Quantization of spectral curves via topological recursion

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General position of the talk

General problem

How to quantize a "**classical spectral curve**" ($[y, x] = 0$)

$$P(x, y) = 0 \ , \ P \text{ rational in x, monic polynomial in y}$$

into a linear differential equation ($[\hbar\partial_x, x] = \hbar$):

$$\hat{P}\left(x, \hbar\frac{d}{dx}\right)\psi(x,\hbar) = 0\,?$$

$\hat{P}$ rational in x with same pole structure as P.

Key ingredients

Key ingredient 1: Topological recursion (Eynard and Orantin [2007]).
Key ingredient 2: Isomonodromic deformations, integrable systems, Lax pairs:

$$\hbar\frac{\partial}{\partial x}\Psi(x,\hbar,t) = L(x,\hbar,t)\Psi(x,\hbar,t) \ , \ \hbar\frac{\partial}{\partial t}\Psi(x,\hbar,t) = R(x,\hbar,t)\Psi(x,\hbar,t)$$

Motivation using Matrix Models

Eigenvalues correlation functions

- Let $Z_N = \int_{\mathcal{H}_N} dM_N e^{-N \operatorname{Tr} V(M_N)}$ with $V(z)$ monic polynomial potential of even degree.
- **Eigenvalues correlation functions** (Stieltjes transforms):

$$
\begin{aligned}
W_1(x) &= \left\langle \sum_{i=1}^N \frac{1}{x-\lambda_i} \right\rangle_N \\
W_2(x_1,x_2) &= \left\langle \sum_{i,j=1}^N \frac{1}{(x_1-\lambda_i)(x_2-\lambda_j)} \right\rangle_N - W_1(x_1)W_1(x_2) \\
W_p(x_1,\dots,x_p) &= \left\langle \sum_{i_1,\dots,i_p} \frac{1}{x_1-\lambda_{i_1}} \cdots \frac{1}{x_p-\lambda_{i_p}} \right\rangle_{N,\text{cumulant}}
\end{aligned}
$$

- Generating series of joint **moments** $\left\langle \sum_{i=1}^N \lambda_i^k \right\rangle_N$, $\left\langle \sum_{i,j=1}^N \lambda_i^r \lambda_j^s \right\rangle_N$ (Mehta [2004]).
- Hermitian case: Correlation functions satisfy algebraic relations known as loop equations, Schwinger-Dyson equations, Virasoro constraints, etc.

Loop equations

- Let:

$$P_p(x_1;x_2,\dots,x_p) = \left\langle \sum_{i_1,\dots i_p} \frac{V'(x_1)-V'(\lambda_{i_1})}{x_1-\lambda_{i_1}} \frac{1}{x_2-\lambda_{i_2}} \cdots \frac{1}{x_p-\lambda_{i_p}} \right\rangle_{N,\textbf{cumulant}}$$

- Loop equations (notation $L_p = \{x_2,\dots,x_p\}$):

$$-P_1(x) = W_1^2(x) - V'(x)W_1(x) + \frac{1}{N^2}W_2(x,x)$$

$$P_p(x_1;L_p) = (2W_1(x_1)-V'(x_1))W_p(L_p) + \frac{1}{N^2}W_{p+1}(x_1,x_1,L_p)$$

$$+\sum_{I\subset L_p} W_{|I|+1}(x_1,L_I)W_{p-|I|}(x_1,L_{J\setminus I})$$

$$-\sum_{j=2}^{p}\frac{\partial}{\partial x_j}\frac{W_{p-1}(L_p)-W_{p-1}(x_1,L_p\setminus\{x_j\})}{x_1-x_j}$$

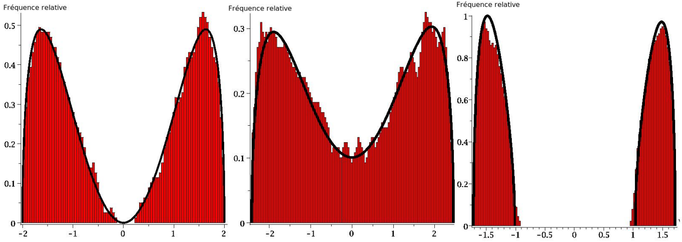
- Property: $x \mapsto P_p(x;L_p)$ is a polynomial. Is it enough to solve the equations and find $(W_p)_{p\geq 1}$?

Limiting eigenvalues density

- Under mild assumptions on the potential V:

$$d\nu_N = \frac{1}{N}\sum_{i=1}^{N}\delta(x-\lambda_i) \underset{N\to\infty}{\overset{\text{law}}{\to}} d\nu_\infty = \rho_\infty(x)dx$$

- ρ_∞ compactly supported on union of intervals.
- Stieljes transform$[\rho_\infty(x)dx] \equiv y(x)dx$ is algebraic: $y^2 = P(x) \Rightarrow$ Provides a classical spectral curve for TR.



- Number of intervals in the support $\Leftrightarrow$ **genus** of the spectral curve
- May be regular or singular

Formal solutions

- $Z_N = \int_{\mathcal{H}_N} dM_N e^{-N \,\mathrm{Tr} V(M_N)}$. **Assume** formal series expansions in $\frac{1}{N}$:

$$F_N \overset{\text{def}}{=} \ln Z_N = \sum_{g=0}^{\infty} F^{(g)} \left(\frac{1}{N}\right)^{2g-2}$$

$$W_p(x_1,\dots,x_p) = \sum_{g=0}^{\infty} \omega_p^{(g)}(x_1,\dots,x_p) \left(\frac{1}{N}\right)^{p+2g-2}$$

- May also work for additional parameters:

$$Z_N[t_4] = \int_{\mathcal{H}_N} dM_N e^{-\frac{N}{2}\mathrm{Tr}(M_N^2) - \frac{t_4}{4} N \,\mathrm{Tr}(M_N^4)}$$

We may consider formal series of the form:

$$\ln Z_N[t_4] = \sum_{g=0}^{\infty}\sum_{v=0}^{\infty} F^{(g,v)}(t_4)^v \left(\frac{1}{N}\right)^{2g-2} + \text{similar dev. for } W_p$$

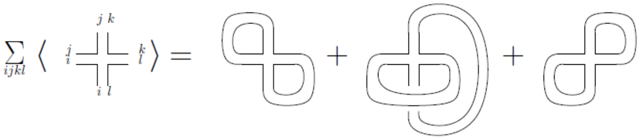
- Allow to solve recursively the loop equations.

Applications in combinatorics

- Interesting in combinatorics:

$$Z_N[t_4] = \int_{\mathcal{H}_N} dM_N e^{-\frac{N}{2}\mathrm{Tr}(M_N^2) - \frac{t_4}{4} N \,\mathrm{Tr}(M_N^4)}$$

Perturbative series expansion in $t_4 \Rightarrow$ enumeration of **fat ribbon graph (similar to Feynman expansion)**:



$F^{(g,v)}$ count the number of such connected graphs with v vertices (4 legs) and of genus g:

$$\ln Z_N[t_4] = \sum_{\mathcal{G} = 4-\text{ribbon graph}} \frac{1}{|\mathrm{Aut}\ \mathcal{G}|} t_4^{\#v(\mathcal{G})} \left(\frac{1}{N}\right)^{-\chi(\mathcal{G})}$$

Applications in geometry

- **Kontsevich integral**: Intersection theory of Riemann surfaces moduli spaces (Kontsevich [1992]):

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle = \int_{\bar{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} \,,\; F[t_0, t_1, \dots] = \sum_{(\mathbf{k})} \left\langle \tau_0^{k_0} \tau_1^{k_1} \dots \right\rangle \prod_{i=0}^{\infty} \frac{t_i^{k_i}}{k_i!}$$

may be computed through the **formal expansion** of the Kontsevich integral of $F = \ln Z$ with:

$$Z[t_0, t_1, \dots] \propto \int dM \exp\left(-\frac{1}{2} \mathrm{Tr}(M\Lambda M) + \frac{i}{3!} \mathrm{Tr}(M^3) \right)$$

and $t_i = -(2i-1)!! \, \mathrm{Tr}(\Lambda^{-(2i-1)})$, Λ positive definite Herm. matrix.

- Remark: $F[t_0, t_1, \dots]$ in connection with KdV equation:

$$u \overset{\text{def}}{=} \frac{\partial^2 F}{\partial t_0^2} \text{ satisfies } \frac{\partial u}{\partial t_1} = u \frac{\partial u}{\partial t_0} + \frac{1}{12} \frac{\partial^3 u}{\partial t_0^3}$$

Generalization: Kontsevich-Penner model - Open intersection numbers (Alexandrov [2015], Safnuk [2016]):

$$Z[Q, t_i] = (\det \Lambda)^Q \int dM \exp\left(-\frac{1}{2} \mathrm{Tr}(M\Lambda M) + \frac{1}{3} \mathrm{Tr}(M^3) - Q \ln M \right)$$

Orthogonal polynomials and RHP formulation

- Define P_n the **monic orthogonal polynomials**:

$$\int_{\mathbb{R}} P_m(x)P_n(x)e^{-\frac{V(x)}{2}} = h_n\delta_{n,m} \ , \ V(x) = \sum_{j=0}^{r} u_j x^j$$

and $\psi_n(x) = \frac{1}{\sqrt{h_n}}P_n(x)e^{\frac{V(x)}{2}}$ and $\tilde{\psi}_n = \text{Cauchy}(\psi_n)$

- Matrix $\Psi_n(x) = \begin{pmatrix} \psi_n & \tilde{\psi}_n \\ \psi_{n-1} & \tilde{\psi}_{n-1} \end{pmatrix}$ satisfies

$$\partial_x \Psi_n(x,\mathbf{u}) = \mathcal{D}_n(x,\mathbf{u})\Psi_n(x,\mathbf{u}) \ , \ \partial_{u_j}\Psi_n(x,\mathbf{u}) = \mathcal{U}_{n,j}(x,\mathbf{u})\Psi_n(x,\mathbf{u})$$

with $\mathcal{D}_n$ **and** $\mathcal{U}_{n,j}$ **polynomials in** x.

- Ψ_n has a **Riemann-Hilbert-Problem** characterization: analytic properties and jump discontinuity, asymptotics at ∞ in complement of the previous differential systems.

Key ingredients

- Christoffel-Darboux kernel: $K(z_1, z_2) = \frac{\psi_{n-1}(z_1)\tilde{\psi}_n(z_2) - \psi_n(z_1)\tilde{\psi}_{n-1}(z_2)}{z_1 - z_2}$.
- Hermitian matrix integrals may be rewritten as Fredholm determinants of integral operators of the kernel (Tracy and Widom [1994]).
- Specific cases (double-scaling limits) include: Airy kernel, Sine kernel, Pearcey kernel, etc.
- Large N asymptotics $\Leftrightarrow$ Large N asymptotics of Fredholm determinants $\Leftrightarrow$ Large N asymptotics of RHP (steepest descent method).
- Well-known generalization for two-matrix models: $P(x, y) = 0$ with arbitrary degree in y, bi-orthogonal polynomials, $d \times d$ RHP problems.
- Generalization when potentials are rational functions: $V \in \mathbb{C}(X)$.
- Generalization for hard edges (constrained eigenvalues support).

Facing both methods

- **Common starting point**: limiting eigenvalues density $\rho_\infty \Leftrightarrow$ Classical spectral curve $P(x, y) = 0$
- **Analytic (RHP) solutions vs Formal (Top. Rec.) solutions**
- Can we built linear differential equations using only the topological recursion approach: $\frac{1}{N}\partial_x \Psi_N = \mathcal{D}_N \Psi_N$?
- Would give a quantum curve ($\hbar \leftrightarrow \frac{1}{N}$): $\hat{P}(\hbar\partial_x, x)\Psi_{1,1} = 0$.
- Some known examples: Airy curve $y^2 = x$, semi-circle: $y^2 = x^2 - 1$ (Dumitrescu and Mulase [2016]).
- Relation with Painlevé equations and exact WKB expansions (Iwaki and Saenz [2016],Takei)
- Description of the integrable structure (Lax formulation) and the RHP problem?

Topological Recursion

Initial data

- Initial data: **"classical spectral curve"**:
  1. Σ Riemann surface of genus g.
  2. Symplectic basis of non-trivial cycles $(\mathcal{A}_i, \mathcal{B}_i)_{i \leq g}$ on Σ.
  3. Two meromorphic functions $x(z)$ et $y(z)$, $z \in \Sigma$ such that: $\Rightarrow$ $P(x, y) = 0$, with P monic polynomial in y, rational in x
  4. A symmetric bi-differential form $\omega_{0,2}$ on $\Sigma \times \Sigma$ such that $\omega_{0,2}(z_1, z_2) \underset{z_2 \to z_1}{\sim} \frac{dz_1\, dz_2}{(z_1 - z_2)^2} + \text{reg}$ with vanishing $\mathcal{A}$-cycles integrals.
- Regularity conditions:
  1. Ramification points $(dx(a_i) = 0)$ are simple zeros of dx. $\Rightarrow$ existence of a local involution σ such that $x(z) = x(\sigma(z))$ around any ramification points.
  2. Ramification points are not finite poles of P.
- Topological Recursion gives by recursion n-**forms** $(\omega_{h,n})_{n \geq 1, h \geq 0}$ **(known as "Eynard-Orantin differentials") and numbers** $(\omega_{h,0})_{h \geq 0}$ **(known as "free energies" or "symplectic invariants")**.

Topological recursion 2

- Recursion formula: $((a_i)_{1\leq i\leq r}$ ramification points)

$$\omega_{h,n+1}(z,\mathbf{z_n}) = \sum_{i=1}^{r} \operatorname*{Res}_{q\to a_i} \frac{dE_q(z)}{(y(q)-y(\bar{q}))dx(q)} \Big[\omega_{h-1,n+2}(q,q,\mathbf{z_n}) + \sum_{\substack{m\in[\![0,h]\!]\,,\, I\subset \mathbf{z_n} \\ (m,|I|)\neq(0,1)}} \omega_{m,|I|+1}(q,I)\,\omega_{g-m,|\mathbf{z_n}\setminus I|+1}(q,\mathbf{z_n}\setminus I)\Big]$$

where $dE_q(z) = \frac{1}{2}\int_q^{\bar{q}} \omega_{0,2}(q,z)$.

- "Free energies" $(\omega_{h,0})_{h\geq 2}$ given by:

$$\omega_{h,0} = \frac{1}{2-2h}\sum_{i=1}^{r} \operatorname*{Res}_{q\to a_i} \Phi(q)\,\omega_{h,1}(q) \;\text{ where } \Phi(q) = \int^q ydx$$

- Specific formulas for $\omega_{0,0}$ and $\omega_{1,0}$

Remarks and properties of TR

- Initially designed to provide **formal solutions in Hermitian RMT** but sufficient conditions (Borot and Guionnet [2011], Borot et al. [2014]) are known to provide exact asymptotics solutions.
- Only valid for regular spectral curves
- **Many existing generalizations**: blobbed (Borot and Shadrin [2015]), irregular curves (Do and Norbury [2018]), Lie algebras (Belliard et al. [2018]), Airy structures (Kontsevich and Soibelman [2017]), etc.
- Many applications in enumerative geometry (Eynard [2016]), RMT (Eynard et al. [2018]), Toeplitz determinants (Marchal [2019]), etc.
- Initial Eynard-Orantin formulation is sufficient for our purpose.

Quantization of hyper-elliptic spectral curves

Literature on quantization of spectral curves via TR

- Conditions on linear differential systems to be reconstructed from TR: Bergère and Eynard [2009], Bergère et al. [2015]
- Examples for genus 0 cases: Painlevé equations: Iwaki and Marchal [2014], Iwaki et al. [2018]
- General genus 0 case: Marchal and Orantin [2020]
- Examples of quantum curves and exact WKB: Iwaki and Saenz [2016], Bouchard and Eynard [2017]
- **General hyper-elliptic case, arbitrary genus**: Marchal and Orantin [2019]
- In progress with B. Eynard, E.Garcia-Failde and N. Orantin: Arbitrary degree, arbitrary genus.

Quadratic differentials with prescribed pole structure

Definition

Let $n \geq 0$ and let $(X_\nu)_{\nu=1}^n$ be a set of distinct points on $\Sigma_0 = \mathbb{P}^1$ with $X_\nu \neq \infty$, for $\nu = 1, \ldots, n$. We define the divisor

$$D = \sum_{\nu=1}^{n} r_\nu (X_\nu) + r_\infty (\infty)$$

Let $\mathcal{Q}(\mathbb{P}^1, D)$ **be the space of quadratic differentials on** $\mathbb{P}^1$ **such that any** $\phi \in \mathcal{Q}(\mathbb{P}^1, D)$ **has a pole of order** $2r_\nu$ **at the finite pole** $X_\nu \in \mathcal{P}^{\text{finite}}$ **and a pole of order** $2r_\infty$ **or** $2r_\infty - 1$ **at infinity**.

Remark

Up to reparametrization, ∞ **is always part of the divisor**. *Infinity may be a pole of odd degree (i.e. a ramification point in what to follow) but all other finite poles are even degree.*

Quadratic differentials with prescribed pole structure 2

$\mathcal{Q}(\mathbb{P}^1, D)$

Let x be a coordinate on $\mathbb{C} \subset \mathbb{P}^1$. Any quadratic differential $\phi \in \mathcal{Q}(\mathbb{P}^1, D)$ defines a compact Riemann surface Σ_ϕ by

$$\Sigma_\phi := \left\{(x, y) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}} \;/\; y^2 = \frac{\phi(x)}{(dx)^2}\right\}$$

$\frac{\phi(x)}{(dx)^2}$ is a meromorphic function on $\mathbb{P}^1$, i.e. a rational function of x.

Classical spectral curve associated to ϕ

For any $\phi \in \mathcal{Q}(\mathbb{P}^1, D)$, we shall call "**classical spectral curve**" associated to ϕ the Riemann surface Σ_ϕ defined as a two-sheeted cover $x : \Sigma_\phi \to \mathbb{P}^1$. Generically, it has genus $g(\Sigma_\phi) = r - 3$ where

$$r = \sum_{\nu=1}^{n} r_\nu + r_\infty$$

Quadratic differentials with prescribed pole structure 3

Branchpoints

Σ_ϕ is branched over the odd zeros of ϕ and ∞ if ∞ is a pole of odd degree. We define:

$$\begin{array}{rcl} \left\{b_\nu^+, b_\nu^-\right\} & := & x^{-1}(X_\nu) \text{ for } \nu = 1, \dots, n \\ \left\{b_\infty^+, b_\infty^-\right\} & := & x^{-1}(\infty) \text{ if } \infty \text{ pole of even degree} \\ \text{or } \left\{b_\infty\right\} & := & x^{-1}(\infty) \text{ if } \infty \text{ pole of odd degree} \end{array}$$

Filling fractions

Let $\eta = \phi^{\frac{1}{2}}$. We define the vector of filling fractions ϵ:

$$\forall j \in [\![1, g]\!] \,:\, \epsilon_j = \oint_{\mathcal{A}_j} \eta.$$

and its dual ϵ^* by:

$$\forall j \in [\![1, g]\!] \,:\, \epsilon_j^* = \frac{1}{2\pi i} \oint_{\mathcal{B}_j} \eta.$$

Spectral Times

Definition (Spectral Times)

Given a divisor D, a *singular type* $\mathbf{T}$ is the data of

- a *formal residue* T_p at each finite pole and at $p = b_\nu^\pm$ satisfying $T_{b_\nu^+} = -T_{b_\nu^-}$;
- an *irregular type* given by a vector $(T_{p,k})_{k=1}^{r_p-1}$ at each pole $p \in \mathcal{P}$ satisfying $T_{b_\nu^+,k} = -T_{b_\nu^-,k}$.

For such a singular type $\mathbf{T}$, let $\mathcal{Q}(\mathbb{P}^1, D, \mathbf{T}) \subset \mathcal{Q}(\mathbb{P}^1, D)$ be the space of quadratic differentials $\phi \in \mathcal{Q}(\mathbb{P}^1, D)$ such that $\eta = \phi^{\frac{1}{2}}$ satisfies

$$\forall\, b_\nu^\pm\,,\ \eta = \sum_{k=1}^{r_{b_\nu}} T_{b_\nu^\pm,k} \frac{dx}{(x - X_\nu)^k} + O\left(dx\right)$$

$$\eta = \sum_{k=1}^{r_\infty} T_{b_\infty^\pm,k} (x^{-1})^{-k} d(x^{-1}) + O(d(x^{-1})) = -\sum_{k=1}^{r_\infty} T_{b_\infty^\pm,k} x^{k-2} dx + O(x^{-2}dx)$$

if ∞ pole of even degree or

$$\eta = \sum_{k=1}^{r_\infty} T_{b_\infty,k} x^{k-1} d(x^{-\frac{1}{2}}) = -\sum_{k=1}^{r_\infty} \frac{T_{b_\infty,k}}{2} x^{k-\frac{5}{2}} dx$$

if ∞ pole of odd degree.

Decomposition on $\mathcal{Q}(\mathbb{P}^1, D, \mathbf{T})$: Notation

- We denote $[f(x)]_{\infty,+}$ (resp. $[f(x)]_{X_\nu,-}$) the positive part of the expansion in x of a function $f(x)$ around ∞, including the constant term, (resp. the strictly negative part of the expansion in $x - X_\nu$ around X_ν) .
- We define $K_\infty = [\![2, r_\infty - 2]\!]$ and for all $k \in K_\infty$:

$$U_{\infty,k}(x) := (k-1) \sum_{l=k+2}^{r_\infty} T_{\infty,l}\, x^{l-k-2}$$

if ∞ pole of even degree and

$$U_{\infty,k}(x) := \left(k - \frac{3}{2}\right) \sum_{l=k+2}^{r_\infty} T_{\infty,l}\, x^{l-k-2}$$

if ∞ pole of odd degree.
- $K_\nu = [\![2, r_\nu + 1]\!]$ and for all $k \in K_\nu$:

$$U_{\nu,k}(x) := (k-1) \sum_{l=k-1}^{r_\nu} T_{\nu,l}\, (x - X_\nu)^{-l+k-2}$$

Decomposition on $\mathcal{Q}(\mathbb{P}^1, D, \mathbf{T})$

Lemma (Variational formulas)

A quadratic differential $\phi \in \mathcal{Q}(\mathbb{P}^1, D, \mathbf{T})$ *reads* $\phi = f_\phi(x)(dx)^2$ *with*

$$f_\phi = \left[\left(\sum_{k=1}^{r_\infty} T_{\infty,k} x^{k-2}\right)^2\right]_{\infty,+} + \sum_{\nu=1}^{n}\left[\left(\sum_{k=1}^{r_\nu} T_{\nu,k}\frac{dx}{(x-X_\nu)^k}\right)^2\right]_{X_\nu,-} + \sum_{k\in K_\infty} U_{\infty,k}(x)\frac{\partial\omega_{0,0}}{\partial T_{\infty,k}} + \sum_{\nu=1}^{n}\sum_{k\in K_\nu} U_{\nu,k}(x)\frac{\partial\omega_{0,0}}{\partial T_{\nu,k}}$$

if ∞ *pole of even degree and*

$$f_\phi = \left[\left(\sum_{k=2}^{r_\infty} \frac{T_{\infty,k}}{2} x^{k-\frac{5}{2}}\right)^2\right]_{\infty,+} + \sum_{\nu=1}^{n}\left[\left(\sum_{k=1}^{r_\nu} T_{\nu,k}\frac{dx}{(x-X_\nu)^k}\right)^2\right]_{X_\nu,-} + \sum_{k\in K_\infty} U_{\infty,k}(x)\frac{\partial\omega_{0,0}}{\partial T_{\infty,k}} + \sum_{\nu=1}^{n}\sum_{k\in K_\nu} U_{\nu,k}(x)\frac{\partial\omega_{0,0}}{\partial T_{\nu,k}}$$

if ∞ *pole of odd degree*

Perturbative partition function

Definition (Perturbative partition function)

Given a classical spectral curve Σ, one defines the **perturbative partition function** as a function of a formal parameter $\hbar$ as

$$Z^{\text{pert}}(\hbar, \Sigma) := \exp\left(\sum_{h=0}^{\infty} \hbar^{2h-2}\omega_{h,0}(\Sigma)\right).$$

where $\omega_{h,0}(\Sigma)$ are the Eynard-Orantin free energies associated to Σ.

Perturbative wave functions 1

Definition ($(F_{h,n})_{h\geq 0, n\geq 1}$ by integration of the correlators)

For $n \geq 1$ and $h \geq 0$ such that $2h-2+n \geq 1$, let us define

$$F_{h,n}(z_1,\dots,z_n) = \frac{1}{2^n}\int_{\sigma(z_1)}^{z_1}\dots\int_{\sigma(z_n)}^{z_n}\omega_{h,n}$$

where one integrates each of the n variables along paths linking two Gallois conjugate points inside a fundamental domain cut out by the chosen symplectic basis $(\mathcal{A}_j,\mathcal{B}_j)_{1\leq j\leq g}$.
For $(h,n)=(0,1)$ we define:

$$F_{0,1}(z) := \frac{1}{2}\int_{\sigma(z)}^{z}\eta$$

For $(h,n)=(0,2)$ regularization is required:

$$F_{0,2}(z_1,z_2) := \frac{1}{4}\int_{\sigma(z_1)}^{z_1}\int_{\sigma(z_2)}^{z_2}\omega_{0,2} - \frac{1}{2}\ln\left(x(z_1)-x(z_2)\right)$$

Perturbative wave functions 2

Definition (Definition of the perturbative wave functions)

We define first:

$$\begin{aligned} S_{-1}^{\pm}(\lambda) &:= \pm F_{0,1}(z(\lambda)) \\ S_0^{\pm}(\lambda) &:= \frac{1}{2} F_{0,2}(z(\lambda), z(\lambda)) \\ \forall\, k \geq 1\,,\ S_k^{\pm}(\lambda) &:= \sum_{\substack{h \geq 0, n \geq 1 \\ 2h-2+n=k}} \frac{(\pm 1)^n}{n!} F_{h,n}(z(\lambda), \dots, z(\lambda)) \end{aligned}$$

where for $\lambda \in \mathbb{P}^1$, we define $z(\lambda) \in \Sigma_\phi$ as the unique point such that $x(z(\lambda)) = \lambda$ and $y(z(\lambda))dx(z(\lambda)) = \sqrt{\phi(\lambda)}$. The perturbative wave functions $\psi_\pm$ by:

$$\psi_\pm(\lambda, \hbar, \Sigma) := \exp\left(\sum_{k \geq -1} \hbar^k S_k^{\pm}(\lambda) \right)$$

Remarks

- Standard definitions used by K. Iwaki for Painlevé 1.
- Formulas do not require restriction to $\mathcal{Q}(\mathbb{P}^1, D, \mathbf{T})$ but are well-defined for any classical spectral curve.
- $S^{\pm} = \ln(\psi_{\pm})$ are somehow more natural than $\psi_{\pm}$.
- **$\psi_{\pm}$ do not have nice monodromy properties**
  1. For $i \in [\![1, g]\!]$, the function $\psi_{\pm}(\lambda, \hbar, \boldsymbol{\epsilon})$ has a formal monodromy along $\mathcal{A}_i$ given by

$$\psi_{\pm}(\lambda, \hbar, \boldsymbol{\epsilon}) \mapsto e^{\pm 2\pi i \frac{\epsilon_i}{\hbar}} \psi_{\pm}(\lambda, \hbar, \boldsymbol{\epsilon}).$$

  2. For $i \in [\![1, g]\!]$, the function $\psi_{\pm}(\lambda, \hbar, \boldsymbol{\epsilon})$ has a formal monodromy along $\mathcal{B}_i$ given by

$$\psi_{\pm}(\lambda, \hbar, \boldsymbol{\epsilon}) \mapsto \frac{Z^{\text{pert}}(\hbar, \boldsymbol{\epsilon} \pm \hbar\, \mathbf{e}_i)}{Z^{\text{pert}}(\hbar, \boldsymbol{\epsilon})} \psi_{\pm}(\lambda, \hbar, \boldsymbol{\epsilon} \pm \hbar\, \mathbf{e}_i)$$

- Necessity of non-perturbative corrections (already known in the exact WKB literature).

Non-perturbative quantities

Definition

We define the non-perturbative partition function via **discrete Fourier transform**:

$$Z(\hbar,\Sigma,\boldsymbol{\rho}) := \sum_{\mathbf{k}\in\mathbb{Z}^g} e^{\frac{2\pi i}{\hbar}\sum_{j=1}^{g} k_j\rho_j}\, Z^{pert}(\hbar,\boldsymbol{\epsilon}+\hbar\mathbf{k})$$

and the non-perturbative wave function:

$$\Psi_\pm(\lambda,\hbar,\Sigma,\boldsymbol{\rho}) := \frac{\sum_{\mathbf{k}\in\mathbb{Z}^g} e^{\frac{2\pi i}{\hbar}\sum_{j=1}^{g} k_j\rho_j}\, Z^{pert}(\hbar,\boldsymbol{\epsilon}+\hbar\mathbf{k})\,\psi_\pm(\lambda,\hbar,\boldsymbol{\epsilon}+\hbar\mathbf{k})}{Z(\hbar,\Sigma,\boldsymbol{\rho})}$$

Remarks

- Definitions similar to those of K. Iwaki for Painlevé 1 (genus 1)
- Discrete Fourier transforms of perturbative quantities
- **Provide good monodromy properties** (see next slide)
- **Dependence in $\hbar$ is no longer WKB**: **trans-series in $\hbar$**:

$$Z(\hbar,\Sigma,\boldsymbol{\rho}) = Z^{pert}(\hbar,\Sigma)\sum_{m=0}^{\infty}\hbar^m\Theta_m(\hbar,\Sigma,\boldsymbol{\rho})$$

$$\Psi_{\pm}(\lambda,\hbar,\Sigma,\boldsymbol{\rho}) = \psi_{\pm}(\lambda,\hbar,\Sigma)\frac{\sum_{m=0}^{\infty}\hbar^m\,\Xi_m(\lambda,\hbar,\Sigma,\boldsymbol{\rho})}{\sum_{m=0}^{\infty}\hbar^m\,\Theta_m(\hbar,\Sigma,\boldsymbol{\rho})}$$

Coefficients $\Theta_m(\hbar,\Sigma,\boldsymbol{\rho})$, $\Xi_m(\lambda,\hbar,\Sigma,\boldsymbol{\rho})$ finite linear combinations of derivatives of Theta functions.

Monodromy properties

- For $j \in [\![1, g]\!]$, $\Psi_\pm(\lambda, \Sigma, \boldsymbol{\rho})$ has a formal monodromy along $\mathcal{A}_j$ given by
$$\Psi_\pm(\lambda, \mathbf{T}, \boldsymbol{\epsilon}, \boldsymbol{\rho}) \mapsto e^{\pm 2\pi i \frac{\epsilon_j}{\hbar}} \Psi_\pm(\lambda, \Sigma, \boldsymbol{\rho}).$$

- For $j \in [\![1, g]\!]$, $\Psi_\pm(\lambda, \Sigma, \boldsymbol{\rho})$ has a formal monodromy along $\mathcal{B}_j$ given by
$$\Psi_\pm(\lambda, \mathbf{T}, \boldsymbol{\epsilon}, \boldsymbol{\rho}) \mapsto e^{\mp 2\pi i \frac{\rho_j}{\hbar}} \Psi_\pm(\lambda, \Sigma, \boldsymbol{\rho}).$$

Wronskian

Wronskian

Let $\phi \in \mathcal{Q}(\mathbb{P}^1, D, \mathbf{T})$ defining a classical spectral curve Σ_ϕ. Then, the Wronskian $W(\lambda;\hbar) = \hbar(\Psi_- \partial_\lambda \Psi_+ - \Psi_+ \partial_\lambda \Psi_-)$ is a rational function of the form:

$$W(\lambda;\hbar) = w(\mathbf{T},\hbar) \frac{P_g(\lambda)}{\prod_{\nu=1}^{n} (\lambda - X_\nu)^{r_{b_\nu}}} = w(\mathbf{T},\hbar) \frac{\prod_{i=1}^{g} (\lambda - q_i)}{\prod_{\nu=1}^{n} (\lambda - X_\nu)^{r_{b_\nu}}}$$

with P_g a monic polynomial of degree g.

Remark

We denote $(q_i)_{i \leq g}$ the simple zeros of the Wronskian and $(p_i)_{i \leq g}$ by:

$$\forall\, i \in [\![1, g]\!] \;:\; p_i = \left.\frac{\partial \log \Psi_+}{\partial \lambda}\right|_{\lambda = q_i} = \left.\frac{\partial \log \Psi_-}{\partial \lambda}\right|_{\lambda = q_i}.$$

Quantum curve

Quantum Curve

The non-perturbative wave functions $\Psi_\pm$ satisfy a linear second order PDE with **rational coefficients**:

$$\left[\hbar^2 \frac{\partial^2}{\partial \lambda^2} - \hbar^2 R(\lambda)\frac{\partial}{\partial \lambda} - \hbar Q(\lambda) - \mathcal{H}(\lambda)\right]\Psi_\pm(\lambda;\hbar) = 0$$

with $R(\lambda) = \frac{\partial \log W(\lambda)}{\partial \lambda}$ and

$$\begin{aligned}
\mathcal{H}(\lambda) &= \left[\hbar^2 \sum_{k\in K_\infty} U_{\infty,k}(\lambda)\frac{\partial}{\partial T_{b_\infty,k}} + \hbar^2 \sum_{\nu=1}^{n}\sum_{k\in K_{b_\nu}} U_{b_\nu,k}(\lambda)\frac{\partial}{\partial T_{b_\nu,k}}\right] \\
&\quad \left[\log Z(\mathbf{T},\epsilon,\boldsymbol{\rho}) - \hbar^{-2}\omega_{0,0}\right] + y^2(\lambda) \\
Q(\lambda) &= \sum_{j=1}^{g}\frac{p_j}{\lambda - q_j} + \frac{\hbar}{2}\left[\sum_{k\in K_\infty} U_{\infty,k}(\lambda)\frac{\partial(S_+(\lambda) - S_-(\lambda))}{\partial T_{\infty,k}}\right]_{\infty,+} \\
&\quad + \frac{\hbar}{2}\sum_{\nu=1}^{n}\left[\sum_{k\in K_\nu} U_{\nu,k}(\lambda)\frac{\partial(S_+(\lambda) - S_-(\lambda))}{\partial T_{\nu,k}}\right]_{X_\nu,-}
\end{aligned}$$

Quantum curve 2

Additional relations

The pairs (q_i, p_i) satisfy $\forall\, i \in [\![1, g]\!]$:

$$p_i^2 = \mathcal{H}(q_i) - \hbar p_i \left[\sum_{j\neq i} \frac{1}{q_i - q_j} - \sum_{\nu=1}^{n} \frac{r_\nu}{q_i - X_\nu}\right] \left.\frac{\partial \log \Psi_+(\lambda)}{\partial \lambda}\right|_{\lambda=q_j} + \left[\frac{\partial \left(Q(\lambda) - \frac{p_i}{\lambda - q_i}\right)}{\partial \lambda}\right]_{\lambda=q_i}$$

Asymptotics $S_\pm(\lambda)$ are given by:

$$S_\pm(\lambda) = \mp \frac{1}{\hbar} \sum_{k=2}^{r_{b_\nu}} \frac{T_{b_\nu,k}}{k-1} \frac{1}{(\lambda - X_\nu)^{k-1}} \pm \frac{1}{\hbar} T_{b_\nu,1} \log(\lambda - X_\nu) + \sum_{k=0}^{\infty} A^{\pm}_{\nu,k} (\lambda - X_\nu)^k$$

$$S_\pm(\lambda) = \mp \frac{1}{\hbar} \sum_{k=2}^{r_\infty} \frac{T_{b_\infty,k}}{k-1} \lambda^{k-1} \mp \frac{1}{\hbar} T_{b_\infty,1} \log(\lambda) - \frac{\log \lambda}{2} + \sum_{k=0}^{\infty} A^{\pm}_{\infty,k} \lambda^{-k}$$

or

$$S_\pm(\lambda) = \mp \frac{1}{\hbar} \sum_{k=2}^{r_\infty} \frac{T_{b_\infty,k}}{2k-3} \lambda^{\frac{2k-3}{2}} \mp \frac{1}{\hbar} T_{b_\infty,1} \log(\lambda) - \frac{\log \lambda}{4} + \sum_{k=0}^{\infty} A^{\pm}_{\infty,k} \lambda^{-\frac{k}{2}}$$

Thus,

$$Q(\lambda) = \sum_{j=1}^{g} \frac{p_j}{\lambda - q_j} + \sum_{k=0}^{r_\infty - 4} Q_{\infty,k} \lambda^k + \sum_{\nu=1}^{n} \sum_{k=1}^{r_\nu + 1} \frac{Q_{\nu,k}}{(\lambda - X_\nu)^k}$$

Linearization and $\hbar$-deformed spectral curve

- Linearize the quantum curve, i.e. choose $\vec{\Psi}_\pm = \begin{pmatrix} \Psi_\pm \\ \alpha(\lambda)\Psi_\pm + \beta(\lambda)\partial_\lambda \Psi_\pm \end{pmatrix}$ to have a 2×2 system

$$\hbar\partial_\lambda \vec{\Psi}_\pm(\lambda) = L(\lambda)\vec{\Psi}_\pm(\lambda) = \begin{pmatrix} P(\lambda) & M(\lambda) \\ W(\lambda) & -P(\lambda) \end{pmatrix} \vec{\Psi}_\pm(\lambda)$$

- Define the $\hbar$-deformed spectral curve: $\det(y\, I_2 - L(\lambda)) = 0$

$$\begin{aligned} y^2(\lambda) \quad = \quad & \mathcal{H}(\lambda) + \hbar \sum_{j=1}^{g} \frac{p_j}{\lambda - q_j} + \frac{\hbar^2}{2}\left[\sum_{k\in K_\infty} U_{\infty,k}(\lambda)\frac{\partial(S_+(\lambda)+S_-(\lambda))}{\partial T_{\infty,k}}\right]_{\infty,+} \\ & + \frac{\hbar^2}{2}\sum_{\nu=1}^{n}\left[\sum_{k\in K_\nu} U_{\nu,k}(\lambda)\frac{\partial(S_+(\lambda)+S_-(\lambda))}{\partial T_{\nu,k}}\right]_{X_\nu,-} + \hbar\frac{\partial P(\lambda)}{\partial \lambda} \\ & - \hbar \frac{\partial \log W(\lambda)}{\partial \lambda} P(\lambda) \end{aligned}$$

Additional material 2

- Write the time differential systems

$$\partial_{T_{\nu,k}} \vec{\Psi}_\pm = R_{\nu,k}(\lambda)\vec{\Psi}_\pm$$

- Define isomonodromic times $t_{\nu,k}$ and the map $(T_{\nu,k})_{\nu,k} \to (t_{\nu,k})_{\nu,k}$ and the differential systems $\partial_{t_{\nu,k}} \vec{\Psi}_\pm = L_{\nu,k}(\lambda)\vec{\Psi}_\pm$
- Connected to the problem isospectral $\to$ isomonodromic: Existence of times t such that $\frac{\delta L(\lambda)}{\delta t} = \frac{\partial L_t}{\partial \lambda}$ where δ is the variation to explicit dependence on t only.
- Define g Hamiltonians $(H_j(q_1, \dots, q_g, p_1, \dots, p_g, \hbar))_{1\leq j\leq g}$ so that $\hbar$-deformed Hamilton's equations are satisfied:

$$\forall\, (i,j) \in [\![1,g]\!]^g \; : \; \hbar\, \partial_t q_i = \frac{\partial H_j}{\partial p_i} \text{ and } \hbar\, \partial_t p_j = -\frac{\partial H_j}{\partial q_i}$$

- Apply to all Painlevé equations and their hierarchies.

Example on Painlevé 2

- Corresponds to $n = 0$, $n_\infty = 0$ **and** $r_\infty = 4$: Family of classical spectral curves:

$$y^2 = T_{\infty,4}^2 x^4 + 2T_{\infty,3}T_{\infty,4}x^3 + \left(T_{\infty,3}^2 + 2T_{\infty,4}T_{\infty,2}\right)x^2 + \left[2T_{\infty,3}T_{\infty,2} + 2T_{\infty,4}T_{\infty,1}\right]x + H_0$$

- Quantum curve reads:

$$0 = \Big[\hbar^2\frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{x-q}\frac{\partial}{\partial x} - \frac{\hbar p}{x-q} - T_{\infty,4}^2 x^4 - 2T_{\infty,3}T_{\infty,4}x^3 - \left(T_{\infty,3}^2 + 2T_{\infty,4}T_{\infty,2}\right)x^2 - \left[2T_{\infty,3}T_{\infty,2} + 2T_{\infty,4}T_{\infty,1}\right]x - H_0 - \hbar T_{\infty,4}q\Big]\Psi_\pm(x)$$

Example on Painlevé 2

- Hamiltonian H_0 is given by:

$$H_0 = p^2 - T_{\infty,4}^2 q^4 - 2T_{\infty,3}T_{\infty,4}q^3 - \left(T_{\infty,3}^2 + 2T_{\infty,4}T_{\infty,2}\right)q^2 - \left[2T_{\infty,3}T_{\infty,2} + 2T_{\infty,4}\left(T_{\infty,1} + \frac{\hbar}{2}\right)\right]q$$

- Define $t_{\infty,1} = 2T_{\infty,2}$, Darboux coordinates (q,p) satisfy Hamiltonian equations:

$$2\hbar\frac{\partial p}{\partial t_{\infty,1}} = \frac{\partial H_0}{\partial q} \quad \text{and} \quad 2\hbar\frac{\partial q}{\partial t_{\infty,1}} = -\frac{\partial H_0}{\partial p}.$$

Recovers the standard Painlevé 2 by setting $T_{\infty,4} = 1$, $T_{\infty,3} = 0$, $T_{\infty,1} = -\theta$

$$\hbar^2\frac{\partial^2 q}{\partial t_{\infty,1}^2} = 2q + t_{\infty,1}q + \frac{\hbar}{2} - \theta.$$

Remarks and open questions

Summary

1. Given classical spectral curve $\Rightarrow$ Top. Rec. $(\omega_{h,n}(z_1,\dots,z_n))_{h\geq 0,n\geq 0}$
2. $(\omega_{n,h})_{n\geq 0,h\geq 0}(z_1,\dots,z_n) \Rightarrow$ Wave function

$$\psi^{\text{pert}} = e^{\int^z \dots \int^z \frac{(-1)^n}{2^n}\omega_{n,h}\hbar^{n+2h-2}}$$

3. Discrete Fourier transform of $\psi^{\text{pert}} \Rightarrow \psi^{\text{non-pert}}$
4. Define $\vec{\Psi} = (\psi^{\text{non-pert}}, \partial_\lambda\psi^{\text{non-pert}},\dots,(\partial_\lambda)^{d-1}\psi^{\text{non-pert}})$. Satisfy a companion-like linear differential system $\Rightarrow$ Quantum spectral curve.
5. Fix the gauge to remove apparent pole singularities
6. Connect deformations of the coefficients of the classical spectral curves (family of spectral curves with prescribed pole structure and fixed genus) with isomonodromic deformations.

Remarks for the hyper-elliptic case

- Genus 0 curves $\Rightarrow$ No Fourier transform (standard WKB expansions) $\Rightarrow$ Reconstruction via Topological Type property possible.
- Isomonodromic times differ from spectral times (naturally arising in the spectral curve and topological recursion). $\Rightarrow$ creates technical complications.
- Standard gauge choice for the 2×2 matrix system is usually not companion-like to avoid apparent singularities.
- These technical issues have been solved in the hyper-elliptic case.

Open questions and future works

- Connection with isomonodromic deformations is missing in the general setting (non-hyper-elliptic).
- Deformations of the classical spectral curves with fixed genus are problematic in the loop equations.
- Technical assumptions like non-degenerate ramification points, poles $\neq$ ramification points should be lifted but requires technical computations.
- Analytic properties of Ψ: Description of the associated RHP to be done.
- Connection with orthogonal polynomials in the case of RMT ?

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