Tensor models, Kronecker coefficients and bipartite ribbon graphs

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Virtual Tensor Journal Club.

Based on

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P. Mattioli and S. Ramgoolam, "Permutation centralizer algebras and multi-matrix invariants ," Phys. Rev. D 93, 065040 (2016), arXiv:1601.06086v1 [hep-th]
J. Ben Geloun and S. Ramgoolam, "Tensor Models, Kronecker coefficients and Permutation Centralizer Algebras ," arXiv:1708.03524v2 [hep-th], JHEP 11 (2017) 092
G. Kemp and S. Ramgoolam ,"BPS states and central charges " JHEP01(2020)146, arXiv:1911.11649v2[hep-th]
J. Ben Geloun and S. Ramgoolam, "Quantum mechanics of bipartite ribbon graphs: Integrality, Lattices and Kronecker coefficients" arXiv:2010.04054v1 [hep-th]

Matrix models and tensor models

Matrix models have been actively studied in string theory and quantum gravity since the nineties, e.g. 1-hermitian matrix model with cubic potential.

Double scaling limits have led to early precursors of gauge-string duality (AdS/CFT correspondence): pure 2D gravity or 2D gravity coupled to CFTs as duals of matrix models in different limits.

These have inspired important developments on tensor models and large N scaling limits - starting in 90’s and a lot of activity in the last 10 years or so.

Matrix models and 2d YM

The early examples of gauge-string duality also included the string theory dual of 2d Yang Mills theory (e.g. $U(N)$ at large N).

Gross and Taylor , 1992-1993

Exactly solvable theory - also related to unitary matrix models - has large N expansion - expressible in terms of counting of branched covers of surfaces.

Links between symmetric groups and matrix models played an important role in this toy model of gauge-string duality.

Intro: Matrix models

In the AdS/CFT correspondence : algebras related to symmetric groups appearing in the combinatorics of complex matrix models have played an important role in understanding BPS-sectors of AdS/CFT and their non-BPS excitations.

A simple 1-matrix example: Complex matrix variables Z^i_j, with i, j ranging over 1 to N. Gaussian model with partition function

$$\mathcal{Z} = \int [dZ] \; e^{-\mathrm{tr} Z Z^\dagger}$$

Invariant under $U(N)$, $Z \to UZU^\dagger$. Matrix observables are holomorphic polynomial $U(N)$ invariants $\mathcal{O}_a(Z)$. Correlators are of interest

$$\langle \mathcal{O}_a(Z) \mathcal{O}_b(Z^\dagger) \rangle = \frac{1}{\mathcal{Z}} \int [dZ] e^{-\mathrm{tr} Z Z^\dagger} \mathcal{O}_a(Z) \mathcal{O}_b(Z^\dagger)$$

Intro: Matrix observables and partitions

Polynomial invariants of degree n are traces and products of traces, e.g. at degree 3

$$\mathrm{tr}Z^3, \mathrm{tr}Z^2\mathrm{tr}Z, (\mathrm{tr}Z)^3$$

For $N \geq 3$, these are all linear independent. Often (but not always) we are interested in questions where $N >> n$.

At degree n, the number of such multi-traces is the number of partitions of n, denoted $p(n)$. For the $n = 3$ example, $p(3) = 3$ and

$$3 = 3;$$
$$3 = 2 + 1$$
$$3 = 1 + 1 + 1$$

Intro : Matrix observables and permutations

$p(n)$ is also the number of conjugacy classes in the symmetric group S_n. There is a concrete and very useful link between matrix (and tensor) invariants and permutations :

$$
\begin{aligned}
(\mathrm{tr} Z)^2 &= \sum_{i_1, i_2} Z^{i_1}_{i_1} Z^{i_2}_{i_2} \\
(\mathrm{tr} Z^2) &= \sum_{i_1, i_2} Z^{i_1}_{i_2} Z^{i_2}_{i_1}
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{tr} Z)^2 &= \sum_{i_1, i_2} Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \qquad \sigma = (1)(2) \\
(\mathrm{tr} Z^2) &= \sum_{i_1, i_2} Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \qquad \sigma = (1, 2)
\end{aligned}
$$

Intro: Matrix observables and permutations

In general at degree n,

$$\mathcal{O}_\sigma(Z) = Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_1}_{i_{\sigma(n)}}$$

2-line proof shows :

$$\mathcal{O}_\sigma(Z) = \mathcal{O}_{\gamma\sigma\gamma^{-1}} \qquad \text{for any } \gamma \in S_n$$

For $n \leq N$

$$\begin{aligned} &\text{Dimension of degree } n \text{ polynomials} \\ = \; &\text{Number of conjugacy classes of } S_n \\ = \; &\text{Dimension of } \mathcal{Z}(\mathbb{C}(S_n)) \; = p(n) \end{aligned}$$

For partition p, let $\mathcal{C}_p$ be the conjugacy class of permutations with cycle structure p

$$\mathcal{O}_p(Z) = \frac{1}{|\mathcal{C}_p|} \sum_{\sigma \in \mathcal{C}_p} \mathcal{O}_\sigma(Z)$$

Intro: Correlators and permutations

Correlators $\langle \mathcal{O}_{\sigma_1}(Z)\mathcal{O}_{\sigma_2}(Z^\dagger)\rangle$ can be expressed in terms of permutation products.

$$\langle \mathcal{O}_{p_1}(Z)\mathcal{O}_{p_2}(Z^\dagger)\rangle = \frac{n!}{|\mathcal{C}_{p_1}||\mathcal{C}_{p_2}|} \sum_{\sigma_1 \in \mathcal{C}_{p_1}} \sum_{\sigma_2 \in \mathcal{C}_{p_2}} \sum_{\sigma_3 \in S_n} \delta(\sigma_1\sigma_2\sigma_3) N^{C_{\sigma_3}}$$

C_{σ_3} is the number of cycles in σ_3. $\delta(\sigma)$ is 1 if $\sigma = id$ and zero otherwise.

Useful to re-write by defining

$$T_p = \sum_{\sigma \in \mathcal{C}_p} \sigma$$

$T_p \in \mathbb{C}(S_n)$. In fact T_p are in the centre $\mathcal{Z}(\mathbb{C}(S_n))$ of $\mathbb{C}(S_n)$ - the sub-algebra which commutes with all $\mathbb{C}(S_n)$.

Intro: Correlators and permutations : Branched covers

$$\frac{1}{n!}\sum_{\sigma_1\in T_1}\sum_{\sigma_2\in T_2}\sum_{\sigma_3\in T_3}\delta(\sigma_1\sigma_2\sigma_3)=\sum_{f:\Sigma_h\to P^1}\frac{1}{|\mathrm{Aut}(f)|}$$

de Mello Koch, Ramgoolam, “Matrix models ... Hurwitz space ...” https://arxiv.org/abs/1002.1634;

Tom Brown, “Complex matrix model duality,” https://arxiv.org/abs/1009.0674

A mathematical model of gauge-string duality : 1-matrix model and branched covers.

Similar to 2d Yang Mills - string theory duality from the 1990’s.
Gross, Taylor, “Two-dimensional QCD is a string theory,” 1993

Intro: Correlators and permutations : Belyi maps

$$(2h-2) = n - C_{\sigma_1} - C_{\sigma_2} - C_{\sigma_3}$$

These permutation triples describe Belyi maps -branched covers of the sphere over 3 branch points
These maps are very interesting in number theory

See for example : Graphs on surfaces and their applications, Lando and Zvonkin.

Intro: Correlators and permutation algebras

$$\langle \mathcal{O}_{p_1}(Z)\mathcal{O}_{p_2}(Z^\dagger)\rangle = \frac{n!}{|\mathcal{C}_{p_1}||\mathcal{C}_{p_2}|}\sum_{p_3 \vdash n} \delta(T_{p_1} T_{p_2} T_{p_3}) N^{C_{p_3}}$$

Correlator expressible in terms of products in $\mathcal{Z}(\mathbb{C}(S_n)) \subset \mathbb{C}(S_n)$.

$$\delta(\sum_\sigma a_\sigma \sigma) = a_{id}$$

Intro: At finite N and Young diagrams

For $n > N$, these trace-structure are not linearly independent, because of the Cayley-Hamilton relations.

$$\mathrm{tr} Z^{N+1}$$

can be expressed in terms of products of traces.

A basis is instead labelled by Young diagrams
$R = (r_1, r_2, \cdots, r_N)$ with row lengths $(r_1 \geq r_2 \geq r_3 \cdots \geq r_N)$

$$\mathcal{O}_R(Z) = \sum_{\sigma \in S_n} \chi^R(\sigma)\, \mathcal{O}_\sigma(Z)$$

Young diagrams with no more than N rows.

This basis is very useful for the physics of AdS/CFT.

Corley, Jevicki, Ramgoolam, "Exact correlators of giant gravitons in AdS/CFT" 2001

Intro: Matrix models and Fourier transforms on algebras

The change of basis from $\mathcal{O}_p$ to $\mathcal{O}_R$ is a change of basis in the centre of the group algebra of $\mathbb{C}(S_n)$.

The centre $\mathbb{Z}(\mathbb{C}(S_n))$ has a basis labelled by conjugacy classes (cycle structures)

$$T_p = \sum_{\sigma \in \mathcal{C}_p} \sigma$$

These form an algebra

$$T_{p_1} T_{p_2} = \sum_q C^q_{p_1 p_2} T_q$$

The correlators are expressed in terms of the structure constants of this algebra.

Intro: Matrix models and Fourier transforms on algebras

Another basis for $\mathcal{Z}(\mathbb{C}(S_n))$ is given by linear combinations labelled by Young diagrams

$$P_R = \sum_{\sigma \in S_n} \chi^R(\sigma)\sigma = \sum_{p \vdash n} \chi^R_p T_p$$

Recall

$$\mathcal{O}_R(Z) = \sum_{\sigma} \chi^R(\sigma)\mathcal{O}_\sigma(Z)$$

$$P_R P_S \propto \delta_{RS} P_R$$

There is an inner product in $\mathbb{C}(S_n)$ and hence in $\mathcal{Z}(\mathbb{C}(S_n))$

$$g(\sigma_1, \sigma_2) = \delta(\sigma_1 \sigma_2^{-1})$$

$$g(P_R, P_S) \propto \delta_{RS}$$

Intro: Tensor models and permutations

Algebras related to $\mathbb{C}(S_n)$, and Fourier transforms therein, are useful in enumerating tensor model observables and calculating their correlators.

Z_{ijk} is a 3-index tensor variable. $\bar{Z}^{ijk}$ is the conjugate

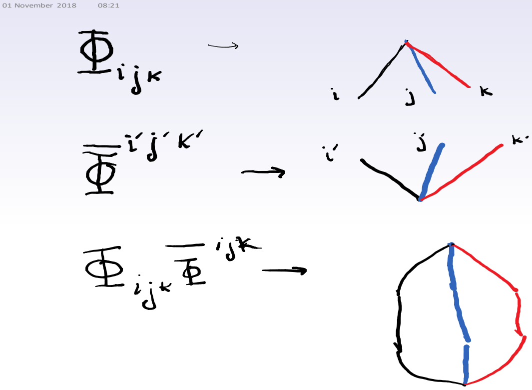
Transform as $V_N \otimes V_N \otimes V_N$ of $U(N) \times U(N) \times U(N)$. And $\bar{V}_N \otimes \bar{V}_N \otimes \bar{V}_N$.

$$
\begin{aligned}
\mathcal{Z} &= \int DZ e^{-\sum_{i,j,k} Z_{i,j,k} \bar{Z}^{i,j,k}} \\
\langle \mathcal{O}(Z,\bar{Z}) \rangle &= \frac{1}{\mathcal{Z}} \int DZ e^{-\sum_{i,j,k} Z_{i,j,k} \bar{Z}^{i,j,k}} \mathcal{O}(Z,\bar{Z})
\end{aligned}
$$

The observables are polynomial functions of $Z, \bar{Z}$ invariant under $U(N) \times U(N) \times U(N)$. Can be classified by degree n (number of $Z, \bar{Z}$).

Models of this type, with particular added quartic term, and double-scaled large N limit have been actively studied in the quantum gravity since 2009/2010- Gurau, Rivasseau, ...

Gauge-invariant observables correspond to colored graphs.



contractions-diagrams Page 1

General gauge invariant observables can be labelled by a triple of permutations

$$\mathcal{O}_{\sigma_1,\sigma_2,\sigma_3}(Z,\bar{Z}) = Z^{i_1 j_1 k_1}\cdots Z^{i_n j_n k_n}\bar{Z}_{i_{\sigma_1(1)} j_{\sigma_2(1)} k_{\sigma_3(1)}}\cdots\bar{Z}_{i_{\sigma_1(n)} j_{\sigma_2(n)} k_{\sigma_3(n)}}$$

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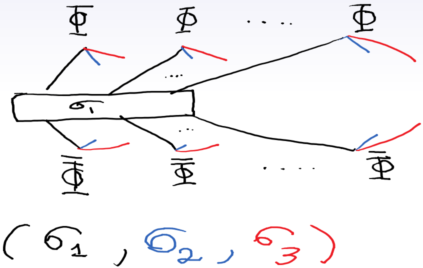

There are equivalences

$$\mathcal{O}_{\sigma_1,\sigma_2,\sigma_3} = \mathcal{O}_{\gamma_1\sigma_1\gamma_2,\gamma_1\sigma_2\gamma_2,\gamma_1\sigma_3\gamma_2}$$

for $\gamma_1, \gamma_2 \in S_n$: Bosonic $Z \to \gamma_1$, $\bar{Z} \to \gamma_2$

These equivalence classes form a double coset.

$$Diag(S_n) \setminus (S_n \times S_n \times S_n)/Diag(S_n)$$

The space of functions on the double coset forms an algebra $\mathcal{K}(n)$. Concretely it is a subspace of

$$\mathbb{C}(S_n) \times \mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$$

Intro: $\mathcal{K}(n)$, Dessins and Belyi maps

The formulation of counting in terms of 3-colored graphs and permutation triples, subject to an equivalence relation, can be reformulated in two ways:

(a) Permutation pairs with another equivalence relation : Related to a known combinatoric object - bipartite ribbon graphs; Dessins d'Enfants (Grothendieck).

(b) Another permutation triple formulation : Related to branched covers of the sphere with 3 branch points. Also called Belyi maps - which can be interpreted as string worldsheets mapping to target sphere. encountered in matrix model correlators earlier, here in counting observables.

Intro : Kronecker coefficients and tensor model observables

How many observables at degree n ?

$$\mathcal{N}(n) = \sum_{p \vdash n} Sym(p) = \sum_{R_1, R_2, R_3 \vdash n} (C(R_1, R_2, R_3))^2$$

J. Ben Geloun and S. Ramgoolam "Counting Tensor Model Observables and Branched Covers of the 2-Sphere," arXiv:1307.6490v1 [hep-th] , Ann.Inst.H.Poincare Comb.Phys.Interact. 1 (2014) 1, 77-138

P. Mattioli and S. Ramgoolam, "Permutation centralizer algebras and multi-matrix invariants ," Phys. Rev. D (2016).

p refers to cycle structures of permutations. The R_1, R_2, R_3 are Young diagrams with n boxes. $C(R_1, R_2, R_3)$ is the Kronecker coefficient for the triple of Young diagrams.

Intro: Kronecker coefficients

Many quantities in symmetric group representation theory are combinatorially constructible.

E.g. dimensions of irreps (standard tableaux) ; characters (Murnaghan-Nakayama Lemma) ; Little-wood Richardson coefficients : composing Young diagrams with some labelling rules). Not known for Kronecker coefficients in the general case of a Young diagram triple (R_1, R_2, R_3).

For Kronecker coefficients, we know from rep theory interpretation that they are non-negative. But there is no manifestly positive construction or formula.

$$C(R_1, R_2, R_3) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^{R_1}(\sigma) \chi^{R_2}(\sigma) \chi^{R_3}(\sigma)$$

Is there a manifestly positive construction ?
Discussed in Stanley (1990s) , Pak-Panova (recent)

Intro: Kronecker coefficients - Hint from tensor models

The sum of Kroneckers is constructible in terms of graphs.

The number of observables is also the dimension of $\mathcal{K}(n)$ - an algebra related to the permutation description of observables - which is also useful in computing correlators of the model.

J. Ben Geloun and S. Ramgoolam, "Tensor Models, Kronecker coefficients and Permutation Centralizer Algebras ,"

arXiv:1708.03524v2 [hep-th], JHEP 11 (2017) 092

Is a particular Kronecker coefficient, for a given triple (R_1, R_2, R_3), constructible in terms of graphs ?

Intro: Fourier transform on $\mathcal{K}(n)$

In the recent paper,

J. Ben Geloun and S. Ramgoolam, "Quantum mechanics of bipartite ribbon graphs: Integrality, Lattices and Kronecker coefficients" arXiv:2010.04054v1 [hep-th]

we used the idea of Fourier transform on $\mathcal{K}(n)$.

$\mathcal{K}(n)$ has a combinatoric basis set $\{E_r\}$. One basis vector E_r for each 3-colored graph (or each bi-partite graph)

And a Fourier basis given in terms of triples of YDs $Q_\alpha^{R_1,R_2,R_3}$. $1 \leq \alpha \leq C(R_1,R_2,R_3)^2$.

Intro : Lattices of ribbon graphs

Consider the space of real linear combinations $\sum_r a_r E_r$.

These a_r define vectors in

$$\mathbb{R}^{\mathcal{N}(n)}$$

Inside this Euclidean space is a lattice formed by the integer a_r

$$\mathbb{Z}^{\mathcal{N}(n)}$$

Intro : Kronecker coefficients and lattices of ribbon graphs

We showed that this integer lattice contains sub-lattices of dimension $C(R_1, R_2, R_3)^2$ (and $C(R_1, R_2, R_3)$) for each triple (R_1, R_2, R_3).

For each triple, a basis for the sub-lattice can be identified by constructing the null vectors of an integer matrix

$$X^{R_1,R_2,R_3} v = 0$$

There are combinatoric algorithms for calculating such integer null spaces of integer matrices, e.g. techniques for Hermite normal forms.

Outline of talk

1. Colored graphs to bipartite graphs.
2. Geometric and Fourier bases for $\mathcal{K}(n)$. Fourier subspace labelled by a Young diagram triple.
3. Operators and Hamiltonians for Fourier subspace using the centre of $\mathcal{Z}(\mathbb{C}(S_n))$.
4. Integer Lattice algorithms for constructing sub-lattices in the lattice of ribbon graphs.
5. C^2 and C.
6. Physics : Quantum experiments and membrane interpretation.

Part 1: Colored graphs to ribbon graphs and Belyi maps

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_1\sigma_1\gamma_2, \gamma_1\sigma_2\gamma_2, \gamma_1\sigma_3\gamma_2)$$

All perms in S_n. These equivalence classes define the double coset

Take $\gamma_2 = \sigma_3^{-1}$:

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\sigma_1\sigma_3^{-1}, \sigma_2\sigma_3^{-1}, 1) \equiv (\tau_1, \tau_2, 1)$$

Apply γ_1, then $\gamma_2 = \sigma_3^{-1}\gamma_1^{-1}$ to get

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_1\sigma_1, \gamma_1\sigma_2, \gamma_1\sigma_3) \sim (\gamma_1\sigma_1\sigma_3^{-1}\gamma_1^{-1}, \gamma_1\sigma_2\sigma_3^{-1}, 1)$$
$$\equiv (\gamma_1\tau_1\gamma_1^{-1}, \gamma_1\tau_2\gamma_1^{-1}, 1)$$

$\tau_1 = \sigma_1\sigma_3^{-1}, \tau_2 = \sigma_2\sigma_3^{-1}$

Equivalence classes in $S_n \times S_n$: Bipartite ribbon graphs

Pairs in S_n

$$(\tau_1, \tau_2) \sim (\gamma\tau_1\gamma^{-1}, \gamma\tau_2\gamma^{-1})$$

These describe bi-partite ribbon graphs embedded on a surface.

A bipartite ribbon graph, also called a hypermap, is a graph embedded on a two-dimensional surface with black and white vertices, such that edges connect black to white vertices and cutting the surface along the edges leaves a disjoint union of regions homeomorphic to open discs. Bipartite ribbon graphs, denoted ribbon graphs for short in this paper, with n edges can be described using permutations of $\{1, 2, \cdots, n\}$ forming the symmetric group S_n.

See for example : Graphs on surfaces and their applications, Lando and Zvonkin.

Permutation pairs and bi-partite ribbon graphs

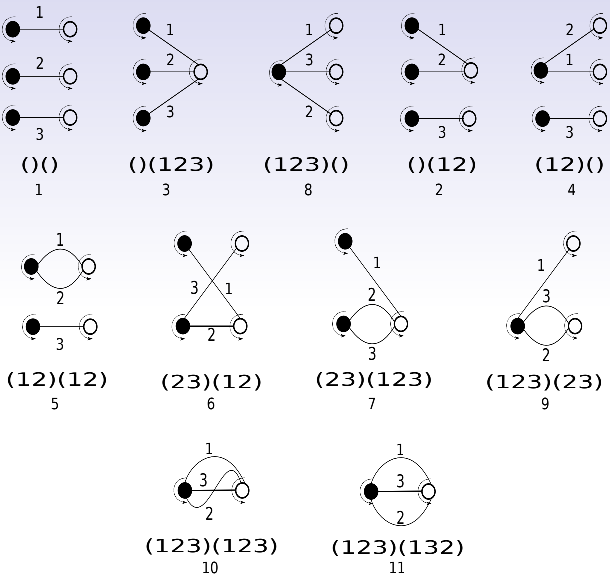

Figure: Bipartite ribbon graphs with $n = 3$ edges

Another triple : Belyi maps

Given the pair (τ_1, τ_2) we can define a third $\tau_3 = (\tau_1\tau_2)^{-1}$ so we have

$$(\tau_1, \tau_2, \tau_3) : \tau_1\tau_2\tau_3 = 1$$

and

$$(\tau_1, \tau_2, \tau_3) \sim (\gamma\tau_1\gamma_1^{-1}, \gamma\tau_2\gamma^{-1}, \gamma\tau_3\gamma^{-1})$$

Such triples classify branched covers of the sphere, branched over three points. (Σ, f) with $f : \Sigma \to \mathbb{P}^1$.

Such pairs (Σ, f) are of interest in number theory and called Belyi maps.

Part 2: Fourier transform on $\mathcal{K}(n)$

The pairs $(\tau_1, \tau_2) \in S_n \times S_n$ can be organised into orbits generated by the simultaneous conjugation

$$(\tau_1, \tau_2) \sim (\gamma\tau_1\gamma^{-1}, \gamma\tau_2\gamma^{-1})$$

with a $\gamma \in S_n$. Each orbit forms an equivalence class - for each equivalence class an observable.

Set of equivalence classes is $\mathrm{Rib}(n)$. Number of equivalence classes is $|\mathrm{Rib}(n)| = \mathcal{N}(n)$, the number of (bi-partite) ribbon graphs

Part 2: Fourier transform on $\mathcal{K}(n)$

Let r label these orbits. So r ranges over $\{1, 2, \cdots, |\mathrm{Rib}(n)|\}$. Pick a representative $(\tau_1^{(r)}, \tau_2^{(r)})$ for each orbit. Consider elements in $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$

$$E_r = \frac{1}{n!} \sum_{\gamma \in S_n} \gamma \tau_1^{(r)} \gamma^{-1} \otimes \gamma \tau_2^{(r)} \gamma^{-1}$$

These span a subspace - in fact a sub-algebra - denoted $\mathcal{K}(n) \subset \mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$.

$\mathcal{K}(n)$ is the subspace which is invariant under conjugation by $\mu \otimes \mu$.

A sub-algebra determined by the equivalence relation coming from counting tensor observables

Part 2: Fourier transform on $\mathcal{K}(n)$

Use the product in $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$ to multiply these. The outcome is within the subspace. $\mathcal{K}(n)$ is a sub-algebra.

$$E_r = \frac{1}{|\mathrm{Orb}(r)|} \sum_{a \in \mathrm{Orb}(r)} \tau_1^{(r)}(a) \otimes \tau_2^{(r)}(a)$$

They are averages of the orbits, i.e averages over distinct labellings of the same ribbon graph.

The E_r form the geometric basis of $\mathcal{K}(n)$.

Part 2: Fourier transform on $\mathcal{K}(n)$

There is also a Fourier basis labelled by triples of Young diagrams.

$$Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} = \frac{d_{R_1} d_{R_2}}{n!^2} \sum_{\sigma_1,\sigma_2 \in S_n} \sum_{i_1,i_2,i_3,j_1,j_2} C^{R_1,R_2;R_3,\tau_1}_{i_1,i_2;i_3} C^{R_1,R_2;R_3,\tau_2}_{j_1,j_2;i_3} D^{R_1}_{i_1 j_1}(\sigma_1) D^{R_2}_{i_2 j_2}(\sigma_2)\, \sigma_1 \otimes \sigma_2$$

Irreps R, orthonormal basis :

Young diagram $R, V^R, D^R(\sigma) : V_R \to V_R; D^R_{ij}(\sigma)$

$V^{R_1} \otimes V^{R_2}$; $D^{R_1}(\sigma) \otimes D^{R_2}(\sigma)$;

$$V^{R_1} \otimes V^{R_2} = \bigoplus_{R_3} V_{R_3} \otimes V^{R_3}_{R_1,R_2}$$

Clebsch-Gordan coefficients $C^{R_1,R_2;R_3,\tau_1}_{i_1,i_2;i_3}$ are inner products $< R_1, i_1, R_2, i_2 | R_3, i_3, \tau >$ where $| R_3, i_3, \tau >$ chosen to be orthonormal basis for $V_{R_3} \otimes V^{R_3}_{R_1,R_2}$

$$1 \leq \tau_1 \leq Dim(V^{R_1,R_2}_{R_3}) = C(R_1, R_2, R_3)$$

With i, j's running over an orthonormal basis, using properties of the D's and Clebsch's, we can show that

$$(\gamma \otimes \gamma) Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} (\gamma^{-1} \otimes \gamma^{-1}) = Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$$

$$Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} Q^{R'_1,R'_2,R'_3}_{\tau'_2,\tau_3} = \delta_{R_1 R'_1} \delta_{R_2 R'_2} \delta_{R_3 R'_3} \delta_{\tau_2 \tau'_2} Q^{R_1,R_2,R_3}_{\tau_1,\tau_3}$$

This gives the explicit decomposition into simple matrix algebras (as expected according to Wedderburn-Artin theorem). Blocks labelled by triples (R_1, R_2, R_3).

Part 3: Operators and Hamiltonians for Fourier subspace of triple

We define the Fourier subspace of $\mathcal{K}(n)$ for a triple (R_1, R_2, R_3) as

$$V^{R_1,R_2,R_3} = \bigoplus_{\tau_1,\tau_2} Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$$

These subspaces can be characterised by using operators $T_k^{(i)} \subset \mathcal{K}(n)$.

We define

$$T_k = \sum_{\sigma \in \mathcal{C}_k} \sigma$$

These are sums of permutations with the cycle structure $[k, 1^{n-k}]$.

There exist $k_* < n$ for $n > 2$, such that $\{T_2, T_3, \cdots, T_{k_*}\}$

- generate the centre $\mathcal{Z}(\mathbb{C}(S_n))$.
- the list of normalized characters

$$\{\frac{\chi^R(T_2)}{d_R}, \frac{\chi^R(T_3)}{d_R}, \cdots, \frac{\chi^R(T_{k_*})}{d_R}\}$$

uniquely determine the Young diagram R.

For example. T_2 generates the centre of $\mathcal{Z}(\mathbb{C}(S_n))$ for $n = 2, 3, 4, 5, 7$. T_2, T_3 generate the centre for all n up 14. $(T_2 ... T_6)$ up to $n = 79$.

G. Kemp and S. Ramgoolam ,"BPS states and central charges " JHEP01(2020)146, arXiv:1911.11649v2[hep-th]

Recall that $Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} \subset \mathcal{K}(n) \subset \mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$.

From each $T_k \in \mathcal{Z}(\mathbb{C}(S_n))$ we define three linear operators on $\mathcal{K}(n)$, acting by left multiplication :

$$T_k^{(1)} = T_k \otimes 1 = \sum_{\sigma \in \mathcal{C}_k} \sigma \otimes 1$$
$$T_k^{(2)} = 1 \otimes T_k = \sum_{\sigma \in \mathcal{C}_k} 1 \otimes \sigma$$
$$T_k^{(3)} = \Delta(T_k) = \sum_{\sigma \in \mathcal{C}_k} \sigma \otimes \sigma$$

We find that

$$T_k^{(1)} Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} = \frac{\chi_{R_1}(T_k)}{d_{R_1}} Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$$
$$T_k^{(2)} Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} = \frac{\chi_{R_2}(T_k)}{d_{R_2}} Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} ,$$
$$T_k^{(3)} Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} = \frac{\chi_{R_3}(T_k)}{d_{R_3}} Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$$

$T_k^{(i)}$ are central operators in $\mathcal{K}(n)$, and their eigenvalues only depend on the R_i labels of the Fourier subspace V^{R_1,R_2,R_3}.

The Fourier subspace V^{R_1,R_2,R_3} is uniquely characterised by using the eigenvalues of

$$\{T_2^{(1)},\cdots,T_{k_*}^{(1)};T_2^{(2)},\cdots,T_{k_*}^{(2)};T_2^{(3)},\cdots,T_{k_*}^{(3)}\}$$

which are the normalized characters

$$\{\widetilde{\chi}^{R_1}(T_2),\cdots,\widetilde{\chi}^{R_1}(T_{k_*});\widetilde{\chi}^{R_2}(T_2),\cdots,\widetilde{\chi}^{R_2}(T_{k_*});\widetilde{\chi}^{R_3}(T_2),\cdots,\widetilde{\chi}^{R_3}(T_{k_*})\}$$

$$\widetilde{\chi}^{R}(T_k)=\frac{\chi^{R}(T_k)}{d_R}$$

In addition to distinguishing the Fourier subspaces with these lists, we can also distinguish them using linear combinations

$$\mathcal{H} = \sum_{i=1}^{3} \sum_{k=2}^{k_*} a_{i,k} T_k^{(i)}$$

For appropriate choices of integers $a_{i,k}$.

The corresponding eigenvalues are :

$$\omega_{R_1,R_2,R_3} = \sum_{i=1}^{3} \sum_{k=2}^{k_*} a_{i,k} \widetilde{\chi}_{R_i}(T_k)$$

Explicitly constructing the D^R_{ij} and the Clebsch's is hard - and not obviously a combinatoric operation.

But we can construct the subspace V^{R_1,R_2,R_3} using the geometric basis.

$$T_k^{(i)} E_r = \sum_s (\mathcal{M}_k^{(i)})_r^s E_s$$

with

$(\mathcal{M}_k^{(i)})_r^s =$ Number of times the multiplication of elements in the sum $T_k^{(i)}$ with a fixed element in orbit r to the right produces an element in orbit s.

$\mathcal{M}_k^{(i)}$ is an integer matrix (entries are either zero or positive integer).

Fiding the eigenvalues and eigenvectors of $\mathcal{H}$ amounts to finding the eigenvalues/eigenvectors of

$$X = \sum_{k,i} a_{i,k} \mathcal{M}_k^{(i)}$$

The eigenvalues are known combinatorially constructible (Murnaghan-Nakayama Lemma) quantities. The eigenvectors in V^{R_1,R_2,R_3} for fixed triple (R_1, R_2, R_3) obey the equation

$$Xv = \omega_{R_1,R_2,R_3} v$$

Equivalently the vectors in V^{R_1,R_2,R_3} are vectors obeying

$$(X - \omega_{R_1,R_2,R_3} 1) v = 0$$

They are the null vectors of the matrix $(X - \omega_{R_1,R_2,R_3} 1)$.

X is an integer matrix. For each Young diagram triple, ω_{R_1,R_2,R_3} is a linear combination of constructible normalized characters (which are rational numbers) with integer coefficients. Hence it is rational.

Part 4: Integer matrix algorithms for null spaces

The null space of the rational matrix $X_{R_1,R_2,R_3} = (X - \omega_{R_1,R_2,R_3} 1)$ has a basis given by integer null vectors.

This can be found by taking $(X^T_{R_1,R_2,R_3})$ and finding its hermite normal form.

This amount to finding a unimodular matrix U (an integer matrix with dterminant± 1) and a matrix h with special triangular form.

$$UX^T = H$$

There are integer algorithms for doing this. First we clear the denominators of the rational matrix by scaling up with the least common multiple of the denominators. Then we implement a sequence of steps involving :

- Swop two rows.
- Multiply a row by -1.
- Add an integer multiple of a row to another row of A.

The sequence of operations determines U.

- H is upper triangular (that is, $H_{ij} = 0$ for $i > j$), and any rows of zeros are located below any other row.
- The leading coefficient (the first non-zero entry from the left, also called the pivot) of a non-zero row is always strictly to the right of the leading coefficient of the row above it; moreover, it is positive.
- The elements below pivots are zero and elements above pivots are non-negative and strictly smaller than the pivot.

Example :

$$H = \begin{pmatrix} 1 & 0 & 40 & -11 \\ 0 & 3 & 27 & -2 \\ 0 & 0 & 61 & -13 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 3 & 2 & 5 \end{pmatrix}$$

$Mv = 0$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 3 & 2 & 5 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(-1)\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + (1)\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The null vectors can be constructed by integer algorithms (e.g. in GAP) for finding Hermite Normal Forms H of matrix M^T

$$H = UM^T$$

U is unimodular - integer matrix with determinant ± 1. In this case

$$H = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

The number of zero rows is equal to the dimension of the null space of X.

The rows of U corresponding to the zero rows of H give integer null vectors of X^{R_1,R_2,R_3}.

The number of these null vectors is equal to $C(R_1, R_2, R_3)^2$.

To summarize

- So we start with an integer matrix.
- Perform integer row operations and arrive at the null vectors.
- We count the null vectors. We obtain C^2.
- The null vectors are a set of vectors in

$$\mathbb{Z}^{|\mathrm{Rib}(n)|}$$

  Taking integer linear combinations of these basis null vectors generates a sub-lattice.

Part 5: Constructing C^2 and constructing C.

We have given a sub-lattice construction of C^2. What about C?

There is a conjugation operation $S : \mathcal{K}(n) \rightarrow \mathcal{K}(n)$ which obeys $S^2 = 1$; acts by inverting the two permutations in $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$.

Acting on the geometric basis, a number of E_r obey

$$S(E_r) = E_r$$

These are self-conjugate ribbons.

For a self-conjugate ribbon represented by (τ_1, τ_2), there exists a γ such that $(\tau_1^{-1}, \tau_2^{-1}) = (\gamma\tau_1\gamma^{-1}, \gamma\tau_2\gamma^{-1})$ For non-self-conjugate (τ_1, τ_2) and $(\tau_1^{-1}, \tau_2^{-1})$ belong to distinct orbits.

Non-self-conjugate ribbons are paired up by S. We have corresponding vectors $\{E_n, S(E_n)\}$.

On the Fourier basis $Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$, the effect of S is to keep R_1, R_2, R_3 unchanged and to swop the τ_1, τ_2. As a result $S = 1$ eigenspace in V^{R_1,R_2,R_3} has dimension

$$C(C+1)/2$$

Integer matrix algorithms can be used to construct a sub-lattice of this dimension. Finding null vectors of

$$\begin{pmatrix} \mathcal{H} - \omega_{R_1,R_2,R_3} \\ S - 1 \end{pmatrix}$$

The dimension of $S = -1$ in V^{R_1,R_2,R_3} is

$$C(C-1)/2$$

Find the sub-lattice basis vectors by finding null vectors of

$$\binom{\mathcal{H} - \omega_{R_1,R_2,R_3}}{S+1}$$

Choose an injection between from the smaller set of sub-lattice generators to the bigger set. The complement of that will have exactly C vectors.

This gives a construction of C.

An interesting corollary of the properties of S is the identity

$$\text{Number of self-conjugate ribbons} = \sum_{R_1, R_2, R_3} C(R_1, R_2, R_3)$$

Take any graph vector E_r

Apply the Hamiltonian $\mathcal{H}$

$$e^{-i\mathcal{H}t}E_r = e^{-iHt} \sum_{R_1,R_2,R_3,\tau_1,\tau_2} (E_r, Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}) Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$$

$$= \sum_{R_1,R_2,R_3,\tau_1,\tau_2} e^{-i\omega_{R_1,R_2,R_3}t}(E_r, Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}) Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$$

Measure a given Fourier component $e^{-i\omega_{R_1,R_2,R_3}t}$in this time-dependent state. This shows that the corresponding C is non-zero.

If the Fourier component is absent, does not mean the Kronecker is zero - since overlap could be zero.

How efficiently can we determine the non-vanishing of C from such experiemnts ? Interesting because the determination of vanishing is an NP-hard problem (Ikenmeyer, Mulmulley, Walter, 2015). An arena to compare classical and quantum computation ...

Part 6: Physics - Membrane interpretation

Quantum mechanics models on $\mathcal{K}(n)$ have an interpretation in terms of time-dependent quantum world-volumes.

Initial state can be a fixed ribbon graph. equivalently Belyi curve/map ; Time evolution produces a superposition of graphs. The Q's are stationary states. But the graphs themselves are complicated linear combinations of the Q's - so not stationary states.

Genus can change ...

An interesting mathematical model for membrane dynamics with finite Hilbert space.

Finite Hilbert spaces with algebra structure have very rich physics.