Large N Matrix and Tensor Models: A Brief Survey

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Talk at virtual

Tensor Journal Club

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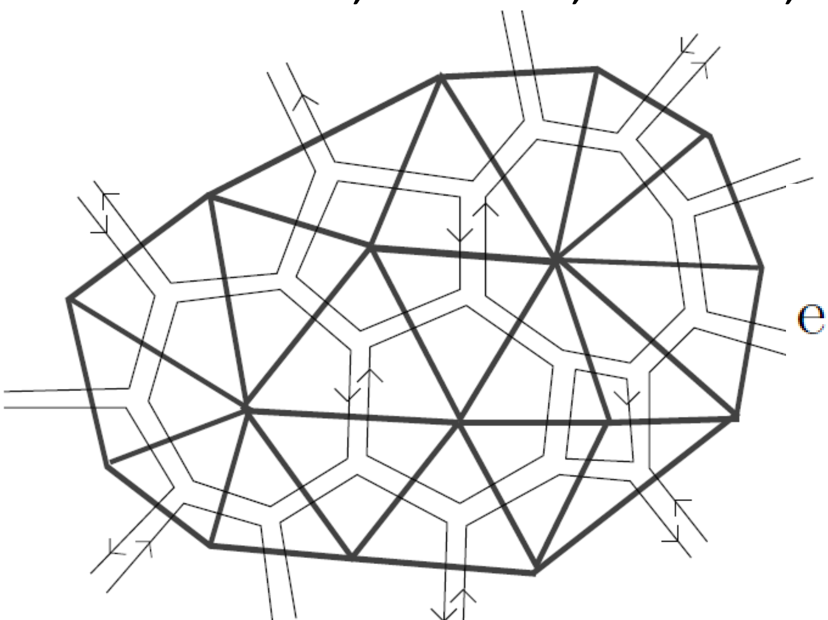
A Bit of History (with apologies)

- 1950s – 60s Hamiltonian of a complex quantum system, e.g. a large nucleus, as a large random Hermitian matrix. The "Wigner semicircle law," the "Dyson gas," etc. Wigner, Mehta, Dyson, …
- 1973 – QCD as the theory of strong interactions. Gross, Wilczek; Politzer; Fritzsch, Gell-Mann, Leutwyler; …
- 1974 – the 't Hooft large N limit; planar Feynman diagrams as discretized random surfaces.
- 1978 – solution of the large N non-Gaussian matrix models. Brezin, Parisi, Itsykson, Zuber

- 1981 -- Sum over continuous surfaces. Non-critical (super)string theory. Quantum (super)Liouville Theory. Polyakov
- 1984-88 -- Matrix Models give ways of solving models of discretized random surfaces in the continuum limit. Kazakov, Kostov, Migdal; David; Ambjorn, Durhuus, Frohlich; Kazakov, Migdal; …



$$e^{Z} = \int dM \, e^{-\frac{1}{2}\mathrm{tr}\, M^2 + \frac{g}{\sqrt{N}}\mathrm{tr}\, M^3}$$

- Late 80s -- First precise connections between Matrix Models and Liouville theory: the KPZ-DDK scaling. Knizhnik, Polyakov, Zamolodhikov; David, Distler, Kawai
- 1989 – The double-scaling limit gives a solution of some matrix models to all orders in the genus expansion. Brezin, Kazakov; Gross, Migdal; Douglas, Shenker; Gross, Miljkovic; Brezin, Kazakov, Zamolodchikov; …
- 1989-94 -- MMMania (Matrix Model Mania) is in full swing.
- 1991-92 – first large N tensor models and Group Field Theory. Ambjorn, Durhuus, Jonsson; Sasakura; Gross; Boulatov; …
- Precise comparisons between Matrix Models and Liouville theory.
- 1994 – Exact 3-point functions in Liouville theory.

Dorn, Otto; Zamolodchikov, Zamolodchikov

Non-Critical String Theory

- Sum over surfaces embedded in D dimension

$$Z \sim \sum_{\text{topologies}} \int \mathcal{D}g \, \mathcal{D}X \; \mathrm{e}^{-S}$$

$$S(X,g) = \frac{1}{2\pi} \int d^2\sigma \sqrt{g} (\mu + g^{ab} \sum_{i=1}^{c} \partial_a X^i \partial_b X^i)$$

- Two-dimensional Quantum Gravity coupled to c massless scalar fields.
- The world sheet cosmological constant is conjugate to the area.

Continuum Approach

- In the conformal gauge $g_{ab} = \mathrm{e}^{\varphi} \hat{g}_{ab}$ the Liouville field acquires induced dynamics and becomes a “large extra dimension.” Polyakov
- The DDK approach to Liouville theory

$$S_L = \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} \left(\hat{g}^{ab} \partial_a \varphi \partial_b \varphi + \frac{1}{4} Q \hat{R} \varphi + \mu e^{\alpha\varphi} \right)$$

- Dilaton linear in the Liouville direction.
- Non-critical string in c dimensions is like critical in c+1. But no full Poincare invariance.

- The complete gauge fixed action includes matter CFT of central charge c, Liouville, and ghosts:

$$\mathcal{S} = \mathcal{S}_L + \mathcal{S}_M + \mathcal{S}_{\text{gh}}$$

- Since the total central charge has to vanish, must choose

$$Q = \sqrt{\frac{25-c}{3}}$$

- Marginality of the “dressed identity operator” requires

$$\alpha_{\pm} = -\frac{Q}{2} \pm \sqrt{\frac{1-c}{12}}$$

- Requiring the operator to be non-normalizable requires choosing the bigger root (the Seiberg bound).

Scaling with Area

- Partition function at fixed area

$$Z(A) = \int \mathcal{D}\varphi\, \mathcal{D}X\, \mathrm{e}^{-S}\, \delta\Big(\int \mathrm{d}^2\xi\, \sqrt{\hat{g}}\, \mathrm{e}^{\alpha_+ \varphi} - A\Big)$$

- Using a shift of integration variable, find

$$Z(A) \sim A^{(\gamma_{\rm str}-2)\chi/2-1}$$

- The “string susceptibility” is $\gamma_{\rm str} = 2 + \frac{Q}{\alpha_+}$
- Agrees with solutions of single-trace Matrix Models where the susceptibility is always negative.

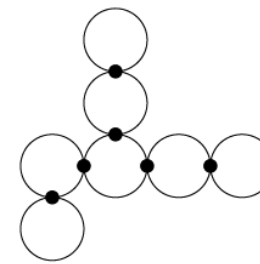
c=1 Barrier

- The dressing exponent becomes complex for c>1.
- The center of mass mode $\mathcal{T}_k = \mathrm{e}^{ikX+\beta(k)\varphi}$ is tachyonic $$m_T^2 = \frac{1-c}{24}$$
- Corresponding problems are observed in discretized formulations (the dominance of branched polymer phase). Weingarten; Eguchi, Kawai; Durhuus, Frohlich, Jonsson;…
- In a sense, AdS/CFT has allowed us to reach above the barrier.

Double-Trace Matrix Models

- Double-trace terms add touching interactions (world sheet wormholes)

$$Z_k = \int \mathcal{D}\Phi \, e^{-N\left[\mathrm{Tr}\, V_k(\Phi) + (c_2 - \lambda)\mathrm{Tr}\,\Phi^4 - \frac{g}{2N}\left(\mathrm{Tr}\,\Phi^4\right)^2\right]}$$



- At leading order in N, find trees of touching spheres.
- For small g, the model is in the same universality class as single-trace. For large g, dominated by double-trace and is in the branched polymer phase with γ=1/2.
- At a special value of g, the susceptibility changes.
  Das, Dhar, Sengupta, Wadia; Korchemsky; Alvarez-Gaume, Barbon, Crnkovic

- The new susceptibility in general explained "phenomenologically" by assuming the other Liouville dressing IRK

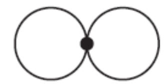
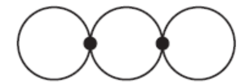
$$\bar{\gamma} = 2 + \frac{Q}{\alpha_-} = \frac{\gamma}{\gamma - 1}$$

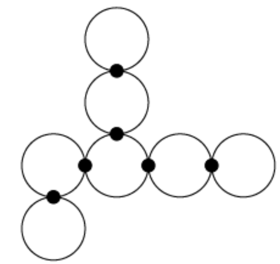
- The double-trace model solved non-perturbatively using the Hubbard-Stratonovich trick IRK, Hashimoto

$$\bar{F}(\bar{t}) = \log \int_{-\infty}^{\infty} dt e^{t\bar{t} + F(t)}$$

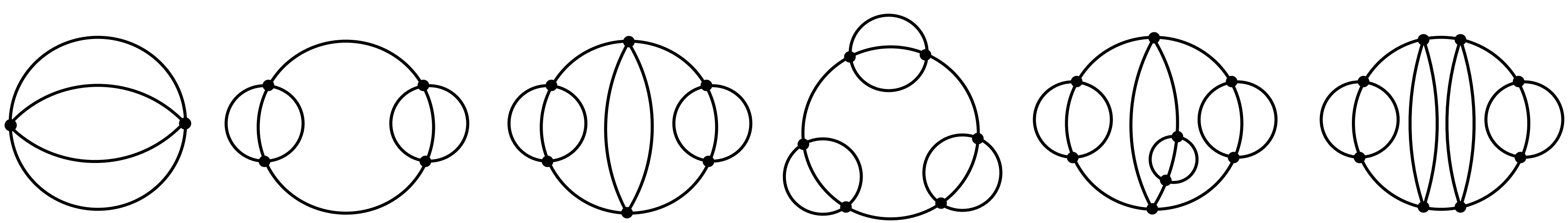
- At leading order related by Legendre transform. Meaning of all this in DOZZ?

Three Large N Limits

- O(N) Vector: solvable because the "cactus" diagrams can be summed.  

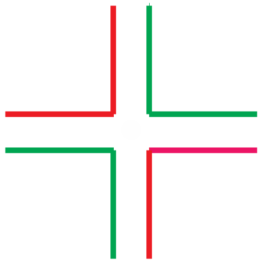


- Matrix ('t Hooft) Limit: planar diagrams. Solvable only in special cases.
- Tensor of rank three and higher. When interactions are specially chosen, dominated by the "melonic" diagrams. Bonzom, Gurau, Riello, Rivasseau; Carrozza, Tanasa; Witten; IK, Tarnopolsky



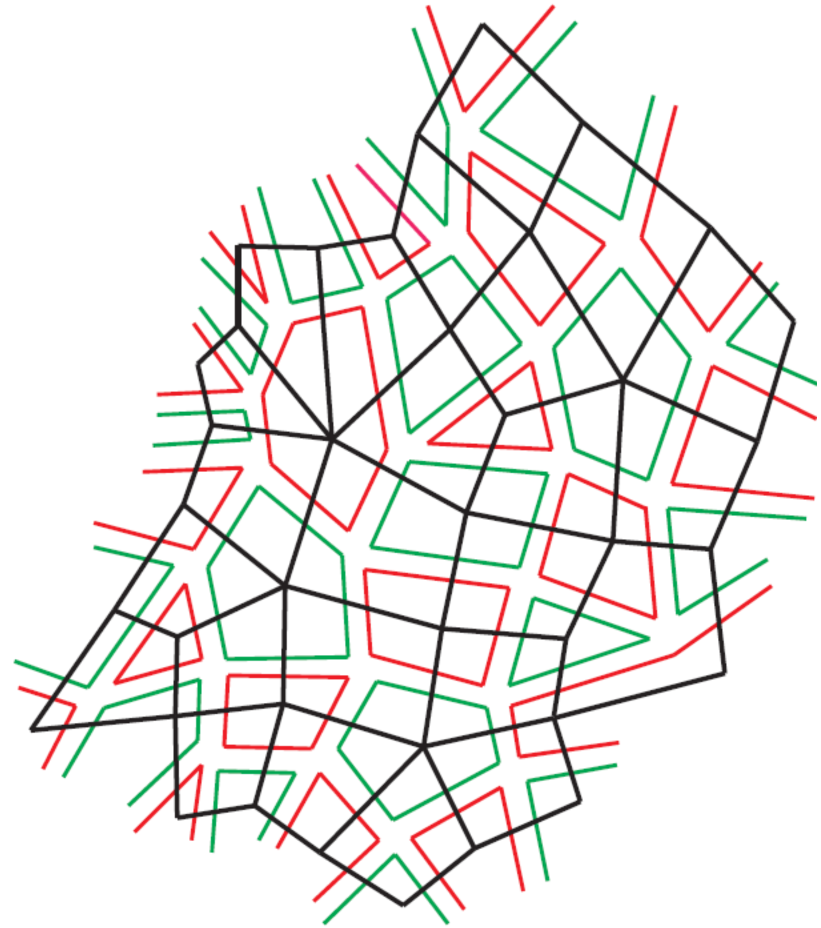
O(N) x O(N) Matrix Model

- Theory of real matrices ϕ^{ab} with distinguishable indices, i.e. in the bi-fundamental representation of $O(N)_a x O(N)_b$ symmetry.
- The interaction is at least quartic: $g \, \mathrm{tr}\, \phi\phi^T\phi\phi^T$
- Propagators are represented by colored double lines, and the interaction vertex is



- In d=0 or 1 special limits describe two-dimensional quantum gravity.

- In the large N limit where gN is held fixed we find planar Feynman graphs, and each index loop may be red or green.
- The dual graphs shown in black may be thought of as random surfaces tiled with squares whose vertices have alternating colors (red, green, red, green).



From Bi- to Tri-Fundamentals

- For a 3-tensor with distinguishable indices the propagator has index structure

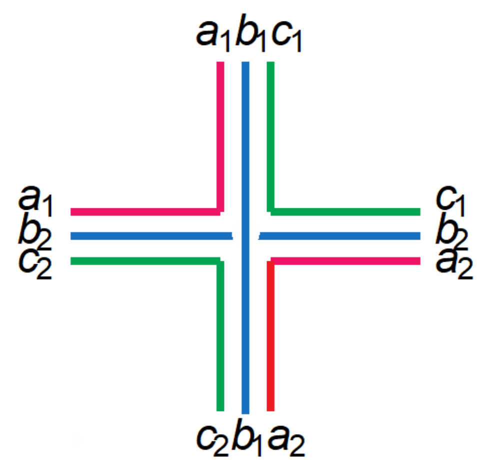
$$\langle \phi^{abc} \phi^{a'b'c'} \rangle = \delta^{aa'} \delta^{bb'} \delta^{cc'}$$

- It may be represented graphically by 3 colored wires 

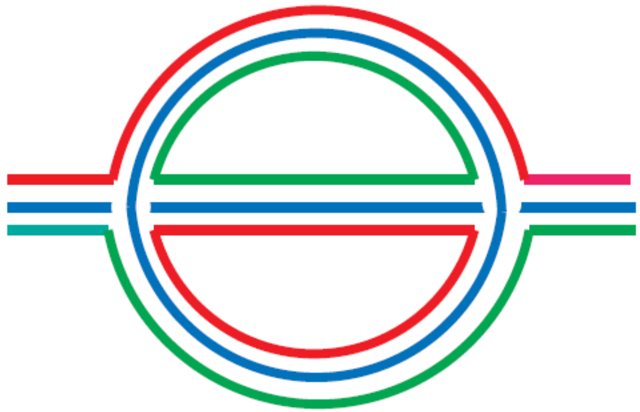
- Tetrahedral interaction with $O(N)_a \times O(N)_b \times O(N)_c$ symmetry

Carrozza, Tanasa; IK, Tarnopolsky

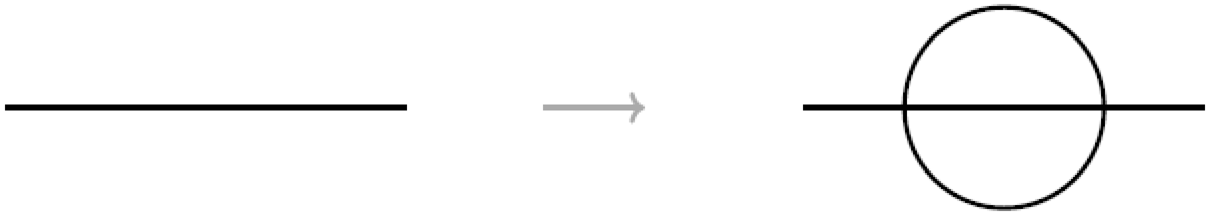
$$\frac{1}{4} g \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1}$$



- Leading correction to the propagator has 3 index loops

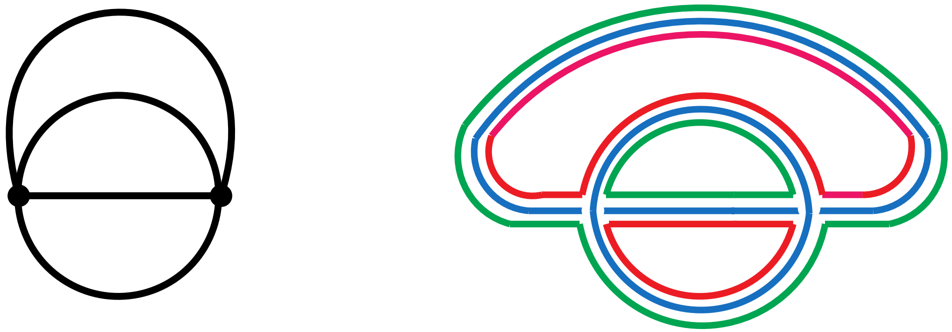


- Requiring that this "melon" insertion is of order 1 means that $\lambda = gN^{3/2}$ must be held fixed in the large N limit.
- Melonic graphs obtained by iterating

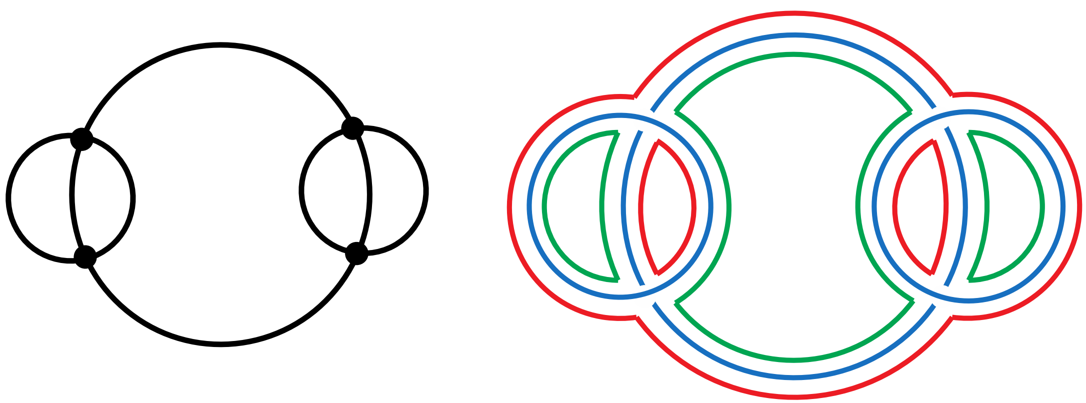


Cables and Wires

- The Feynman graphs of the quartic field theory may be resolved in terms of the colored wires (triple lines)



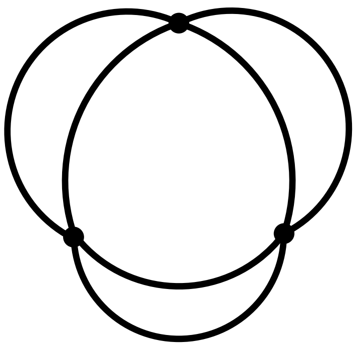
$$g^2 N^6 \sim N^3 \lambda^2$$

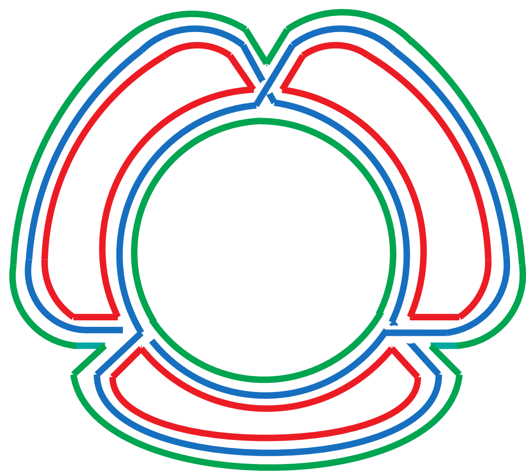


$$g^4 N^9 \sim N^3 \lambda^4$$

Non-Melonic Graphs

- Most Feynman graphs in the quartic field theory are not melonic are therefore subdominant in the new large N limit, e.g.





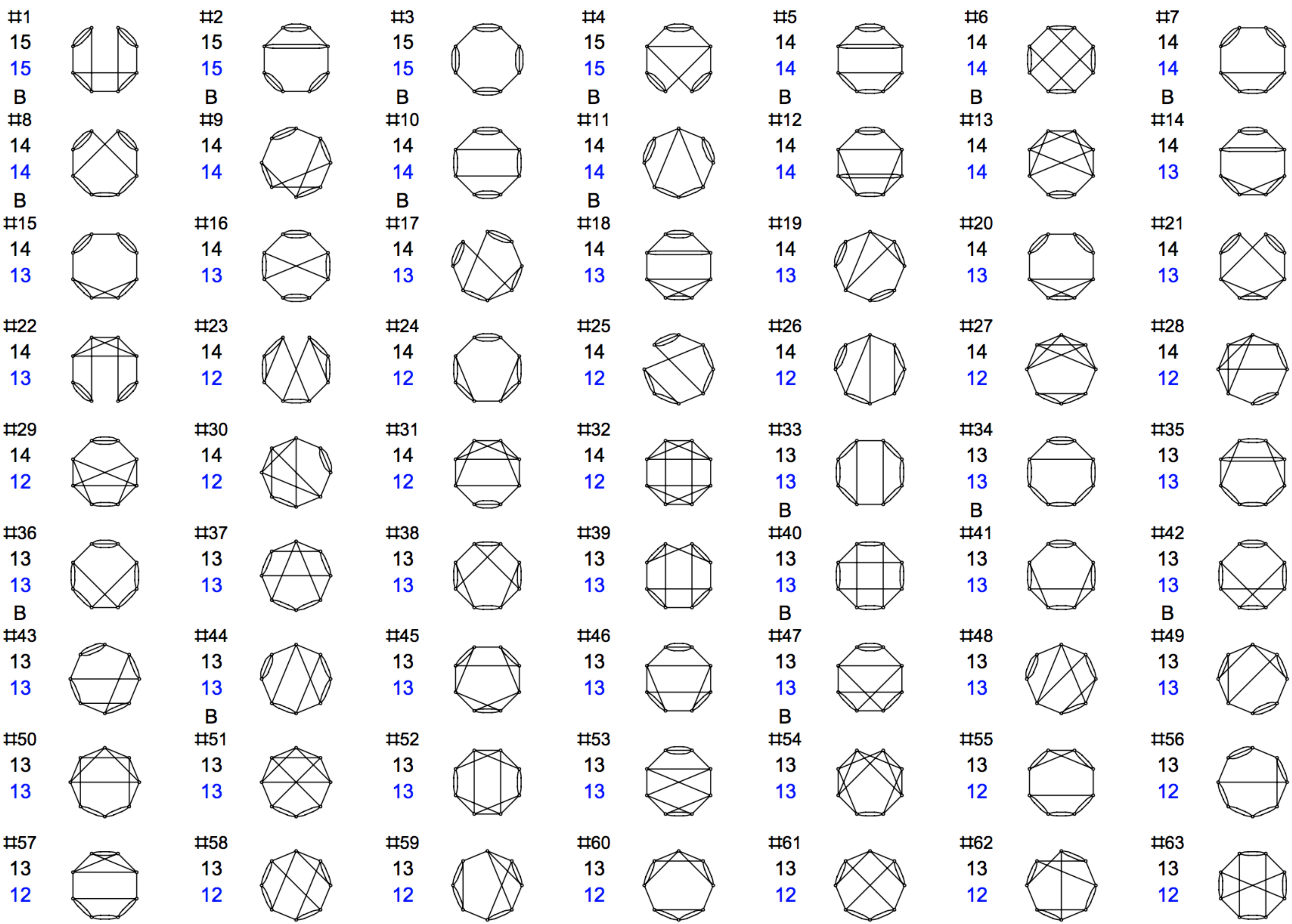
- Scales as $g^3 N^6 \sim N^3 \lambda^3 N^{-3/2}$
- None of the graphs with an odd number of vertices are melonic.

Bosonic Symmetric Traceless Tensors

- Consider a symmetric traceless bosonic tensor of O(N) with tetrahedron interaction: IK, Tarnopolsky

$$V = \phi_{abc}\phi_{ab'c'}\phi_{a'bc'}\phi_{a'b'c}$$

- Similar to the models considered in the early 90's but the tracelessness condition is crucial. IK, Tarnopolsky; Azeyanagi, Ferrari, Gregori, Leduc, Valette
- Explicit checks of combinatorial factors up to 8^{th} order show that they do dominate. There are 177 diagrams without "snails."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- The propagator has the more complicated index structure IK, Tarnopolsky

$$\langle \phi^{abc}\phi^{a'b'c'}\rangle_0 = \frac{1}{6}\Big(\delta^{aa'}\delta^{bb'}\delta^{cc'} + \delta^{ab'}\delta^{bc'}\delta^{ca'} + \delta^{ac'}\delta^{ba'}\delta^{cb'} + \delta^{ab'}\delta^{ba'}\delta^{cc'} + \delta^{ac'}\delta^{bb'}\delta^{ca'} + \delta^{aa'}\delta^{bc'}\delta^{cb'} - \frac{2}{N+2}\big(\delta^{ab}\delta^{ca'}\delta^{b'c'} + \delta^{ab}\delta^{cb'}\delta^{a'c'} + \delta^{ab}\delta^{cc'}\delta^{a'b'} + \delta^{ac}\delta^{ba'}\delta^{b'c'} + \delta^{ac}\delta^{bb'}\delta^{a'c'} + \delta^{ac}\delta^{bc'}\delta^{a'b'} + \delta^{bc}\delta^{aa'}\delta^{b'c'} + \delta^{bc}\delta^{ab'}\delta^{a'c'} + \delta^{bc}\delta^{ac'}\delta^{a'b'}\big)\Big)$$

- Similarly, the theory of antisymmetric tensor of O(N) with propagator

$$\langle \phi^{abc}\phi^{a'b'c'}\rangle_0 = \frac{1}{6}\big(\delta^{aa'}\delta^{bb'}\delta^{cc'} + \delta^{ab'}\delta^{bc'}\delta^{ca'} + \delta^{ac'}\delta^{ba'}\delta^{cb'} - \delta^{ab'}\delta^{ba'}\delta^{cc'} - \delta^{ac'}\delta^{bb'}\delta^{ca'} - \delta^{aa'}\delta^{bc'}\delta^{cb'}\big)$$

  is also dominated by the melonic diagrams.

- Combinatorial proof. Benedetti, Carrozza, Tanasa, Kolanowski

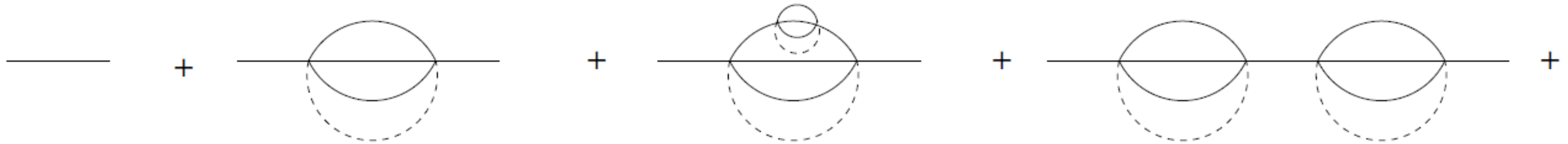
The Sachdev-Ye-Kitaev Model

- Quantum mechanics of a large number N_{SYK} of anti-commuting variables with action

$$I = \int \mathrm{d}t \left(\frac{\mathrm{i}}{2} \sum_i \psi_i \frac{\mathrm{d}}{\mathrm{d}t} \psi_i - \mathrm{i}^{q/2} j_{i_1 i_2 \ldots i_q} \psi_{i_1} \psi_{i_2} \ldots \psi_{i_q} \right)$$

- Random couplings j have a Gaussian distribution with zero mean.
- The model flows to strong coupling and becomes nearly conformal. Sachdev, Ye; Georges, Parcollet, Sachdev; Kitaev; Polchinski, Rosenhaus; Maldacena, Stanford; Jevicki, Suzuki, Yoon; Kitaev, Suh

- The simplest dynamical case is q=4.
- Exactly solvable in the large N_{SYK} limit because only the melonic Feynman diagrams contribute

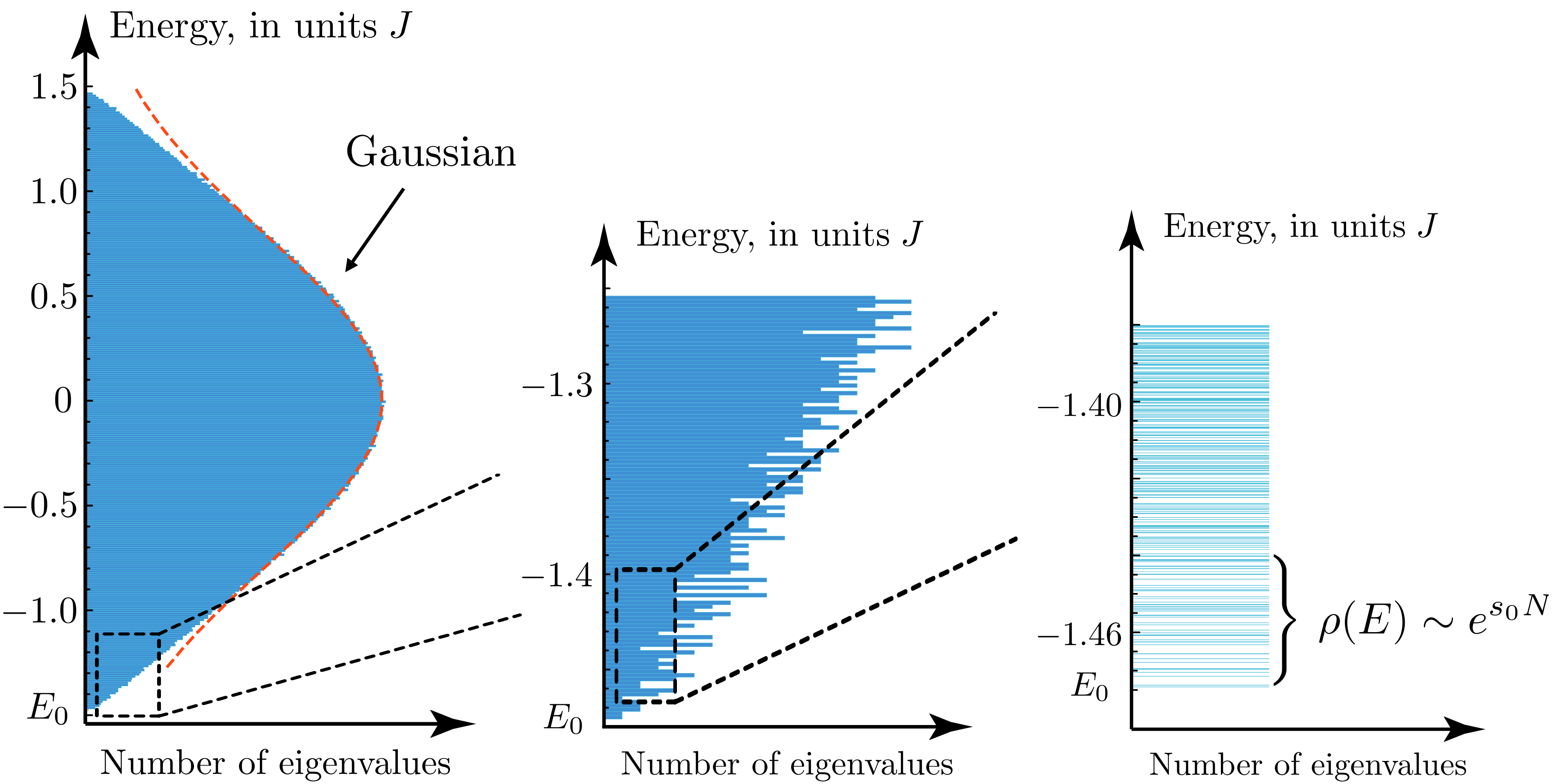


- Solid lines are fermion propagators, while dashed lines mean disorder average.
- The exact solution shows resemblance with physics of certain two-dimensional black holes.

Kitaev; Almheiri, Polchinski; Sachdev; Maldacena, Stanford, Yang; Engelsoy, Mertens, Verlinde; Jensen; Kitaev, Suh; …

Spectrum of the SYK model

- Energy levels for N=32 Majorana q=4 SYK model: 65536 energy levels



- s_0 is zero temperature entropy $s_0 \approx 0.23$

$O(N)^3$ Tensor QM

- Quantum Mechanics of N^3 Majorana fermions
  IK, Tarnopolsky

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'}\delta^{bb'}\delta^{cc'}$$

$$H = \frac{g}{4}\psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c} - \frac{g}{16}N^4$$

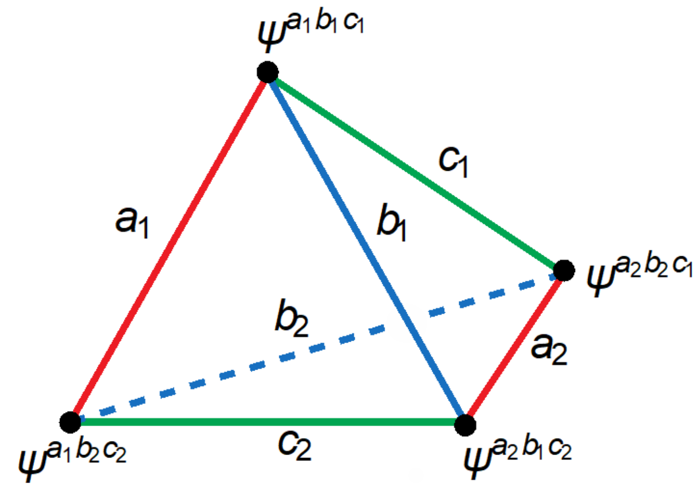
- Has $O(N)_a$x$O(N)_b$x$O(N)_c$ symmetry under

$$\psi^{abc} \to M_1^{aa'}M_2^{bb'}M_3^{cc'}\psi^{a'b'c'}, \quad M_1, M_2, M_3 \in O(N)$$

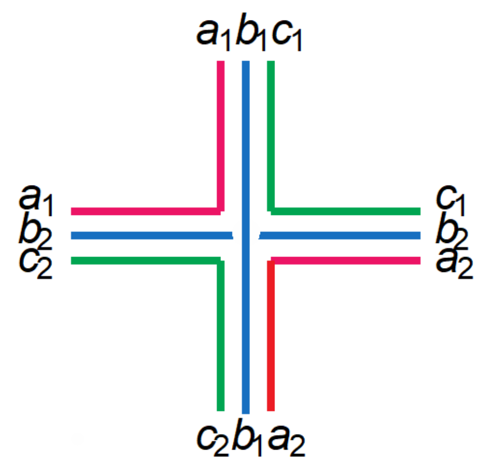
- The SO(N) symmetry charges are

$$Q_1^{aa'} = \frac{i}{2}[\psi^{abc}, \psi^{a'bc}], \qquad Q_2^{bb'} = \frac{i}{2}[\psi^{abc}, \psi^{ab'c}], \qquad Q_3^{cc'} = \frac{i}{2}[\psi^{abc}, \psi^{abc'}]$$

- The 3-tensors may be associated with indistinguishable vertices of a tetrahedron.

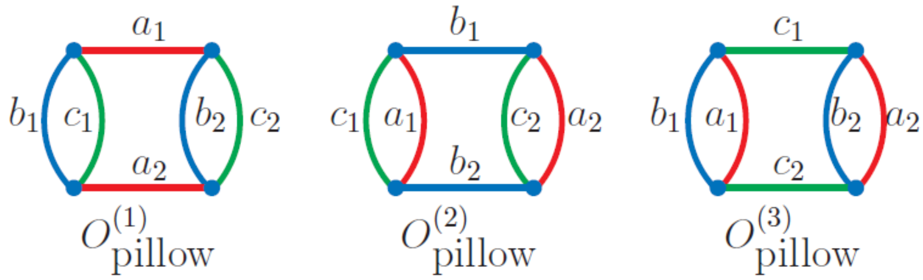


- This is equivalent to



- The triple-line Feynman graphs are produced using the propagator

- The tetrahedral term is the unique dynamical quartic interaction with $O(N)^3$ symmetry.
- The other possible terms are quadratic Casimirs of the three SO(N) groups.



$$O^{(1)}_{\text{pillow}} = \sum_{a_1<a_2} Q_1^{a_1a_2} Q_1^{a_1a_2}\,, \qquad O^{(2)}_{\text{pillow}} = \sum_{b_1<b_2} Q_2^{b_1b_2} Q_2^{b_1b_2}\,, \qquad O^{(3)}_{\text{pillow}} = \sum_{c_1<c_2} Q_3^{c_1c_2} Q_3^{c_1c_2}$$

- In the model where $SO(N)^3$ is gauged, they vanish.

$O(N)^3$ vs. SYK Model

- Using composite indices $I_k = (a_k b_k c_k)$

$$H = \frac{1}{4!} J_{I_1 I_2 I_3 I_4} \psi^{I_1} \psi^{I_2} \psi^{I_3} \psi^{I_4}$$

The couplings take values $0, \pm 1$

$$J_{I_1 I_2 I_3 I_4} = \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_1 b_3} \delta_{b_2 b_4} \delta_{c_1 c_4} \delta_{c_2 c_3} - \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_2 b_3} \delta_{b_1 b_4} \delta_{c_2 c_4} \delta_{c_1 c_3} + 22 \text{ terms}$$

- The number of distinct terms is

$$\frac{1}{4!} \sum_{\{I_k\}} J^2_{I_1 I_2 I_3 I_4} = \frac{1}{4} N^3 (N-1)^2 (N+2)$$

- Much smaller than in SYK model with $N_{\text{SYK}} = N^3$

$$\frac{1}{24} N^3 (N^3 - 1)(N^3 - 2)(N^3 - 3)$$

Gauged Model

- To eliminate large degeneracies, focus on the states invariant under SO(N)^3.
- Their number can be found by gauging the free theory IK, Milekhin, Popov, Tarnopolsky

$$L = \psi^I \partial_t \psi^I + \psi^I A_{IJ} \psi^J$$

$$A = A^1 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes A^2 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes A^3$$

$$\#\text{singlet states} = \int d\lambda_G^N \prod_{a=1}^{M/2} 2\cos(\lambda_a/2)$$

$$d\lambda_{SO(2n)} = \prod_{i<j}^{n} \sin\left(\frac{x_i - x_j}{2}\right)^2 \sin\left(\frac{x_i + x_j}{2}\right)^2 dx_1 \dots dx_n$$

- There are no singlets for odd N due to a QM anomaly for odd numbers of flavors.
- The number grows very rapidly for even N

N	# singlet states
2	2
4	36
6	595354780

Table 1: Number of singlet states in the $O(N)^3$ model

$$\#\text{singlet states} \sim \exp\left(\frac{N^3}{2}\log 2 - \frac{3N^2}{2}\log N + O(N^2)\right)$$

- The large low-temperature entropy suggests tiny gaps for singlet excitations ~ c^{-N^3}

Qubit Hamiltonian

- Convenient to introduce operator basis which breaks the third O(N) to U(N/2)

$$\bar{c}_{abk} = \frac{1}{\sqrt{2}}\left(\psi^{ab(2k)} + i\psi^{ab(2k+1)}\right), \quad c_{abk} = \frac{1}{\sqrt{2}}\left(\psi^{ab(2k)} - i\psi^{ab(2k+1)}\right),$$

$$\{c_{abk}, c_{a'b'k'}\} = \{\bar{c}_{abk}, \bar{c}_{a'b'k'}\} = 0, \quad \{\bar{c}_{abk}, c_{a'b'k'}\} = \delta_{aa'}\delta_{bb'}\delta_{kk'},$$

$$a, b = 0, 1, \ldots, N-1, \text{ and } k = 0, \ldots, \tfrac{1}{2}N - 1$$

- Operators $c_{abk}, \bar{c}_{abk}$ correspond to qubit number $N^2 k + Nb + a$
- The Hamiltonian couples N/2 sets of N^2 qubits

$$H = 2\left(\bar{c}_{abk}\bar{c}_{ab'k'}c_{a'bk'}c_{a'b'k} - \bar{c}_{abk}\bar{c}_{a'bk'}c_{ab'k'}c_{a'b'k}\right)$$

- The Cartan generators of U(N/2) are

$$Q_k = \sum_{a,b} \frac{1}{2}[\bar{c}_{abk}, c_{abk}] \ , \qquad k = 0, \ldots, \frac{1}{2}N - 1$$

- For the oscillator vaccuum

$$c_{abk} \left|\text{vac}\right\rangle = 0 \ , \qquad Q_k \left|\text{vac}\right\rangle = -\frac{N^2}{2} \left|\text{vac}\right\rangle$$

- The gauge singlet states appear in the sector where all these charges vanish: each set of N^2 qubits is at half filling.
- This reduces the number of states but it still grows rapidly. For N=4 there are 165636900, while for N=6 over 7.47 * 10^29

Spectrum of the Gauged N=4 Model

- Studied the system of 32=16+16 qubits

K. Pakrouski, IK, F. Popov and G. Tarnopolsky

- Needed to isolate the 36 states invariant under $SO(4)^3$ out of the 165080390 “half-half-filled” states.
- Diagonalize 4H/g + 100 C where C is the sum of three Casimir operators.
- A Lanczos type algorithm is well suited for this sparse operator.

Discrete Symmetries

- Act within the $SO(N)^3$ invariant sector and can lead to small degeneracies.
- Z_2 parity transformation within each group like

$$\psi^{1bc} \to -\psi^{1bc}$$

- Interchanges of the groups flip the energy

$$P_{23}\psi^{abc}P_{23} = \psi^{acb} \ , \qquad P_{12}\psi^{abc}P_{12} = \psi^{bac}$$

$$P_{23}HP_{23} = -H \ , \qquad P_{12}HP_{12} = -H$$

- Z_3 symmetry generated by $P = P_{12}P_{23}$, $\quad P^3 = 1$

$$P\psi^{abc}P^\dagger = \psi^{cab} \ , \qquad PHP^\dagger = H$$

- At non-zero energy the gauge singlet states transform under the discrete group $A_4 \times Z_2$.
- Spectrum for N=4. Pakrouski, IK, Popov, Tarnopolsky

$$\pm\sqrt{32(447 \pm \sqrt{125601})}$$

$$\pm\sqrt{32(187 \pm \sqrt{11481})}$$

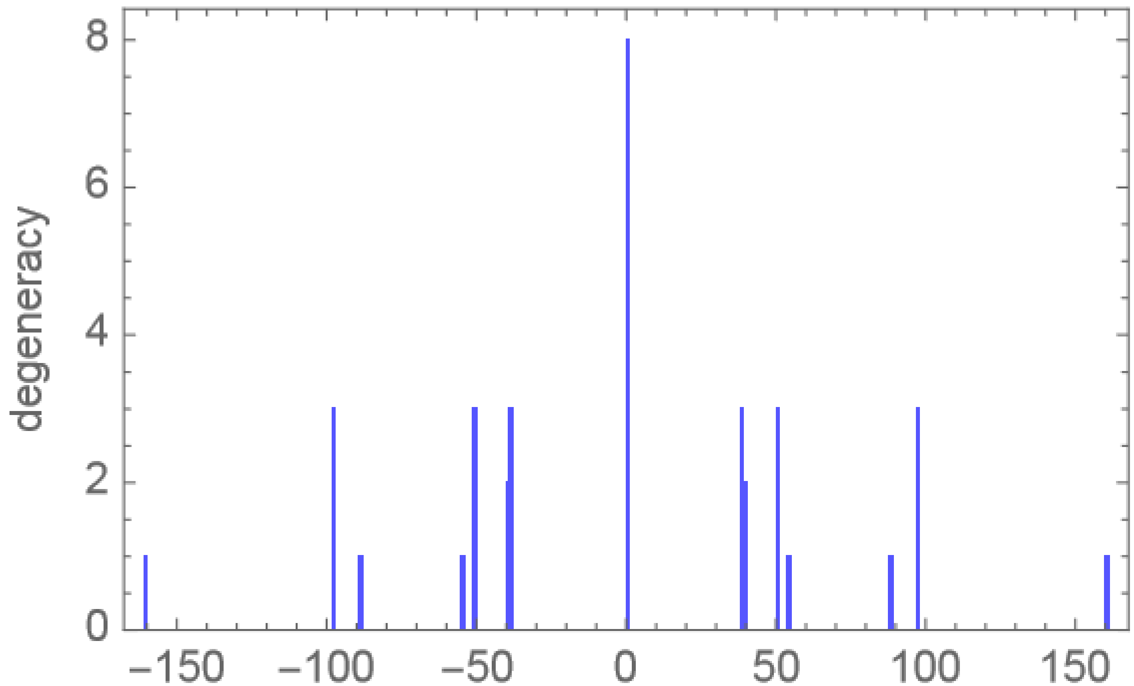
$$8\sqrt{24} =$$

$$8\sqrt{23} =$$

E	P_1	P_2	P_3	E	P_1	P_2	P_3
−160.140170	1	1	1	160.140170	1	1	1
−97.019491	1	1	−1	97.019491	1	1	−1
−97.019491	−1	1	1	97.019491	−1	1	1
−97.019491	1	−1	1	97.019491	1	−1	1
−88.724292	−1	−1	−1	88.724292	−1	−1	−1
−54.434603	1	1	1	54.434603	1	1	1
−50.549167	1	1	−1	50.549167	1	1	−1
−50.549167	−1	1	1	50.549167	−1	1	1
−50.549167	1	−1	1	50.549167	1	−1	1
−39.191836	1	1	1	39.191836	1	1	1
−39.191836	1	1	1	39.191836	1	1	1
−38.366652	1	−1	−1	38.366652	1	−1	−1
−38.366652	−1	1	−1	38.366652	−1	1	−1
−38.366652	−1	−1	1	38.366652	−1	−1	1
0.000000	1	1	1	0.000000	−1	−1	−1
0.000000	−1	1	1	0.000000	1	−1	−1
0.000000	1	−1	1	0.000000	−1	1	−1
0.000000	1	1	−1	0.000000	−1	−1	1

arXiv:1808.07455

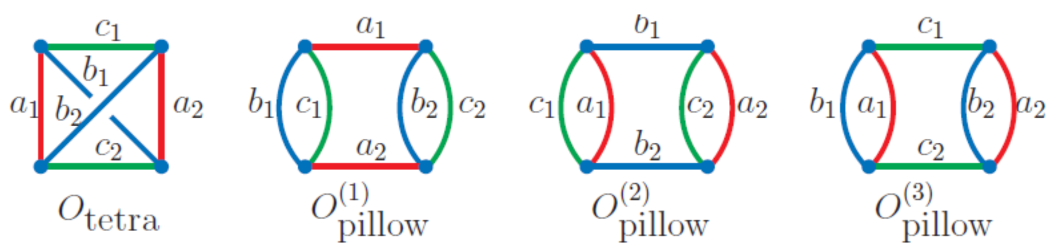
Energy Distribution for N=4



- For N=6 there will be over 595 million states packed into energy interval <1932. So, the gaps should be tiny.

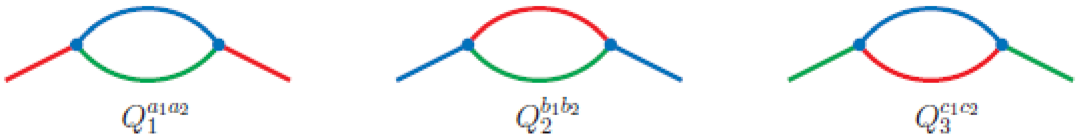
Gauge Invariant Operators

- Bilinear operators $\psi^{abc}\partial_t^{2n+1}\psi^{abc}$
- Related by the EOM to some of the higher particle “single-sum” operators.

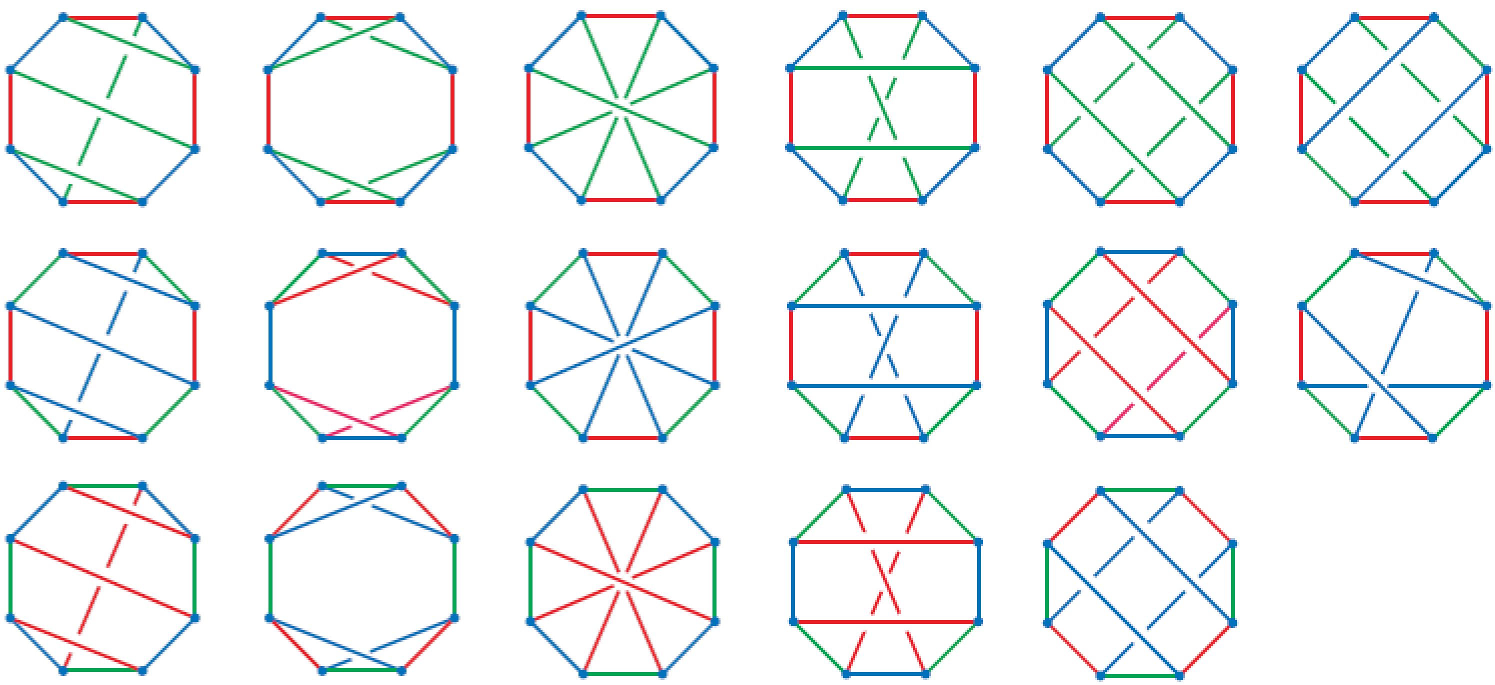


- All the 6-particle operators vanish by the Fermi statistics in the theory of one Majorana tensor.
- For higher number of fields, the number of invariants exhibits rapid, factorial growth. Ben Geloun, Ramgoolam

- The bubbles come from O(N) charges and vanish in the gauged model:



- The 17 single-sum 8-particle operators which do not include bubble insertions are



Factorial Growth

- There are 24 bubble-free 10-particle; 617 12-particle; 4887 14-particle; 82466 16-particle operators; etc.
- The number of (2k)-particle operators grows asymptotically as $k!\,2^k$. Bulycheva, IK, Milekhin, Tarnopolsky
- The Hagedorn temperature of the large N theory vanishes as 1/log N.
- The tensor models seem to lie "beyond string theory."
- Are they related to M-theory?

Tetrahedral Bosonic Tensor Model

- Action with a potential that is not positive definite IK, Tarnopolsky; Giombi, IK, Tarnopolsky

$$S = \int d^d x \Big(\frac{1}{2} \partial_\mu \phi^{abc} \partial^\mu \phi^{abc} + \frac{1}{4} g \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1} \Big)$$

- Schwinger-Dyson equation for 2pt function
Patashinsky, Pokrovsky

$$G^{-1}(p) = -\lambda^2 \int \frac{d^d k d^d q}{(2\pi)^{2d}} G(q) G(k) G(p+q+k)$$

- Has solution

$$G(p) = \lambda^{-1/2} \Big(\frac{(4\pi)^d d \Gamma(\frac{3d}{4})}{4\Gamma(1-\frac{d}{4})} \Big)^{1/4} \frac{1}{(p^2)^{\frac{d}{4}}}$$

Spectrum of two-particle spin zero operators

- Schwinger-Dyson equation

$$\int d^d x_3 d^d x_4 K(x_1, x_2; x_3, x_4) v_h(x_3, x_4) = g(h) v_h(x_1, x_2)$$

$$K(x_1, x_2; x_3, x_4) = 3\lambda^2 G(x_{13}) G(x_{24}) G(x_{34})^2$$

$$v_h(x_1, x_2) = \frac{1}{[(x_1 - x_2)^2]^{\frac{1}{2}(\frac{d}{2} - h)}}$$

$$g_{\text{bos}}(h) = -\frac{3\Gamma\left(\frac{3d}{4}\right)\Gamma\left(\frac{d}{4} - \frac{h}{2}\right)\Gamma\left(\frac{h}{2} - \frac{d}{4}\right)}{\Gamma\left(-\frac{d}{4}\right)\Gamma\left(\frac{3d}{4} - \frac{h}{2}\right)\Gamma\left(\frac{d}{4} + \frac{h}{2}\right)}$$

- In d<4 the first solution is complex $\frac{d}{2} + i\alpha(d)$

Complex Fixed Point in 4-ε Dimensions

- The tetrahedron

$$O_t(x) = \phi^{a_1b_1c_1}\phi^{a_1b_2c_2}\phi^{a_2b_1c_2}\phi^{a_2b_2c_1}$$

mixes at finite N with the pillow and double-sum operators

$$O_p(x) = \frac{1}{3}\left(\phi^{a_1b_1c_1}\phi^{a_1b_1c_2}\phi^{a_2b_2c_2}\phi^{a_2b_2c_1} + \phi^{a_1b_1c_1}\phi^{a_2b_1c_1}\phi^{a_2b_2c_2}\phi^{a_1b_2c_2} + \phi^{a_1b_1c_1}\phi^{a_1b_2c_1}\phi^{a_2b_1c_2}\phi^{a_2b_2c_2}\right),$$

$$O_{ds}(x) = \phi^{a_1b_1c_1}\phi^{a_1b_1c_1}\phi^{a_2b_2c_2}\phi^{a_2b_2c_2}$$

- The renormalizable action is

$$S = \int d^dx \left(\frac{1}{2}\partial_\mu\phi^{abc}\partial^\mu\phi^{abc} + \frac{1}{4}\left(g_1 O_t(x) + g_2 O_p(x) + g_3 O_{ds}(x)\right)\right)$$

- The large N scaling is

$$g_1 = \frac{(4\pi)^2 \tilde{g}_1}{N^{3/2}}, \quad g_2 = \frac{(4\pi)^2 \tilde{g}_2}{N^2}, \quad g_3 = \frac{(4\pi)^2 \tilde{g}_3}{N^3}$$

- The 2-loop beta functions and fixed points:

$$\tilde{\beta}_t = -\epsilon \tilde{g}_1 + 2\tilde{g}_1^3 ,$$

$$\tilde{\beta}_p = -\epsilon \tilde{g}_2 + \left(6\tilde{g}_1^2 + \frac{2}{3}\tilde{g}_2^2\right) - 2\tilde{g}_1^2 \tilde{g}_2 ,$$

$$\tilde{\beta}_{ds} = -\epsilon \tilde{g}_3 + \left(\frac{4}{3}\tilde{g}_2^2 + 4\tilde{g}_2\tilde{g}_3 + 2\tilde{g}_3^2\right) - 2\tilde{g}_1^2(4\tilde{g}_2 + 5\tilde{g}_3)$$

$$\tilde{g}_1^* = (\epsilon/2)^{1/2}, \quad \tilde{g}_2^* = \pm 3i(\epsilon/2)^{1/2}, \quad \tilde{g}_3^* = \mp i(3 \pm \sqrt{3})(\epsilon/2)^{1/2}$$

- The scaling dimension of $\phi^{abc}\phi^{abc}$ is

$$\Delta_O = d - 2 + 2(\tilde{g}_2^* + \tilde{g}_3^*) = 2 \pm i\sqrt{6\epsilon} + \mathcal{O}(\epsilon)$$

- Spectrum in d=1 again includes scaling dimension h=2, suggesting the existence of a gravity dual.
- However, the leading solution is complex, which suggests that the large N CFT is unstable Giombi, IK, Tarnopolsky $h_0 = \frac{1}{2} + 1.525i$
- It corresponds to the operator $\phi^{abc}\phi^{abc}$

- The dual scalar field in AdS violates the Breitenlohner-Freedman bound.

A Richer Set of Tensor Models

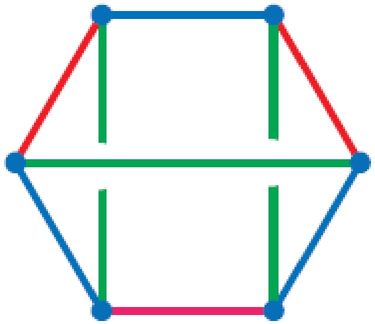
- The tetrahedral interaction is the simplest possibility of obtaining a solvable large N tensor model.
- There are many others!
- For the interaction of order 2n the maximal tensor rank is 2n-1. When it is lower, the theory may be called "subchromatic." Prakash, Sinha
- It is helpful to chose the dominant interaction to be Maximally Single Trace (MST). Ferrari, Rivasseau, Valette; IK, Pallegar, Popov

Prismatic Bosonic QFT

- Large N limit dominated by the positive sextic "prism" interaction Giombi, IK, Popov, Prakash, Tarnopolsky

$$S=\int d^dx\left(\frac{1}{2}(\partial_\mu\phi^{abc})^2+\frac{g_1}{6!}\phi^{a_1b_1c_1}\phi^{a_1b_2c_2}\phi^{a_2b_1c_2}\phi^{a_3b_3c_1}\phi^{a_3b_2c_3}\phi^{a_2b_3c_3}\right)$$

- It is subchromatic and MST (erasing any color leaves the diagram connected).
- To obtain the large N solution it is convenient to rewrite



$$S=\int d^dx\left(\frac{1}{2}(\partial_\mu\phi^{abc})^2+\frac{\lambda}{3!}\phi^{a_1b_1c_1}\phi^{a_1b_2c_2}\phi^{a_2b_1c_2}\chi^{a_2b_2c_1}-\frac{1}{2}\chi^{abc}\chi^{abc}\right)$$

- Tensor counterpart of a bosonic SYK-like model.

  Murugan, Stanford, Witten

- The IR solution in general dimension:

$$3\Delta_\phi + \Delta_\chi = d \ , \qquad d/2 - 1 < \Delta_\phi < d/6$$

$$\frac{\Gamma(\Delta_\phi)\Gamma(d-\Delta_\phi)}{\Gamma(\frac{d}{2}-\Delta_\phi)\Gamma(-\frac{d}{2}+\Delta_\phi)} = 3\frac{\Gamma(3\Delta_\phi)\Gamma(d-3\Delta_\phi)}{\Gamma(\frac{d}{2}-3\Delta_\phi)\Gamma(-\frac{d}{2}+3\Delta_\phi)}$$

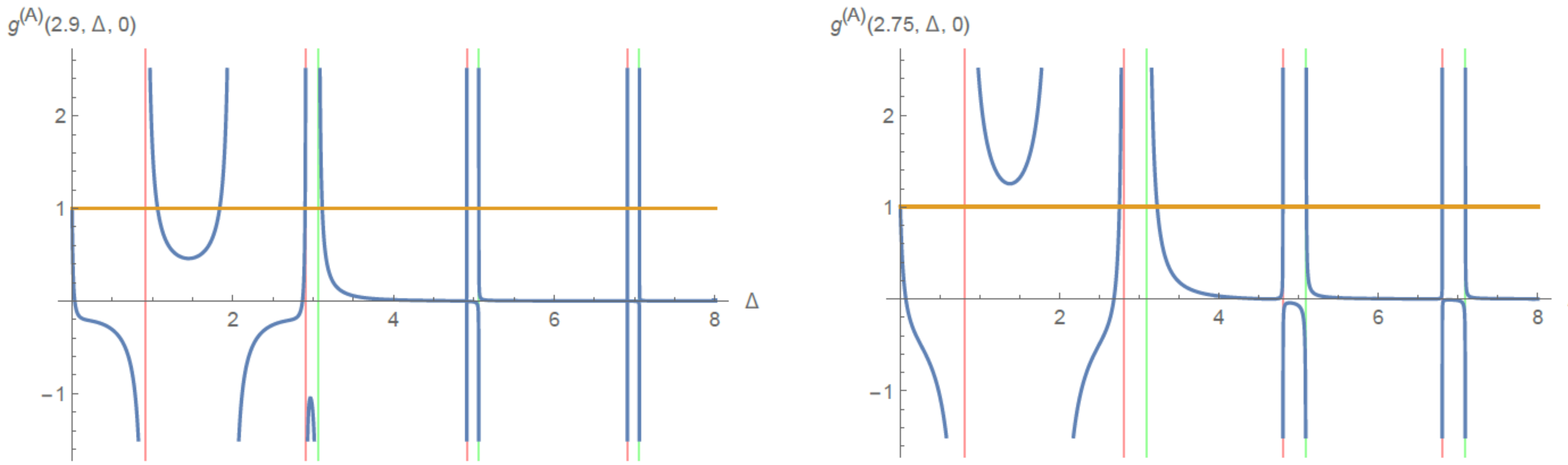
- In $d = 3 - \epsilon$

$$\Delta_\phi = \frac{1}{2} - \frac{\epsilon}{2} + \epsilon^2 - \frac{20\epsilon^3}{3} + \left(\frac{472}{9} + \frac{\pi^2}{3}\right)\epsilon^4 + \left(7\zeta(3) - \frac{12692}{27} - \frac{56\pi^2}{9}\right)\epsilon^5 + O\left(\epsilon^6\right)$$

- For d=2.9 find numerically

$$\Delta_\phi = 0.456264 \ , \qquad \Delta_\chi = 1.53121$$

- Dimensions of bilinear operators in d=2.9 and 2.75



- The first root has expansion

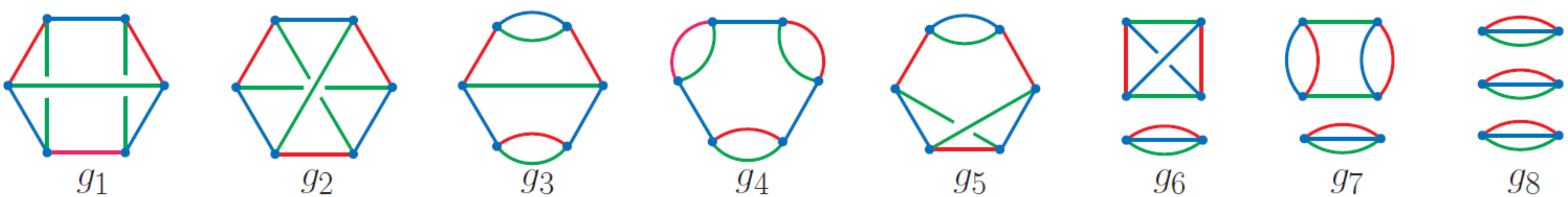
$$\Delta_{\phi^2} = 1 - \epsilon + 32\epsilon^2 - \frac{976\epsilon^3}{3} + \left(\frac{30320}{9} + \frac{32\pi^2}{3}\right)\epsilon^4 + O\left(\epsilon^5\right)$$

- For $1.6799 < d < 2.8056$ Δ_{ϕ^2} becomes complex

$$\frac{d}{2} + i\alpha(d)$$

Finite N

- The 3-ε expansion at finite N may be generated using standard perturbation theory.
- Need to include 7 more $O(N)^3$ invariant operators.



- The 8 beta functions have a "prismatic" fixed point" for N>53. At large N the scaling dimensions there agree with the Schwinger-Dyson results, including 4-loop corrections to beta functions (in preparation with C. Jepsen and F. Popov).

Many Questions Remaining

- What is the precise holographic setting of tensor models in view of the factorial growth of the number of gauge invariant operators.
- What is the list of “stable” melonic or generalized melonic large N theories in d>1.
- Coupled tensor or SYK models can exhibit interesting dynamical phenomena, such as symmetry breaking. Applications to 2-d wormholes. Maldacena, Qi; Kim, IRK, Tarnopolsky, Zhao; IRK, Milekhin, Tarnopolsky, Zhao
- Applications of melonic models to condensed matter physics?
- Applications to quantum information theory? Milekhin