lambda^{\frac{\infty}{2}} = \text{span}(e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \\ \mid i_0 > i_1 > \cdots \quad \& \quad i_n = -n \text{ for } n \gg 0)$$

- (**Semi**-infinite) + (**frozen** $-\infty$) = Fredholm condition
- **Sato Grassmannian**: factorized vector in $\mathbb{P}\Lambda^{\frac{\infty}{2}}$

Maya diagrams

Questions: Facts 1 and 2

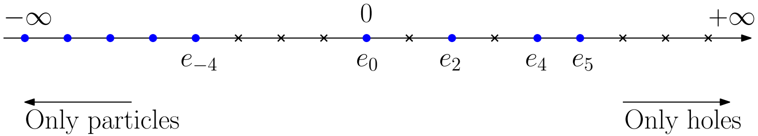
- Orbit under $GL(\infty)$?
- Plücker relations?

Maya diagrams

- **Semi-infinite wedge**: all linear superpositions of such vectors

$$\Lambda^{\frac{\infty}{2}} = \mathrm{span}(e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \quad | \; i_0 > i_1 > \cdots \quad \& \quad i_n = -n \text{ for } n \gg 0)$$

- e_i appears = **particle** at slot i
- e_i absent = **hole** at slot i



Operations for Plücker relations

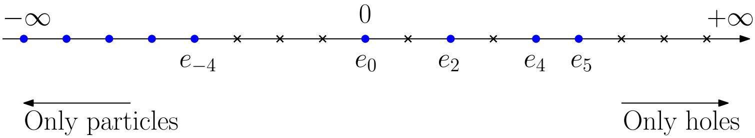
- **Wedging** adds a particle at slot i if possible

$$\psi_i v = e_i \wedge v$$

- **Contraction** removes particle at slot i if possible

$$\psi_i^* v = e_i^*(v_{i_0})\, v_{i_1} \wedge \cdots - v_{i_0}\, e_i^*(v_{i_1}) \wedge \cdots + \cdots$$

- Test for a particle at slot i: $\psi_i \psi_i^*$
- Move a particle from i to j: $\psi_j \psi_i^*$
- Relation to $GL(\infty)$: $E_{ij} = \psi_i \psi_j^*$



Free fermions

Clifford algebra $\{\psi_i, \psi_j\} = \{\psi_i^*, \psi_j^*\} = 0$, $\{\psi_i, \psi_j^*\} = \delta_{ij}$

Subtleties – Dirac sea phenomenon

- Charge $=$ # particles? $= \sum_{i \in \mathbb{Z}} \psi_i \psi_i^*$
- Charge $=$ # particles - # holes
- Insert a separator and count # particles to its right and # number of holes to its left
- Need a reference vector $\rightarrow$ **vacuum** state $=$ Dirac sea

$$|0\rangle = e_0 \wedge e_{-1} \wedge e_{-2} \wedge \cdots$$

- **Normal ordering** w.r.t. $|0\rangle$

$$: \psi_i \psi_j^* := \psi_i \psi_j^* - \langle 0 | \psi_i \psi_j^* | 0 \rangle = \psi_i \psi_j^* - \delta_{ij} \delta_{i \leq 0}$$

- Charge $= \sum_{i \in \mathbb{Z}} : \psi_i \psi_i^* := \sum_{i > 0} \psi_i \psi_i^* - \sum_{i \leq 0} \psi_i^* \psi_i$
- Energy $= \sum_{i \in \mathbb{Z}} i : \psi_i \psi_i^* :$

Plücker relations

Theorem

$\tau \in \Lambda^{\frac{\infty}{2}}, \tau \neq 0$. Then

$$\tau \text{ factorizable} \quad \Leftrightarrow \quad \underbrace{\left(\sum_{i \in \mathbb{Z}} \psi_i \otimes \psi_i^*\right)}_{\text{Casimir } C} \tau \otimes \tau = 0$$

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`Vaznag Rurau`: What the heck is this seminar?

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`Vaznag Rurau`: He pretends talking about PDEs, talks about silly fermions instead

RHS is KP hierarchy in a fermionic disguise!

Plücker relations

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`Vaznag Rurau`: What the heck is this seminar?

- Prove the theorem?
- Is it useful in practice?

$GL(\infty)$ is the solution!

$GL(\infty)$

- Lie algebra $gl(n)$ = all matrices of size $n \times n$
- Group $GL(n) = \{e^X, X \in gl(n)\}$
- In infinite dimension, need boundary condition at ∞

$$gl(\infty) = \{(a_{ij})_{i,j\in\mathbb{Z}}, \qquad a_{ij} = 0 \quad \text{for } |i-j| \gg 0\}$$

- Finite linear combinations of k-th diagonal matrices

$$A_k = \sum_{i\in\mathbb{Z}} a_i \; E_{i,i+k}$$

- Representation on $\Lambda^{\frac{\infty}{2}}$

$$Av = Av_0 \wedge v_1 \wedge v_2 \wedge \cdots + v_0 \wedge Av_1 \wedge v_2 \wedge \cdots + v_0 \wedge v_1 \wedge Av_2 \wedge \cdots$$

- Group representation $G = e^A$

$$Gv = Gv_0 \wedge Gv_1 \wedge Gv_2 \wedge \cdots$$

With fermions and normal ordering

- Relation to fermions $E_{ij} = \psi_i \psi_j^*$, gives correct commutator

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$$

- Divergences due to main diagonal $A_0 = \sum_{i \in \mathbb{Z}} a_i E_{ii}$

$$A_0 |0\rangle = \left(\sum_{i \leq 0} a_i \right) |0\rangle$$

- Remove the vacuum expectation

$$\hat{E}_{ij} =: \psi_i \psi_j^* := E_{ij} - \delta_{ij} \delta_{i \leq 0}$$

- Produces central extension of $gl(\infty)$

$$\hat{gl}(\infty) = gl(\infty) \oplus \mathbb{C} c$$

- Group elements $G = e^{\text{const}} e^{\sum_{i,j} a_{ij} : \psi_i \psi_j^* :}$

$\hat{G}L(\infty)$ and Sato Grassmannian

- $\hat{G}L(\infty)$ is transitive on Sato Grassmannian
- Fermions as vectors

$$G^{-1}\psi_i G = \sum R_{ij}(G)\ \psi_j \qquad G^{-1}\psi_i^* G = \sum_j R_{ji}(G^{-1})\ \psi_j^*$$

Proof that τ in Sato's $\Rightarrow \sum_i \psi_i\tau \otimes \psi_i^*\tau$ vanishing

- Start with the vacuum: $\sum_{i\in\mathbb{Z}} \psi_i|0\rangle \otimes \psi_i^*|0\rangle = 0$
- Use transitivity $\tau = G|0\rangle$
- C is a Casimir

$$\Big[\sum_{i\in\mathbb{Z}} \psi_i \otimes \psi_i^*, G \otimes G\Big] = 0$$

- Immediate result

$$C\ (\tau \otimes \tau) = C\ (G \otimes G)|0\rangle \otimes |0\rangle = (G \otimes G)\ C|0\rangle \otimes |0\rangle = 0$$

Heisenberg algebra

- Define $J_n = \sum_{i\in\mathbb{Z}} E_{i,i+n}$

$$\text{Abelian algebra} \qquad [J_n, J_m] = 0$$

- Re-define $J_n = \sum_{i\in\mathbb{Z}} \hat{E}_{i,i+k}$

$$\text{Heisenberg algebra} \qquad [J_n, J_m] = n\,\delta_{n,-m}$$

- Bosonic representation on functions of $\vec{p} = (p_1, p_2, \dots)$

$$J_n F(\vec{p}) = \begin{cases} n\frac{\partial}{\partial p_n} F(\vec{p}) & \text{for } n > 0 \\ p_{-n} F(\vec{p}) & \text{for } n < 0 \end{cases}$$

- Introduce $\Gamma(\vec{p}) = \exp \sum_{n>0} \frac{p_n}{n} J_n$. Then

$$\langle 0|\Gamma(\vec{p}) J_n = \begin{cases} n\frac{\partial}{\partial p_n} \langle 0|\Gamma(\vec{p}) & \text{for } n > 0 \\ p_{-n} \langle 0|\Gamma(\vec{p}) & \text{for } n < 0 \end{cases}$$

Boson-fermion correspondence

Theorem

- The representations of Heisenberg algebra on bosons and fermions are equivalent
- Isomorphism between the bosonic and fermionic Fock spaces

$$|v\rangle \in \Lambda^{\frac{\infty}{2}} \quad \mapsto \quad F(\vec{p}) = \langle 0|\Gamma(\vec{p})|v\rangle$$

- Represent fermionic operators and $GL(\infty)$ as differential operators on bosonic Fock space

Corollary

- Rewrite $\sum_i \psi_i \tau \otimes \psi_i^* \tau = 0$ in bosonic space
- This is the KP hierarchy! (a bit lengthy)
- Call tau-function = solution of KP hierarchy
- Tau-function ⇔ Point in Sato Grassmannian

Other consequences

- Nice geometric picture!
- Prove that some function satisfies KP $\to$ prove that it can be seen in Sato's Grassmannian?

Use $GL(\infty)$?

- Prove that a function lies in the orbit of $GL(\infty)$ under boson-fermion correspondence?
- Recognize an element of $gl(\infty)$ when disguised as differential op?

Examples

- Multiplication by p_n, diff $p_n^\perp = n\frac{\partial}{\partial p_n} \Leftrightarrow J_n = \sum_{i\in\mathbb{Z}} \hat{E}_{i,i+n}$
- Multiplication by $p_n p_m$?
- Virasoro generator at $c = 1$

$$L_{-1} = \sum_{i>0} p_{i+1} p_i^\perp$$

More examples

- Virasoro generators $c = 1$, $L_n = \frac{1}{2}\sum_{i+j=n} : J_i J_j :$?

$$L_{n>0} = \frac{1}{2}\sum_{\substack{i,j>0 \\ i+j=n}} p_i^{\perp} p_j^{\perp} + \sum_{i>0} p_i p_{i+n}^{\perp}$$

More examples

- Virasoro generators $c = 1$, $L_n = \frac{1}{2}\sum_{i+j=n} : J_i J_j :$?

$$L_{n>0} = \frac{1}{2}\sum_{\substack{i,j>0 \\ i+j=n}} p_i^\perp p_j^\perp + \sum_{i>0} p_i p_{i+n}^\perp$$

$$L_n = \sum_{i\in\mathbb{Z}} \left(i + \frac{n-1}{2}\right) E_{i,i+n} \in gl(\infty)$$

- W-operators
  $W_{-1} = \sum_{n>0} p_{n+1} L_n = \sum_{i,j>0} p_{i+j+1} p_i^\perp p_j^\perp + p_i p_j p_{i+j-1}^\perp$?

$$W_{-1} = \sum_i (i-1)^2 E_{i,i-1} \in gl(\infty)$$

- W-operators

$$W_n^{(k)} = \sum_{i_1+\cdots+i_k=n} : J_{i_k} \cdots J_{i_1} :$$

How to prove that?

- Express everything with fermions, act on Maya diagrams, use Wick theorem and good luck
- Vertex algebra trick?

Lemma

X is determined by its commutators with J_i, $i \in \mathbb{Z}$, up to a central term

- Use bosonic representation to evaluate $[L_n, J_m] = -mJ_{n+m}$
- Find fermionic operators L_n satisfying same commutators

Cut-and-join

- Map (non-necessarily connected) with labeled edges
- Last edge decomposition à la Tutte
- Remove the last edge $\partial_t \tau$

$$\partial_t \tau =$$

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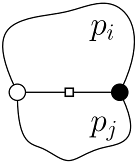

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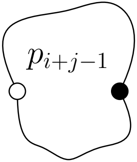

$$\partial_t \tau = \Big(\sum_{i,j>0} p_i p_j p^{\perp}_{i+j-1}$$

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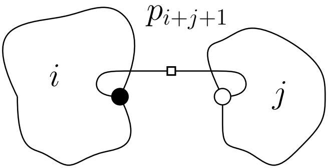

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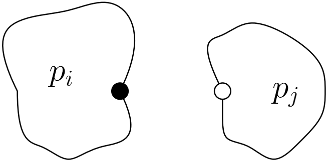

$$\partial_t \tau = \Big(\sum_{i,j>0} p_i p_j p^{\perp}_{i+j-1} + \sum_{i,j>0} p_{i+j+1} p^{\perp}_i p^{\perp}_j$$

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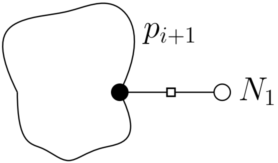

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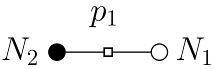

$$\partial_t \tau = \Big(\sum_{i,j>0} p_i p_j p^{\perp}_{i+j-1} + \sum_{i,j>0} p_{i+j+1} p_i^{\perp} p_j^{\perp} + (N_1 + N_2) \sum_{i>0} p_{i+1} p_i^{\perp}$$

Cut-and-join

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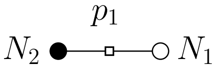
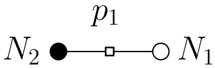
- Last edge decomposition à la Tutte
- Remove the last edge $\partial_t \tau$



$$\partial_t \tau = \Big(\sum_{i,j>0} p_i p_j p^{\perp}_{i+j-1} + \sum_{i,j>0} p_{i+j+1} p^{\perp}_i p^{\perp}_j + (N_1 + N_2) \sum_{i>0} p_{i+1} p^{\perp}_i + N_1 N_2 p_1 \Big) \tau$$

Cut-and-join

- Map (non-necessarily connected) with labeled edges
- Last edge decomposition à la Tutte
- Remove the last edge $\partial_t \tau$



$$\partial_t \tau = \Big(\sum_{i,j>0} p_i p_j p^{\perp}_{i+j-1} + \sum_{i,j>0} p_{i+j+1} p^{\perp}_i p^{\perp}_j + (N_1 + N_2) \sum_{i>0} p_{i+1} p^{\perp}_i + N_1 N_2 p_1 \Big) \tau$$

Recognize some operators

$$\partial_t \tau = \big(W_{-1} + (N_1 + N_2) L_{-1} + N_1 N_2 J_{-1} \big) \tau$$

Proof of KP hierarchy for bipartite maps

- $CJ(\vec{p}; N_1, N_2) = W_{-1} + (N_1 + N_2)L_{-1} + N_1 N_2 J_{-1} \ \in gl(\infty)$
- Its exponential is in $GL(\infty)$
- Explicit solution to CJ equation

$$\tau(\vec{p}; N_1, N_2, t) = e^{tCJ(\vec{p}; N_1, N_2)} 1$$

- 1 is tau-function
- τ is a tau-function [Kazarian-Zograf]

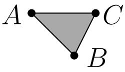
Constellations

- Bipartite maps $\sigma_\circ \sigma_\bullet = \phi \quad \in S_n$

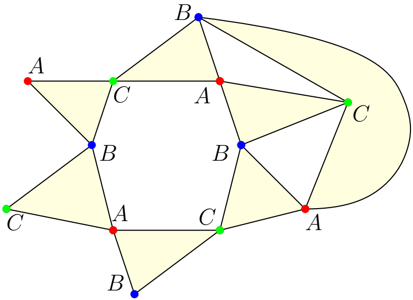
Permutations	Maps
$\sigma_\circ$	White v.
$\sigma_\bullet$	Black v.
S_n	n edges between black and white

- m-constellations $\sigma_1 \sigma_2 \cdots \sigma_m = \phi \quad \in S_n$

Permutations	Constellations
σ_i	Vertices of color $i = 1, \ldots, m$
S_n	n m-hyperedges

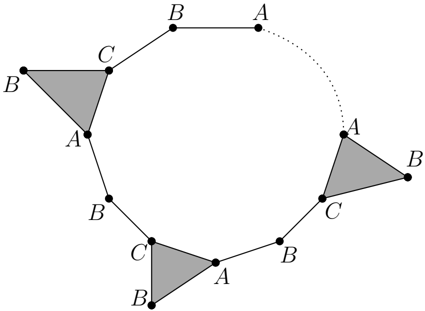
- 3-hyperedge 

Example of 3-constellation



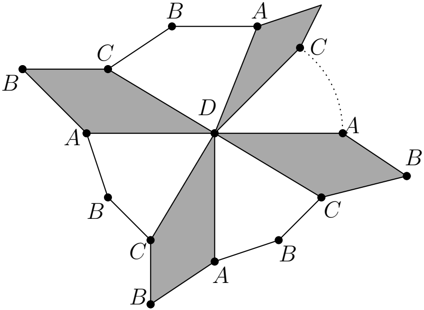
Trick

- Consider white faces and add vertex of new color in each



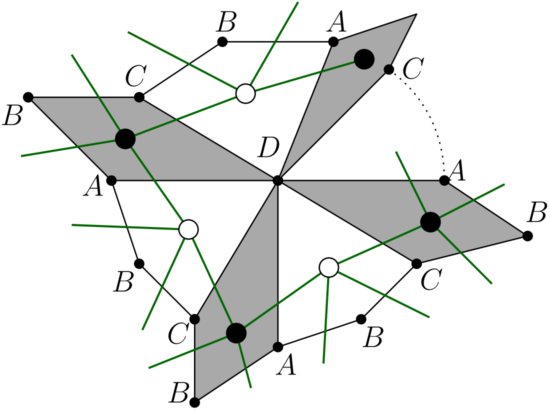
Trick

- Consider white faces and add vertex of new color in each



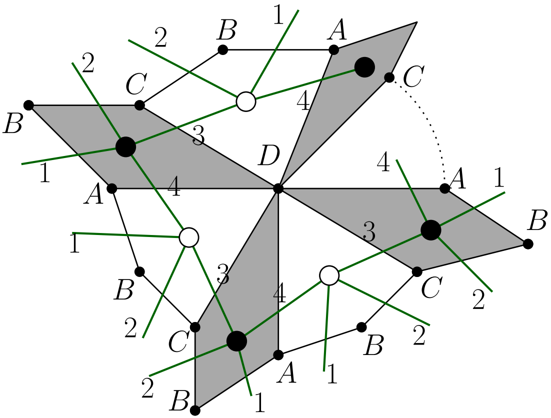
- Black and white faces have degree $m + 1$

Take the dual map



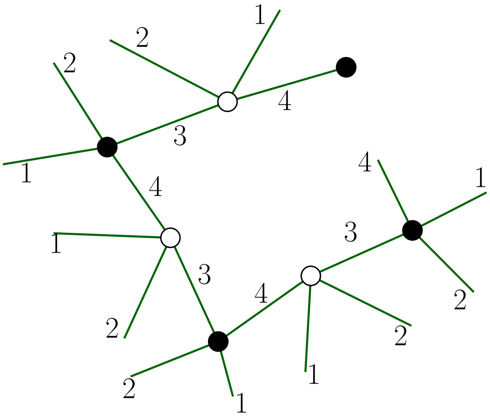
- Two types of vertices
- Having degree $m + 1$

Take the dual map



- Two types of vertices
- Having degree $m+1$
- Label the dual edges: $AB = 1, BC = 2, CD = 3, \ldots$

Take the dual map



- Two types of vertices
- Having degree $m + 1$
- Label the dual edges: $AB = 1, BC = 2, CD = 3, \ldots$
- Jacket!

What about them?

Ambjørn-Chekhov matrix model

- Hermitianized multimatrix model

$$\sum_{i=1}^{m} \operatorname{tr} M_i M_i^\dagger + \sum_k \frac{p_k}{k} \operatorname{tr}\left(M_1 M_2 \cdots M_m M_m^\dagger \cdots M_2^\dagger M_1^\dagger\right)^k$$

- Kind of useless?
- Know from algebraic combinatorics that KP satisfied
- Gaussian model satifies the topological recursion [Alexandrov-Chapuy-Eynard-Harnad]
- Derive Schwinger-Dyson equations [Dartois-Forrester]

Some progress

Chapuy-Dolega

- Generating function of β-deformed constellations with weight p_i on faces of degree i, q_i on vertices of color 0 of degree i, and N_c per vertex of color $c \neq 0$, t per edge

Corollaries [w.i.p. w/ S. Dartois]

- Cut-and-join equation $F = e^{tCJ_m(N_0,\dots,N_m)}1$ with
- Bosonic representation

$$CJ_m(N_0,\dots,N_m) = \sum_{C\subset\{0,\dots,m\}} \Bigg(\prod_{c\in\{0,\dots,m\}\setminus C} N_c \Bigg) \times \sum_{\substack{(i_1,\dots,i_{|C|})\\ i_1>0\\ i_1+i_2>0\\ \vdots \\ i_1+\dots+i_{|C|}>0}} p_{i_1+\dots+i_{|C|}+1} J_{i_{|C|}} \cdots J_{i_1}$$

Some progress

Chapuy-Dolega

- Generating function of β-deformed constellations with weight p_i on faces of degree i, q_i on vertices of color 0 of degree i, and N_c per vertex of color $c \neq 0$, t per edge

Corollaries [w.i.p. w/ S. Dartois]

- Cut-and-join equation $F = e^{tCJ_m(N_0,\dots,N_m)}1$ with
- Fermionic representation

$$CJ_m(N_0,\dots,N_m) = \sum_{i\in\mathbb{Z}} \prod_{c=0}^{m} (i-1+N_c)\ E_{i,i-1} \quad \in gl(\infty)$$

Conclusion & perspectives

- Awesome geometric framework
- Same set of equations for the all-genus generating functions of maps, Hurwitz numbers, etc.
- Some recent applications
- Anything for higher dim. or tensor models?

Tensor models

- Intermediate field for quartic melonic model

$$\sum_{c=1}^{d} \operatorname{tr} M_c^2 + \operatorname{tr} \ln\big(1 - M_1 \otimes 1^{d-1} - 1 \otimes M_2 \otimes 1^{d-2} - \dots\big)$$
$$= \sum_{c=1}^{d} \operatorname{tr} M_c^2 + \sum_{a_1,\dots,a_d} t_{a_1\cdots a_d} \operatorname{tr}(M_1)^{a_1} \cdots \operatorname{tr}(M_d)^{a_d}$$

- Generate the multitrace from single traces [Dartois]

$$Z = e^{\sum t_{a_1\cdots a_d} p^{(1)\perp}_{a_1} \cdots p^{(d)\perp}_{a_d}} \prod_{c=1}^{d} \int dM_c \, e^{-\operatorname{tr} M_c^2 + \sum \frac{p_k^{(c)}}{k} \operatorname{tr} M_c^k}$$

- $p^{(1)\perp}_{a_1} \cdots p^{(d)\perp}_{a_d} \in gl^{(1)}(\infty) \otimes \cdots \otimes gl^{(d)}(\infty)$
- Not a Lie algebra