Long-range multi-scalar models at three loops.

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Based on arXiv:2007:04603 with D.Benedetti, R.Gurau and K.Suzuki

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Long-range models

- Kinetic term of the form $\phi(\partial^2)^\zeta\phi$ with $0 < \zeta < 1$
- Vast array of applications [Campa, Dauxois, Ruffo, 2009]
- Admit phase transition [Dyson]
- One-parameter families of universality classes: ζ
- Study transition between short-range and long-range universality classes [Angelini et al., Brezin et al.,...]
- Rigorous renormalization group in $d = 3$ [Brydges et al., Abdesselam,...]

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Challenges

- No local energy momentum tensor: conformal invariance ? [Paulos, Rychkov, van Rees, Zan]
- Analytic evaluation of Feynman integrals: only up to two loops

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$\Rightarrow$ Here up to three loops

Outline

1. The long-range multi-scalar model
   - Two and four-point functions
   - Beta functions up to three loops
2. Applications
   - Long-range Ising model
   - Long-range $O(N)$ vector model
   - Long-range cubic model
   - Long-range $O(M) \times O(N)$ bi-fundamental model
3. Conclusion and further work

Long-range multi-scalar model

$$S[\phi] = \int d^d x \left[\frac{1}{2} \phi_{\mathbf{a}}(x)(-\partial^2)^{\zeta} \phi_{\mathbf{a}}(x) + \frac{1}{2} \kappa_{\mathbf{ab}} \phi_{\mathbf{a}}(x) \phi_{\mathbf{b}}(x) + \frac{1}{4!} \lambda_{\mathbf{abcd}} \phi_{\mathbf{a}}(x) \phi_{\mathbf{b}}(x) \phi_{\mathbf{c}}(x) \phi_{\mathbf{d}}(x) \right]$$

- Indices take values from 1 to $\mathcal{N}$
- Mass parameter κ treated as a perturbation
- $d < 4$ fixed

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$$\left. + \frac{1}{4!}\,\lambda_{\mathbf{abcd}}\phi_{\mathbf{a}}(x)\phi_{\mathbf{b}}(x)\phi_{\mathbf{c}}(x)\phi_{\mathbf{d}}(x) \right]$$

- Indices take values from 1 to $\mathcal{N}$
- Mass parameter κ treated as a perturbation
- $d < 4$ fixed
- Canonical dimension of the field: $\Delta_\phi = \frac{d-2\zeta}{2}$
- Weakly relevant case: $\zeta = \frac{d+\epsilon}{4}$ with small ϵ
- UV dimension of the field $\Delta_\phi = \frac{d-\epsilon}{4}$

Divergences and regularization

Theory at $\epsilon = 0$:

- With only λ vertices: 2-point graphs power divergent, 4-point graphs log divergent, higher orders convergent
- Local power divergence subtracted: convergent 2-point graphs
- No wave function renormalization
- κ vertices: ϕ^2 insertions $\Rightarrow$ quartic vertices with two external half-edges

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With power counting, the only superficially divergent graphs are:

- Four-point graphs with only λ vertices
- Two-point graphs with exactly one κ vertex

Choice of scheme

Logarithmically divergent $\Rightarrow$ need both UV and IR regularization:

- UV regularization: $\epsilon > 0$
- IR regularization: mass regulator $\mu > 0$ with modified covariance

$$C_\mu(p) = \frac{1}{(p^2 + \mu^2)^\zeta} = \frac{1}{\Gamma(\zeta)} \int_0^\infty da\, a^{\zeta-1} e^{-ap^2 - a\mu^2}$$

- Zero momentum BPHZ subtraction scheme

Two and four point functions: amplitude

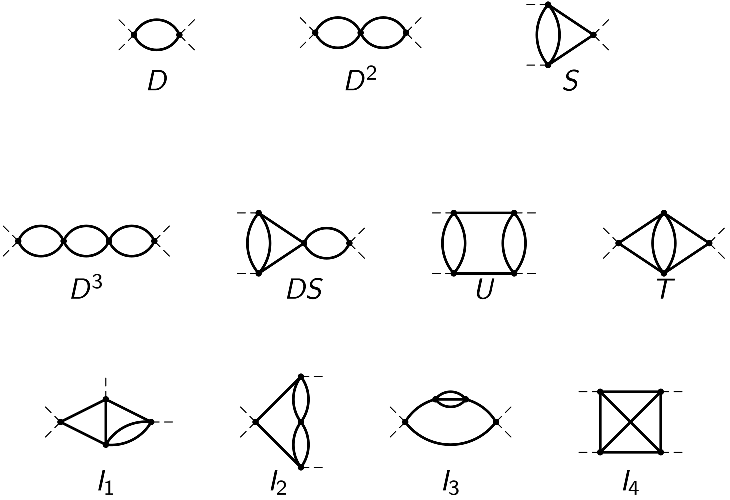
$\Gamma^{(2)}_{\mathbf{ab}}$ and $\Gamma^{(4)}_{\mathbf{abcd}}$: one-particle irreducible two and four-point functions at zero external momentum

Amplitude of a Feynman graph G in Schwinger parametrization:

$$\mathcal{A}(G) = \mu^{(d-4\zeta)(V-1)}\, \hat{\mathcal{A}}(G)\,,$$
$$\hat{\mathcal{A}}(G) = \frac{1}{\left[(4\pi)^{d/2}\Gamma(\zeta)^2\right]^{V-1}} \int_0^\infty \prod_{e\in G} da_e \; \frac{\prod_{e\in G} a_e^{\zeta-1}\; e^{-\sum_{e\in G} a_e}}{\left(\sum_{\mathcal{T}\in G}\prod_{e\notin\mathcal{T}} a_e\right)^{d/2}}$$

- V: number of vertices
- e runs over the edges of G
- $\mathcal{T}$: spanning trees in G

Four-point function: contributions up to three loops

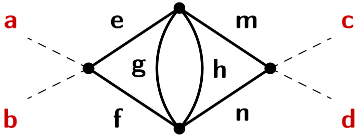


Four-point function: bare expansion

$$\Gamma^{(4)}_{\mathbf{abcd}} = \lambda_{\mathbf{abcd}} + \text{one-loop term} + \text{two-loop terms} + \text{three-loop terms}$$

For example, the three-loop diagram T contributes to:

$$-\frac{1}{4}\left(\lambda_{\mathbf{abef}}\lambda_{\mathbf{eghm}}\lambda_{\mathbf{fghn}}\lambda_{\mathbf{mncd}} + 2\ \text{terms}\right)\mu^{-3\epsilon}\, T$$



- Why "+ 2 terms " ? To conserve the permutation symmetry

Two-point function: bare expansion

$$\Gamma^{(2)}_{\mathbf{cd}} = \kappa_{\mathbf{cd}} + \text{one-loop term} + \text{two-loop terms} + \text{three-loop terms}$$

- Contributions from diagrams having at least one vertex with two external half-edges
- Substitute one of this vertices by a κ vertex
- Broken symmetry: only one term for each contribution

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- Contributions from diagrams having at least one vertex with two external half-edges
- Substitute one of this vertices by a κ vertex
- Broken symmetry: only one term for each contribution
- Only interested in fixed points with $\kappa = 0$: means to obtain the scaling dimension of quadratic operators at the fixed points
- No diagrams with more insertions of κ vertices in $\Gamma^{(2)}_{\mathbf{cd}}$
- No κ contributions to $\Gamma^{(4)}_{\mathbf{abcd}}$

Beta functions

The dimensionless four-point function at zero external momenta is identified with the running coupling:

$$g_{\mathbf{abcd}} = \mu^{-\epsilon}\,\Gamma^{(4)}_{\mathbf{abcd}}\,, \qquad r_{\mathbf{cd}} = \mu^{-(d-2\Delta_\phi)}\,\Gamma^{(2)}_{\mathbf{cd}}\,.$$

Beta functions: scale derivatives of the running coupling at fixed bare couplings

$$\beta^{(4)}_{\mathbf{abcd}} = \mu\partial_\mu g_{\mathbf{abcd}}\,, \qquad \beta^{(2)}_{\mathbf{cd}} = \mu\partial_\mu r_{\mathbf{cd}}\,.$$

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Method:

- Derive the bare expansion with respect to μ
- Invert the bare expansion to obtain the renormalized series
- Substitute the bare coupling by its expression in terms of the running coupling

Three-loop beta function

$$\begin{aligned}
\beta^{(4)}_{\mathbf{abcd}} = -\epsilon\tilde{g}_{\mathbf{abcd}} &+ \alpha_D\left(\tilde{g}_{\mathbf{abef}}\tilde{g}_{\mathbf{efcd}} + 2 \text{ terms}\right) \\
&+ \alpha_S\left(\tilde{g}_{\mathbf{abef}}\tilde{g}_{\mathbf{eghc}}\tilde{g}_{\mathbf{fghd}} + 5 \text{ terms}\right) \\
&+ \alpha_U\left(\tilde{g}_{\mathbf{aefg}}\tilde{g}_{\mathbf{befh}}\tilde{g}_{\mathbf{gmnc}}\tilde{g}_{\mathbf{hmnd}} + 5 \text{ terms}\right) \\
&+ \alpha_T\left(\tilde{g}_{\mathbf{abef}}\tilde{g}_{\mathbf{eghm}}\tilde{g}_{\mathbf{fghn}}\tilde{g}_{\mathbf{mncd}} + 2 \text{ terms}\right) \\
&+ \alpha_{I_1}\left(\tilde{g}_{\mathbf{abef}}\tilde{g}_{\mathbf{eghm}}\tilde{g}_{\mathbf{fgnc}}\tilde{g}_{\mathbf{hmnd}} + 11 \text{ terms}\right) \\
&+ \alpha_{I_2}\left(\tilde{g}_{\mathbf{abef}}\tilde{g}_{\mathbf{eghc}}\tilde{g}_{\mathbf{fmnd}}\tilde{g}_{\mathbf{ghmn}} + 5 \text{ terms}\right) \\
&+ \alpha_{I_3}\left(\tilde{g}_{\mathbf{abef}}\tilde{g}_{\mathbf{hmnf}}\tilde{g}_{\mathbf{hmng}}\tilde{g}_{\mathbf{gecd}} + 2 \text{ terms}\right) \\
&+ \alpha_{I_4}\left(\tilde{g}_{\mathbf{aemh}}\tilde{g}_{\mathbf{befn}}\tilde{g}_{\mathbf{cfmg}}\tilde{g}_{\mathbf{dgnh}}\right)
\end{aligned}$$

where we rescaled the couplings as $g_{\mathbf{abcd}} = (4\pi)^{d/2}\,\Gamma(d/2)\,\tilde{g}_{\mathbf{abcd}}$ and $r_{\mathbf{ab}} = (4\pi)^{d/2}\,\Gamma(d/2)\,\tilde{r}_{\mathbf{ab}}$

Alpha coefficients

Finite quantities thanks to counterterms from the renormalized series

$$\alpha_D = \epsilon Q \frac{D}{2}, \qquad \alpha_S = \epsilon Q^2 \frac{(D^2 - 2S)}{2},$$

$$\alpha_U = \epsilon Q^3 \frac{(D^3 - 4DS + 3U)}{4}, \qquad \alpha_T = \epsilon Q^3 \frac{(3T - 2DS)}{4},$$

$$\alpha_{I_1} = \epsilon Q^3 \frac{(D^3 - 3DS + 3I_1)}{2}, \qquad \alpha_{I_2} = \epsilon Q^3 \frac{(D^3 - 4DS + 3I_2)}{4},$$

$$\alpha_{I_3} = \epsilon Q^3 \frac{I_3}{2}, \qquad \alpha_{I_4} = \epsilon Q^3\, 3I_4 \,.$$

with $Q = (4\pi)^{d/2}\,\Gamma(\frac{d}{2})$

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with $Q = (4\pi)^{d/2}\,\Gamma(\frac{d}{2})$

Method:

- Schwinger parameters and Mellin-Barnes integrals
- Computation only up to the needed powers in ϵ

Example: computation of T integral

Amplitude T:

$$T = \frac{1}{(4\pi)^{3d/2}\Gamma(\zeta)^6} \int_0^\infty \prod_{i=1}^{2} da_i db_i dc_i \frac{(a_1 a_2 b_1 b_2 c_1 c_2)^{\zeta-1} e^{-(a_1+a_2+b_1+b_2+c_1+c_2)}}{\left[(a_1+a_2)(b_1+b_2)(c_1+c_2)+b_1 b_2(a_1+a_2+c_1+c_2)\right]^{d/2}}$$

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Introducing Mellin-parameters:

$$T = \frac{1}{(4\pi)^{3d/2}\Gamma(\zeta)^2\Gamma(2\zeta)^2\Gamma(\frac{d}{2})} \int_{0^-}[dz_1]\int_{0^-}[dz_2]\,\Gamma(\tfrac{d}{2}+z_1+z_2) \frac{\Gamma(\zeta+z_1+z_2)^2}{\Gamma(2\zeta+2z_1+2z_2)}\Gamma(\tfrac{\epsilon}{2}+z_1+z_2)\Gamma(-z_1)\Gamma(-z_2)\Gamma(\tfrac{\epsilon}{2}-z_2)\Gamma(\tfrac{\epsilon}{2}-z_1)\,.$$

Example: computation of T integral

- Move both contours to the right
- Poles in z_1 and z_2 are independent
- Only four poles give singular contributions: $(0, 0), (0, \epsilon/2), (\epsilon/2, 0)$ and $(\epsilon/2, \epsilon/2)$
- Expand the contribution of each pole in ϵ
- Remaining integrals of order $\mathcal{O}(\epsilon^0)$

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- Remaining integrals of order $\mathcal{O}(\epsilon^0)$

$$T = \frac{1}{3(4\pi)^{3d/2}\Gamma(d/2)^3}\left[\frac{8}{\epsilon^3} + \frac{8}{\epsilon^2}\left(2\psi(1) - \psi(\tfrac{d}{4}) - \psi(\tfrac{d}{2})\right) + \frac{1}{3\epsilon}\left(\pi^2 + 12\left(2\psi(1) - \psi(\tfrac{d}{4}) - \psi(\tfrac{d}{2})\right)^2 - 6\psi_1(\tfrac{d}{2})\right)\right] + \mathcal{O}(\epsilon^0)\,.$$

Main result: computation of the alpha coefficients (1)

One and two-loop coefficients:

$$
\begin{aligned}
\alpha_D &= 1 + \frac{\epsilon}{2}\left[\psi(1) - \psi(\tfrac{d}{2})\right] + \frac{\epsilon^2}{8}\left[\left(\psi(1) - \psi(\tfrac{d}{2})\right)^2 + \psi_1(1) - \psi_1(\tfrac{d}{2})\right], \\
\alpha_S &= 2\psi(\tfrac{d}{4}) - \psi(\tfrac{d}{2}) - \psi(1) \\
&\quad + \frac{\epsilon}{4}\Big[\left[2\psi(\tfrac{d}{4}) - \psi(\tfrac{d}{2}) - \psi(1)\right] \left[3\psi(1) - 5\psi(\tfrac{d}{2}) + 2\psi(\tfrac{d}{4})\right] \\
&\quad + 3\psi_1(1) + 4\psi_1(\tfrac{d}{4}) - 7\psi_1(\tfrac{d}{2}) - 4J_0(\tfrac{d}{4})\Big]
\end{aligned}
$$

with ψ_i the polygamma functions of order i and J_0 an indefinite sum.

Main results: computation of the alpha coefficients (2)

Three-loop coefficients:

$$\alpha_U = \alpha_{I_2} = -\psi_1(1) - \psi_1(\tfrac{d}{4}) + 2\psi_1(\tfrac{d}{2}) + J_0(\tfrac{d}{4}) \,,$$

$$\alpha_T = \frac{1}{2}\Big[2\psi(\tfrac{d}{4}) - \psi(\tfrac{d}{2}) - \psi(1)\Big]^2 + \frac{1}{2}\psi_1(1) + \psi_1(\tfrac{d}{4}) - \frac{3}{2}\psi_1(\tfrac{d}{2}) - J_0(\tfrac{d}{4}) \,,$$

$$\alpha_{I_1} = \frac{3}{2}\left[2\psi(\tfrac{d}{4}) - \psi(\tfrac{d}{2}) - \psi(1)\right]^2 + \frac{1}{2}\psi_1(1) - \frac{1}{2}\psi_1(\tfrac{d}{2})$$

$$\alpha_{I_3} = \frac{\Gamma(-\frac{d}{4})\Gamma(\frac{d}{2})^2}{3\,\Gamma(\frac{3d}{4})} \,,$$

$$\alpha_{I_4} = \frac{\Gamma(1+\frac{d}{4})^3\Gamma(-\frac{d}{4})}{\Gamma(\frac{d}{2})}\, 6\left[\psi_1(1) - \psi_1(\tfrac{d}{4})\right] .$$

Long-range Ising Model

$\mathcal{N} = 1$, $\tilde{g}_{\mathbf{abcd}} = \tilde{g}$ and $\tilde{r}_{\mathbf{ab}} = \tilde{r}$
Perturbative fixed point in ϵ:

$$\tilde{g}_\star = \frac{\epsilon}{3} - \left(\frac{3\alpha_{D,1} + 2\alpha_{S,0}}{9}\right)\epsilon^2 - \frac{\epsilon^3}{81}\Big[27\alpha_{D,2} - 24\alpha_{S,0}^2 + 9(2\alpha_{S,1} - 3\alpha_{D,1}^2) - 54\alpha_{D,1}\alpha_{S,0} + 3\alpha_T + 6\alpha_U + 12\alpha_{I_1} + 6\alpha_{I_2} + 3\alpha_{I_3} + \alpha_{I_4}\Big] + \mathcal{O}(\epsilon^4)\,.$$

with

$$\alpha_D = 1 + \alpha_{D,1}\,\epsilon + \alpha_{D,2}\,\epsilon^2 + \mathcal{O}(\epsilon^3)\,,$$
$$\alpha_S = \alpha_{S,0} + \alpha_{S,1}\,\epsilon + \mathcal{O}(\epsilon^2)\,.$$

Critical exponents

- Stability exponents: $\partial_{\tilde{r}} \beta^{(2)}(\tilde{g}_\star)$ and $\partial_{\tilde{g}} \beta^{(4)}(\tilde{g}_\star)$
- Critical exponents: anomalous dimension η, susceptibility exponent γ and correlation length ν
- Related by: $\gamma = (2 - \eta)\nu$

Enough to consider:

$$\nu^{-1} = -\partial_{\tilde{r}} \beta^{(2)}(\tilde{g}_\star)$$

and the correction to scaling exponent:

$$\omega = \partial_{\tilde{g}} \beta^{(4)}(\tilde{g}_\star)$$

Comparison with numerical result from the literature: good consistency

Link between short and long-range Ising model

Transition between short-range and long-range:

- Happens at $2\zeta = 2 - \eta_{SR}$
- In 2d consider ϵ up to 1.5
- Find consistent value with $\nu_{SR} = 1$

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Conjectured relation between short and long range Ising models:

$$\frac{2\zeta}{d} = \frac{2 - \eta_{SR}(d_{SR})}{d_{SR}}$$

[Angelini et al., Defenu et al., Banos et al.]

- True at order ϵ_{SR}
- Fail already at order ϵ_{SR}^2

Long-range $O(N)$ vector model

$\mathcal{N} = N$, $\mathbf{a} = a = 1, \ldots, N$, and so on and:

$$\tilde{g}_{\mathbf{abcd}} = \frac{\tilde{g}}{3}\left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right), \qquad \tilde{r}_{\mathbf{ab}} = \tilde{r}\,\delta_{ab}\,.$$

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Long-range version of the Wilson-Fisher fixed point:

$$\begin{aligned}\tilde{g}_\star = {} & \frac{3\epsilon}{N+8} - \frac{3\epsilon^2}{(N+8)^3}\left[(N+8)^2\alpha_{D,1} + 2(5N+22)\alpha_{S,0}\right] \\ & + \frac{3\epsilon^3}{(N+8)^5}\Big[(N+8)^2\Big(2(5N+22)(3\alpha_{S,0}\alpha_{D,1} - \alpha_{S,1}) - 3(N+2)\alpha_{I_3}\Big) \\ & + (N+8)^4(\alpha_{D,1}^2 - \alpha_{D,2}) + 8(5N+22)^2\alpha_{S,0}^2 \\ & - (N+8)\Big((3N^2+22N+56)(2\alpha_{I_2} + \alpha_T) + (5N+22)\alpha_{I_4} \\ & + 2(N^2+20N+60)(2\alpha_{I_1} + \alpha_U)\Big)\Big] + \mathcal{O}(\epsilon^4)\end{aligned}$$

Long-range $O(N)$ vector model

Critical exponents:

- One and two-loop results agree with literature [Fisher,Ma, Nickel; Yamazaki, Suzuki]
- Three-loop contributions are new
- N_c: critical value at which ω vanishes and the WF fixed-point becomes marginal

$$N_c = -8 \pm 6\sqrt{2|\alpha_{S,0}|}\epsilon^{1/2} + \mathcal{O}(\epsilon)$$

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- Negative at small ϵ
- Quartic operator never becomes relevant for bosonic models
- Can cross marginality for symplectic fermions [Giuliani, Mastropietro, Rychkov]

Large N

Critical exponents up to order N^{-2} after rescaling $\tilde{g} = \bar{g}/N$:

$$\begin{aligned}\nu^{-1} = {} & 2\Delta_\phi + \frac{\epsilon}{N}\Big[6 + 7\alpha_{S,0}\epsilon + \big(2\alpha_U + 2\alpha_{I_1} + 5\alpha_{I_2} \\ & + 7(\alpha_{S,1} - 2\alpha_{D,1}\alpha_{S,0})\big)\epsilon^2\Big] \\ & - \frac{\epsilon}{N^2}\Big[48 + 134\alpha_{S,0}\epsilon + \big(12\alpha_{I_1} + 122\alpha_{I_2} - 5\alpha_{I_4} \\ & + 134(\alpha_{S,1} - 2\alpha_{D,1}\alpha_{S,0}) + 140\alpha_{S,0}^2 + 8\alpha_T + 20\alpha_U\big)\epsilon^2\Big] + \mathcal{O}(N^{-3}, \epsilon^4)\end{aligned}$$

- Leading-order: spherical model result $\gamma = 2\zeta\nu = 2\zeta/(d - 2\zeta)$ [Joyce]
- Order N^{-1} already computed to all orders in ϵ: reproduce it to three-loops [Fisher, Ma, Nickel]
- Order N^{-2} new

Long-range cubic model

Explicit breaking of the $O(N)$ symmetry with an interaction of the form $\sum_{\mathbf{a}} \phi_{\mathbf{a}}^4$. Setting:

$$\tilde{g}_{\mathbf{abcd}} = \frac{\tilde{g}_d}{3} \left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right) + \tilde{g}_c \delta_{ab}\delta_{ac}\delta_{ad} \, .$$

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Four types of fixed points:

- Gaussian fixed point
- Heisenberg fixed point: $\tilde{g}_c^\star = 0$ and $\tilde{g}_d^\star \neq 0$
- Ising fixed point: $\tilde{g}_c^\star \neq 0$ and $\tilde{g}_d^\star = 0$
- Cubic fixed point: both couplings are non-zero

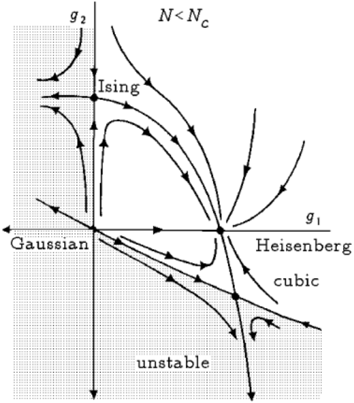
Computation up to ϵ^3, two-loop terms agree with [Yamazaki,Holz;Chen,Li]

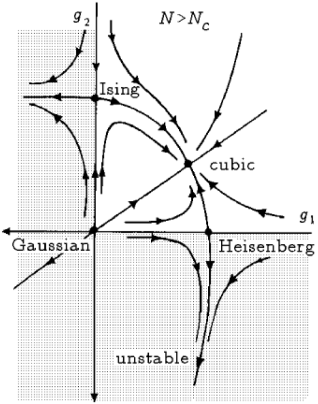
Stability

- Gaussian fixed point: doubly unstable
- Ising fixed point: one stable and one unstable direction
- N_c: critical value at which Heisenberg and cubic fixed points exchange stability
- $N < N_c$: Heisenberg fixed point is stable
- $N > N_c$: Cubic fixed point is stable

$$N_c = 4 + 2\epsilon\alpha_{S,0} + \frac{\epsilon^2}{6}\Bigg(8\alpha_{I_1} + 4\alpha_{I_2} + \frac{5}{4}\alpha_{I_4} + 12(\alpha_{S,1} - 2\alpha_{D,1}\alpha_{S,0}) - 13\alpha_{S,0}^2 - \alpha_T + 7\alpha_U\Bigg).$$

Flow for $N < N_c$ and $N > N_c$





Taken from Kleinert, Schulte and Frohlinde, *Critical properties of ϕ^4 theories.*

Numerical values

N_c at $d = 3$:

ϵ	one-loop	three-loop	PB [1/1]
0.2	4	3.5712	3.500(5)
0.4	4	3.5897	3.171(13)
0.6	4	4.0553	2.926(21)

- $N_c \sim 3$: for $N = 3$ ν_H and ν_C lie very close to each other
- Heisenberg and cubic critical behavior are practically indistinguishable

$O(M) \times O(N)$ bi-fundamental model

$\mathcal{N} = MN$ and

$$
\begin{aligned}
\tilde{g}_{\mathbf{abcd}} \;=\; & \frac{\tilde{g}_s}{6}\big(\delta_{a_1 b_1}\delta_{c_1 d_1}(\delta_{a_2 c_2}\delta_{b_2 d_2} + \delta_{a_2 d_2}\delta_{b_2 c_2}) \\
& \quad + \delta_{a_1 c_1}\delta_{b_1 d_1}(\delta_{a_2 b_2}\delta_{c_2 d_2} + \delta_{a_2 d_2}\delta_{c_2 b_2}) \\
& \quad + \delta_{a_1 d_1}\delta_{c_1 b_1}(\delta_{a_2 c_2}\delta_{b_2 d_2} + \delta_{a_2 b_2}\delta_{d_2 c_2})\big) \\
& + \frac{\tilde{g}_d}{3}\big(\delta_{a_1 b_1}\delta_{a_2 b_2}\delta_{c_1 d_1}\delta_{c_2 d_2} + \delta_{a_1 c_1}\delta_{a_2 c_2}\delta_{b_1 d_1}\delta_{b_2 d_2} \\
& \quad + \delta_{a_1 d_1}\delta_{a_2 d_2}\delta_{c_1 b_1}\delta_{c_2 b_2}\big)\,, \\
\tilde{r}_{\mathbf{ab}} \;=\; & \tilde{r}\,\delta_{a_1 b_1}\delta_{a_2 b_2}\,,
\end{aligned}
$$

where $a_1 = 1, \ldots, M$ and $a_2 = 1, \ldots, N$ and so on.

Fixed points

Four fixed points:

- Gaussian fixed point
- Heisenberg fixed point: $\tilde{g}_s^\star = 0$ and $\tilde{g}_d^\star = \frac{3\epsilon}{MN+8}$
- Chiral and anti-chiral fixed points: both couplings non zero:

$$\tilde{g}_s^\star = \frac{(12-3MN)\epsilon}{4+10(M+N)-MN(M+N+4)\pm 6\sqrt{Q}}$$

$$\tilde{g}_d^\star = \frac{-3(-80+2M+2N+M^2+N^2+2MN\mp 4(M+N+4)\sqrt{Q})\epsilon}{2(464-56(M+N)-16(M^2+N^2+MN)+8MN(M+N)+MN(M+N)^2)}$$

where $Q = 52 - 4(M+N) + (M^2 - 10MN + N^2)$

Stability

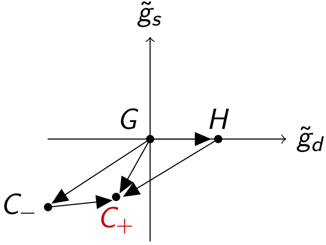
Four regimes of criticality at fixed M:

- $N > N_{c+}$: four real fixed points, and the chiral one is stable
- $N_{c-} < N < N_{c+}$: only the Gaussian and the Heisenberg fixed points are real, both unstable
- $N_H < N < N_{c-}$: four real fixed points, and the chiral one is stable
- $N < N_H$: four real fixed points, but the Heisenberg one is stable

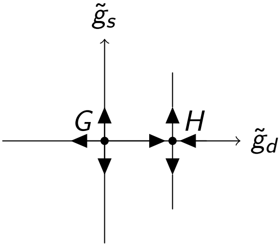
Found by solving:

$$\det \left| \frac{\partial(\beta_s, \beta_d)}{\partial(\tilde{g}_s, \tilde{g}_d)} \right|_{\tilde{g}=\tilde{g}^\star} = 0\,.$$

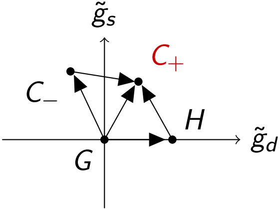
Flows for the four regimes of stability



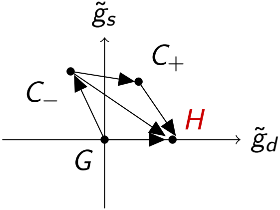
$N > N_{c+}$



$N_{c-} < N < N_{c+}$



$N_H < N < N_{c-}$



$N < N_H$

Numerical values

Critical N's at $d = 3$ and $M = 2$:

ϵ		one-loop	three-loop	PB [1/1]
0.2	N_{c+}	21.8	16.36	15.7(12)
	N_{c-}	2.202	2.120	2.076(35)
	N_H	2	1.806	1.759(12)
0.4	N_{c+}	21.8	15.35	11.6(29)
	N_{c-}	2.202	2.245	2.01(9)
	N_H	2	1.875	1.608(29)
0.6	N_{c+}	21.8	18.76	8(4)
	N_{c-}	2.202	2.578	1.96(14)
	N_H	2	2.207	1.50(5)

- $N = 2$: chiral fixed point might exist
- $N = 3$: chiral fixed point not present and Heisenberg fixed point unstable

Conclusion

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Our work:

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- Renormalization group beta functions for the general multi-scalar model up to three loops
- Provide higher order results for specific models: Ising, $O(N)$, cubic and $O(M) \times O(N)$

Perspective:

- Long-range studies with other methods: Monte-Carlo simulations, bootstrap or functional RG
- Crossover from long-range to short-range for general multi-scalar models

Work in progress: tri-fundamental model

$O(N)^3$ tensor model:

- Complex unstable fixed points in the short-range case [Giombi, Klebanov, Tarnopolsky]
- Real IR attractive fixed points in the long-range case [Benedetti, Gurau,SH]
- What happens at sub-leading order ?

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Idea: use beta functions of general multi-scalar model to study the $1/N$ corrections. Even more general setting: $O(N_1) \times O(N_2) \times O(N_3)$ tri-fundamental model