# Weingarten calculus in a tensor setup

Benoît Collins

Kyoto University

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# Overview

Partly based on joint work with Razvan Gurau and Luca Lionni
**Plan:**

1. Motivations
2. Classical Weingarten calculus.
3. Variants: real, quantum, centered, etc.
4. Asymptotics.
5. Some results

# Motivation

A little bit of down-to-earth free probability

- Let $A, B$ be two selfadjoint matrices in $M_n(\mathbb{C})$. Their eigenvectors are unknown but their eigenvalues ($\lambda_1 \geq \ldots \geq \lambda_n$, resp. $\mu_1 \geq \ldots \geq \mu_n$) are known.
- What are all possible eigenvalues of $A + B$? (say $\nu_1 \geq \ldots \geq \nu_n$)

# Motivation

- This is Horn's problem.
- Some equalities and inequalities are easy to prove. Horn conjectured it to be a polytope that he described
- The prof that it is a polytope was obtained through the help of symplectic geometry (Guillemin, Kirwan, Sternberg)
- The full description of the polytope was solved by Knutson, Tao, etc. (after Klyachko proved an equivalence with a problem in representation theory – the saturation conjecture).

# Motivation

- The above is not probabilistic but the eigenvalues follow a distribution (and in principle the problem boils down to describing its support)
- This is a problem about convolution of orbits.
- Therefore, it can in principle be done with the Harish Chandra Itzyson Zuber HCIZ integral $\int_{U \in U(n)} \exp(Tr(AUBU^*))dU$.
- There exists nice formulas for HCIZ.

# Motivation

- If we take sequences $A_n$, $B_n$ and assume the histogram of the spectrum has a limiting shape (technically: the ESD converges), then the same holds true (with high probability) for $A_n + UB_nU^*$.
- Technically, $n^{-1}Tr(A_n^k) \to a_k$ and $n^{-1}Tr(B_n^k) \to b_k$ for all integers.
- This is a concentration phenomenon and it is a possible definition of free probability (or at least, free additive convolution)

# Motivation

- How about tensors, then? One concrete example: if $A, B \in M_n(\mathbb{C})^{\otimes 2}$, what can be said about

$$A + U \otimes V \cdot B \cdot U^* \otimes V^*$$

- Motivation: bipartite Hamiltonian or states that evolve according to local reversible dynamics (or the typical behavior under this setup). And generalize free probability.
- Remark: we are also interested in (and deal with) $D$-partite
- Difficulty: the eigenvalues are not enough to describe the orbits (they are described by diagrams). The situation is slightly simpler for rang 1 matrices in the bipartite case (pure states, classified by Schmidt coefficients)

# Motivation

- This involves matrix integrals over tensors. Typically:

$$\int_{U\in U(n),V\in U(n)} \exp(Tr(AU \otimes VBU^* \otimes V^*))dUdV.$$

  and the logarithm thereof.
- Problem: there is no HCIZ integral.
- Gaussian variants have been studied for Gaussian vectors. In this case one relies on the Wick theorem, which is simpler. But we are limited on the orbits.
- In our case, we need to extend to unitary integrals and therefore calculate unitary integrals.

# Weingarten calculus

- We want to calculate

$$\int u_{i_1 j_1} \dots u_{i_k j_k} \overline{u_{i'_1 j'_1} \dots u_{i'_k j'_k}}.$$

- Answer (Weingarten theorem): This is

$$\sum_{\sigma,\tau \in S_k} \delta_{i,i',\sigma} \delta_{j,j',\tau} Wg(\sigma\tau^{-1}, n)$$

  where $Wg$ will be defined @next slides.
- Compare with real and GOE, GUE.

# Weingarten calculus: formula

- Set $Wg(\sigma, \tau, n) = Wg(\sigma\tau^{-1}, n)$. $Wg$ is defined as follows: take a $k! \times k!$ matrix indexed by permutations (or pairings).

$$Wg = (Wg(\sigma, \tau, n))_{\sigma,\tau}$$

- Set $G = (n^{\# loops(\sigma\tau^{-1})})_{\sigma,\tau}$. This is a positive (Gram matrix, see later). $Wg$ is characterized by

$$Wg = G^{-1}$$

(pseudo-inverse if needs be)

# Weingarten calculus: proof

- Why is this true? Consider the unitary matrix $Z = U^{\otimes k} \otimes \overline{U}^{\otimes k}$.
- Fact: $P = E(Z)$ is an orthogonal projection onto fixed points of $Z$.
- Schur-Weyl duality (black box): the fixed points are generated by permutations.

# Weingarten calculus: proof

- The canonical matrix coefficients of $P$ are exactly $\int u_{i_1 j_1} \dots u_{i_k j_k} \overline{u_{i'_1 j'_1} \dots u_{i'_k j'_k}}$.
- $G$ is the gram matrix of the canonical fixed point basis, and $\delta_{i,i',\sigma}$ is the scalar product of a canonical matrix $E_{i_1 i'_1} \otimes \dots \otimes E_{i_k i'_k}$ with the canonical element $\sigma$.
- Conclude with formula for a projection when the basis is not orthogonal (remark: that's one key difference with Wick calculus).

# Weingarten calculus: expansion

- How to calculate $Wg$? (the theorem itself is already a formula)
- $$Wg(\sigma, n) = n^{-k} \sum_{path} (-n)^{l(path)}$$

  where a path is a solution $(i_1 j_1) \ldots (i_l j_l) = \sigma$ satisying $i_m < j_m, j_{m-1} \leq j_m$, and $l(path) = l$
- This follows either from Jucys-Murphy theory (Novak-Matsumoto) or from orthogonality relations (Weingarten, C-Matsumoto)
- There is a relation in terms of monotone Hurwitz number.
- Remark: there is also a signed formula in terms of permutation paths.

# Weingarten calculus, asymptotics

- First order asymptotics $Wg$ in dimension.
- $$Wg(\sigma_1 \sqcup \sigma_2, n) = Wg(\sigma_1, n) Wg(\sigma_2, n)(1 + O(n^{-2}))$$
- $$Wg((1, \ldots, k), n) = \frac{(-1)^{k-1} C_{k-1}}{(n-k+1) \ldots (n+k-1)}$$
- These two facts combined basically yield Speicher's non-crossing Moebius function.
- Uniform bounds are achievable (C, Matsumoto)

# Weingarten calculus, orthogonal version

- Complex conjugate not needed in the orthogonal case:

$$\int u_{i_1 j_1} \ldots u_{i_k j_k}$$

  with $k$ even, and permutations become pair partitions.
- The proof is the same (with projections) and leading estimates are the same.
- No more explicit formula or genus expansion.
- Uniform bounds similar to the uniform case.

# Centered Weingarten calculus

- For a random variable $X$, we define $[X] = X - E(X)$ (its centering).
- For a symbol $\varepsilon \in \{\cdot, -\}$ and $z \in \mathbb{C}$, we take the notation that $z^\varepsilon = z$ if $\varepsilon = \cdot$ and $z^\varepsilon = \bar{z}$ if $\varepsilon = -$. We want to to compute for $U = (U_{ij})$ Haar distributed on $\mathbb{U}_n$, expresssions of the form

$$E \prod_{t=1}^{T} [\prod_{l=1}^{k_t} U_{x_{tl} y_{tl}}^{\varepsilon_{tl}}]$$

  in a meaningful way.
- Needed for estimates for tensors (Bordenave, C). .

# Centered Weingarten calculus

- We can write a Weingarten formula

$$E\prod_{t=1}^{T}[\prod_{l=1}^{k_t} U_{x_{tl}y_{tl}}^{\varepsilon_{tl}}] = \sum_{\sigma,\tau\in P_2(k_1+\ldots+k_T)} \delta_{\sigma,x}\delta_{\tau,y} Wg_{centered}(\sigma,\tau,part)$$

- The function $Wg$ depends on the pairings and the partition.

**Theorem**

*$Wg$ decays as $n^{-k}$ where*
$k = (k_1+\ldots+k_T)/2 + d(\sigma,\tau) + 2\#lonesome\ blocks.$

- This estimate is uniform on $k \sim Poly(n)$.

# Connected Weingarten calculus

- Next, we consider the classical cumulants $C_T$. These are symmetric $T$-linear forms polarizing the expansion

$$\log E(\exp X) = \sum_{T\geq 1} C_T(X)/T!.$$

For example $C_1(X) = X$, $C_2(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2)$, $C(X_1, X_2, X_3) = E(X_1X_2X_3) - E(X_1)E(X_2X_3) - E(X_2)E(X_3X_1) - E(X_3)E(X_1X_2) + 2E(X_1)E(X_2)E(X_3)$, etc.
- We want to to compute for $U = (U_{ij})$ Haar distributed on $\mathbb{U}_n$, expressions of the form

$$C_T([\prod_{l=1}^{k_t} U_{x_{tl}y_{tl}}^{\varepsilon_{tl}}]_t)$$

in a meaningful way.

# Connected Weingarten calculus

- We can write a Weingarten formula

$$E\prod_{t=1}^{T}[\prod_{l=1}^{k_t} U_{x_{tl}y_{tl}}^{\varepsilon_{tl}}] = \sum_{\sigma,\tau\in P_2(k_1+\ldots+k_T)} \delta_{\sigma,x}\delta_{\tau,y} Wg_{connected}(\sigma,\tau,part)$$

- The function $Wg$ depends on the pairings and the partition.

**Theorem**

*$Wg_{connected}$ decays as $n^{-k}$ where*
$k = (k_1 + \ldots + k_T)/2 + d(\sigma,\tau) + 2(\#blocks(part,\sigma,\tau) - 1)$.

# Centered and connected leading orders

- In the centered case, we do not know how to interpret leading orders (Bordenave C). Unlikely to have a conceptual formulation
- In the connected case, we have a combinatorial interpretation (following C-03, Zuber O'Brien 81, Bousquet-Melou Schaeffer, etc, with consellations) – left to a further presentation of Luca/Razvan?

# Tensor connected Weingarten calculus: expansion and asymptotics

- Problem: compute $C_n(Tr(AUBU^*))$ where $A, B \in M_n(\mathbb{C})^{\otimes C}$ and $U = U_1 \otimes \ldots \otimes U_D$.
- Thm 4.1 p15

# Thank you!