Permutation Invariant 2-Matrix Model

Adrian Padellaro

Queen Mary University of London

March 24, 2021
Tensor Journal Club

Permutation Invariant 2-Matrix Model

Adrian Padellaro

Queen Mary University of London

March 24, 2021
Tensor Journal Club

Experimental motivation

- Distributional semantics [Harris 1954]
  - Meaning represented by vectors of co-occurrence

$$(\text{box})_i = \text{Frequency of box co-occuring with the } i\text{th context word}$$

  - Successful for words, compositional model needed for more complex semantical structures
- Compositional distributional semantics [Coecke, Sadrzadeh, Clark '10]
  - Tensors act on vectors to form sentences
  - Adjectives act on nouns to give *noun phrases*

$$(\text{red})_{ij}(\text{box})_j = (\text{red box})_i$$

  - Two matrices give *adjective verb combinations*

$$(\text{eat})_{ij}(\text{quickly})_{jk}(\text{cats})_k = (\text{cats eat quickly})_i$$

Experimental motivation

- Statistics of word matrices should be analyzed using permutation invariant matrix theories [Kartsaklis, Ramgoolam, Sadrzadeh '17]
- 5-parameter permutation invariant Gaussian model showed promising results
- General 13-parameter permutation invariant Gaussian 1-matrix model was solved [Ramgoolam '18]
- Expectation values in good agreement with experiment [Ramgoolam, Sadrzadeh, Sword '19]

1-Matrix Model

[Ramgoolam '18]

- Matrix model with D^2 matrix M_{ij}

$$Z = \int \mathrm{d}M \, \mathrm{e}^{-\mathcal{S}(M)}$$

- Free (Gaussian)

$$\mathcal{S}(M) \sim M + MM$$

- Permutation invariant

$$\mathcal{S}(M_{\sigma(i)\sigma(j)}) = \mathcal{S}(M_{ij}), \quad \sigma \in S_D$$

- Observables are permutation invariant polynomials

$$f(M_{\sigma(i)\sigma(j)}) = f(M_{ij})$$

1-Matrix Model

[Ramgoolam '18]

- Permutations act on $V_D = \text{Span}(|i\rangle \mid i = 1, \ldots, D)$

$$\sigma |i\rangle = \left|\sigma^{-1}(i)\right\rangle , \quad \sigma \in S_D$$

- V_D decomposes into irreducible representations

$$V_D \to V_0 \oplus V_H$$

- There is an orthonormal basis for $V_0 \oplus V_H$

$$|E_0\rangle = \sum_{i=1}^{D} C_{0i} |i\rangle = \frac{1}{\sqrt{D}} \sum_{i=1}^{D} |i\rangle$$

$$|E_a\rangle = \sum_{i=1}^{D} C_{ai} |i\rangle , \; a = 1, \ldots, D-1$$

1-Matrix Model

[Ramgoolam '18]

- Linear combinations of M_{ij} form a vector space $V_D \otimes V_D$

$$V_D \otimes V_D = \mathrm{Span}(|i\rangle \otimes |j\rangle \mid i, j = 1, \dots, D)$$

- S_D acts diagonally on $V_D \otimes V_D$

$$\sigma(|i\rangle \otimes |j\rangle) = \sigma\,|i\rangle \otimes \sigma\,|j\rangle$$

1-Matrix Model

[Ramgoolam '18]

- Linear combinations of degree m monomials in M_{ij} also form a vector space
- 1-Matrix monomials are symmetric

$$M_{i_1 j_1} M_{i_2 j_2} = M_{i_2 j_2} M_{i_1 j_1}$$

- Instead of $(V_D \otimes V_D)^{\otimes m}$ we have

$$\mathrm{Sym}^m(V_D \otimes V_D) = \text{Symmetric part of } (V_D \otimes V_D)^{\otimes m}$$

- For example

$$M_{i_1 j_1} M_{i_2 j_2} \longleftrightarrow \Big[(|i_1\rangle \otimes |j_1\rangle) \otimes (|i_2\rangle \otimes |j_2\rangle) + (|i_2\rangle \otimes |j_2\rangle) \otimes (|i_1\rangle \otimes |j_1\rangle) \Big]$$

1-Matrix Model

[Ramgoolam '18]

- Notation
$$W_{m,D} = \mathrm{Sym}^m(V_D \otimes V_D)$$
- $W_{m,D}$ is also a vector space of bosonic oscillators
- For example: $W_{2,D}$

$$M_{i_1 j_1} M_{i_2 j_2} \longleftrightarrow \Big[(|i_1\rangle \otimes |j_1\rangle) \otimes (|i_2\rangle \otimes |j_2\rangle) + (|i_2\rangle \otimes |j_2\rangle) \otimes (|i_1\rangle \otimes |j_1\rangle) \Big] \longleftrightarrow M^{\dagger}_{i_1 j_1} M^{\dagger}_{i_2 j_2} |0\rangle$$

2-Matrix Model

[Barnes, AP, Ramgoolam '21]

- Matrix model with two D^2 matrices M, N.
- Zero-dimensional QFT

$$Z = \int \mathrm{d}M \, \mathrm{d}N \mathrm{e}^{-\mathcal{S}(M,N)}.$$

- Free (Gaussian)

$$\mathcal{S}(M,N) \sim M + N + MM + NN + MN$$

- Permutation invariant

$$\mathcal{S}(M_{\sigma(i)\sigma(j)}, N_{\sigma(k)\sigma(l)}) = \mathcal{S}(M_{ij}, N_{kl}), \quad \sigma \in S_D$$

2-Matrix Model

[Barnes, AP, Ramgoolam '21]

- Observables

$$f(M_{\sigma(i)\sigma(j)}, N_{\sigma(k)\sigma(l)}) = f(M_{ij}, N_{kl}),$$

- Expectation values from Wick's theorem

$$\langle f(M_{ij}, N_{kl})\rangle = \frac{1}{Z}\int \mathrm{d}M\,\mathrm{d}N \mathrm{e}^{-\mathcal{S}(M,N)} f(M_{ij}, N_{kl}),$$

- Degree n monomials in N_{kl} form a vector space

$$\mathrm{Sym}^n(V_D \otimes V_D)$$

- Mixed monomials are not symmetric in M and N

$$M_{i_1 j_1} N_{k_1 l_1} \neq M_{k_1 l_1} N_{i_1 j_1}$$

- Degree $m + n$ monomials form tensor product of symmetric parts

$$W_{m,n,D} = \mathrm{Sym}^m(V_D \otimes V_D) \otimes \mathrm{Sym}^n(V_D \otimes V_D)$$

- Example: $W_{2,2,D}$

$$M_{i_1 j_1} M_{i_2 j_2} N_{k_1 l_1} N_{k_2 l_2}$$

$$\updownarrow$$

$$\Big[(|i_1\rangle \otimes |j_1\rangle) \otimes (|i_2\rangle \otimes |j_2\rangle) + (|i_2\rangle \otimes |j_2\rangle) \otimes (|i_1\rangle \otimes |j_1\rangle)\Big] \otimes$$
$$\Big[(|k_1\rangle \otimes |l_1\rangle) \otimes (|k_2\rangle \otimes |l_2\rangle) + (|k_2\rangle \otimes |l_2\rangle) \otimes (|k_1\rangle \otimes |l_1\rangle)\Big]$$

- $W_{m,n,D}$ is also a vector space of bosonic oscillators

$$M_{i_1 j_1} \dots M_{i_m j_m} N_{k_1 l_1} \dots N_{k_n l_n} \longleftrightarrow M^\dagger_{i_1 j_1} \dots M^\dagger_{i_m j_m} N^\dagger_{k_1 l_1} \dots N^\dagger_{k_n l_n} |0\rangle$$

2-Matrix Observables

- Degree $m+n$ observables form a subspace of S_D-invariants

$$W_{m,n,D}^{V_0} = \{ w \in W_{m,n,D} \,|\, \sigma w = w \}.$$

- The goal is to construct a basis for $W_{m,n,D}^{V_0}$
- For example, a basis for $W_{1,D}^{V_0}$ is

$$\sum_{i=1}^{D} |i\rangle \otimes |i\rangle \longleftrightarrow \sum_{i=1}^{D} M_{ii}, \quad \sum_{i,j=1}^{D} |i\rangle \otimes |j\rangle \longleftrightarrow \sum_{i,j=1}^{D} M_{ij}.$$

- A good start is compute the dimension of $W^{V_0}_{m,n,D}$.
- Use projector to compute dimension of subspace

$$P^{S_D}_{V_0} : W_{m,n,D} \to W^{V_0}_{m,n,D}$$

- Trace gives dimension

$$\dim(W^{V_0}_{m,n,D}) = \mathrm{Tr}_{W_{m,n,D}}\left(P^{S_D}_{V_0}\right).$$

- Notation

$$\mathrm{Dim}(m,n,D) \equiv \dim(W^{V_0}_{m,n,D})$$
$$\mathrm{Dim}(m,D) \equiv \dim(W^{V_0}_{m,0,D})$$

- Average over group gives a projector to invariants

$$P_{V_0}^{S_D} = \frac{1}{|S_D|} \sum_{\sigma \in S_D} \sigma$$

- Check that it is invariant

$$\sigma' P_{V_0}^{S_D} = P_{V_0}^{S_D}.$$

- Compute trace

$$\mathrm{Dim}(m,n,D) = \mathrm{Tr}_{W_{m,n,D}}\left(P_{V_0}^{S_D}\right) =$$

$$\sum_{\substack{p\vdash D\\ q\vdash m\\ r\vdash n}} \frac{1}{\prod_{i=1}^{D} i^{p_i} p_i! \prod_{i=1}^{m} i^{q_i} q_i! \prod_{i=1}^{n} i^{r_i} r_i!} \prod_{i=1}^{m} \Big(\sum_{l|i} l p_l\Big)^{2q_i} \prod_{j=1}^{n} \Big(\sum_{l|j} l p_l\Big)^{2r_j}$$

m	n	$\mathrm{Dim}(m, n, D)$ for $D = 1, \ldots, 2m + 2n + 5$
1	0	1, 2, 2, 2, 2, 2, 2
2	0	1, 6, 10, 11, 11, 11, 11, 11, 11
1	1	1, 8, 14, 15, 15, 15, 15, 15, 15
3	0	1, 10, 31, 47, 51, 52, 52, 52, 52, 52, 52
2	1	1, 20, 70, 107, 116, 117, 117, 117, 117, 117, 117

When $D \geq 2m + 2n$, the dimension stabilizes

$$\mathrm{Dim}(m, n, D \geq 2m + 2n,) = \mathrm{Dim}(m, n, 2m + 2n)$$

1. What happens when $\mathrm{Dim}(m, n, D)$ stabilizes?
   - 1-matrix case [Kartsaklis, Ramgoolam, Sadrzadeh '17]

     $\mathrm{Dim}(m, 0, 2m) =$ # directed graphs with m edges on unlabeled vertices

   - Basis for observables labeled by directed graphs
   - 2-matrix case [Barnes, AP, Ramgoolam '21]

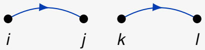
     $\mathrm{Dim}(m, n, 2m + 2n) =$ # directed colored graphs with m blue edges n green edges on unlabeled vertices

   - Basis for observables labeled by directed colored graphs
2. What happens for $D \leq 2m + 2n$?

Observables from graphs

Each blue edge going from vertex i to vertex j is associated with a factor of M_{ij}. Similarly for green edges and N_{kl}. We then sum over the vertex labels from one to D.
For example

$$\sum_{i,j,k,l} M_{ij}M_{kl} \leftrightarrow$$



,

$$\sum_{i,j,k} M_{ij}N_{jk} \leftrightarrow$$

Proposition

$\mathrm{Dim}(m, D)$ is the number of directed graphs with m edges on up to D unlabeled vertices.

Strategy

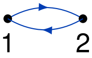
1. Start with labeled vertices
2. Unlabeled graphs correspond to equivalence classes of labeled graphs
3. Count equivalence classes
4. Relate counting to trace of projector

Step 1: Labeled graphs with m edges and k vertices

- Labeled graphs are collections of m ordered pairs of integers

$$\left[(a_1^-, a_1^+), \ldots, (a_m^-, a_m^+)\right], \quad a_i^\pm \in \{1, \ldots, k\}.$$

- Integers can take values in $\{1, \ldots, k\}$
- Each pair is an edge from vertex a_i^- to vertex a_i^+.
- Order of pairs in the collection does not matter.
- Order within pairs is important (directed edges).
- Example: $[(1, 2), (2, 1)]$

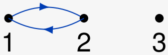


Step 2: Unlabeled graphs

- Two collections describe the same unlabeled graph if one is a re-labeling of the other.

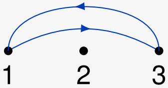
Example

$[(1,2),(2,1)]$



$\sim$

$[(1,3),(3,1)]$



$[(1,2),(2,1)]$

$\not\sim$

$[(1,2),(2,3)]$

- Re-labeling is a S_k action on collections of pairs (labeled graphs)

$$\sigma^{-1} \cdot \left[(a_1^-, a_1^+), \ldots, (a_m^-, a_m^+)\right] = \left[(\sigma(a_1^-), \sigma(a_1^+)), \ldots, (\sigma(a_m^-), \sigma(a_m^+))\right].$$

Definition: Group orbit

If G acts on the set U, we say that $u \in U$ and $u' \in U$ are in the same orbit if there exists $g \in G$ such that $g \cdot u = u'$.

$$O_u = \{g \cdot u \,|\, \forall g \in G\}$$

- Group orbits partition the set U into disjoint subsets
- The set of all orbits is written as a quotient

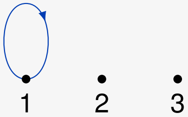
$$G \backslash U$$

Example

Labeled graphs with 1 edge and 3 vertices under action of S_3.

$$U_{1,3} = \{[(1,1)], [(1,2)], [(1,3)], [(2,1)], \\ [(2,2)], [(2,3)], [(3,1)], [(3,2)], [(3,3)]\}$$

$[(1,1)] \longleftrightarrow$



$[(1,2)] \longleftrightarrow$

$$O_{[(1,1)]} = \{[(1,1)], [(2,2)], [(3,3)]\},$$
$$O_{[(1,2)]} = \{[(1,2)], [(1,3)], [(2,1)], [(2,3)], [(3,1)], [(3,2)]\}$$

Step 3: Count Orbits

- Let $U_{m,k}$ be the set of all labeled graphs with m edges and k vertices.
- Proposition is equivalent to the statement

$$\mathrm{Dim}(m,k) = \left| S_k \backslash U_{m,k} \right|.$$

- Burnside's lemma: Number of orbits is equal to average number of fixed points

$$\begin{aligned} \left| S_k \backslash U_{m,k} \right| &= \frac{1}{|S_k|} \sum_{\sigma \in S_k} (\text{\# Elements in } U_{m,k} \text{ fixed by } \sigma) \\ &= \frac{1}{|S_k|} \sum_{\sigma \in S_k} \sum_{u \in U_{m,k}} \delta(\sigma \cdot u = u) \end{aligned}$$

Example

$U_{2,2}$ consists of the 10 elements

$$
\begin{array}{llll}
[(1,1),(1,1)] & [(1,1),(1,2)] & [(1,1),(2,1)] & [(1,1),(2,2)] \\
[(1,2),(1,2)] & [(1,2),(2,1)] & [(1,2),(2,2)] & \\
[(2,1),(2,1)] & [(2,1),(2,2)] & & \\
[(2,2),(2,2)] & & &
\end{array}
$$

$S_2 = \{e, (12)\}$.

$$\left| S_2 \backslash U_{2,2} \right| = \frac{1}{2!}(10+2) = 6.$$

Step 4: Trace of projector

- $U_{m,k}$ defines permutation representation

$$V_{m,k} = \mathrm{Span}(\,|u\rangle \mid u \in U_{m,k})$$

- S_k acts on $V_{m,k}$ by the action on $U_{m,k}$

$$\sigma\,|u\rangle = |\sigma \cdot u\rangle\,.$$

- It is the symmetric part of $(V_k \otimes V_k)^{\otimes m}$

$$V_{m,k} \cong \mathrm{Sym}^m(V_k \otimes V_k)$$

- $V_{m,k}$ is a bosonic oscillator vector space

$$|u\rangle = \left|(a_1^-, a_1^+), \ldots, (a_m^-, a_m^+)\right\rangle = A^\dagger_{a_1^-,a_1^+} \ldots A^\dagger_{a_m^-,a_m^+}\,|0\rangle\,.$$

Step 4: Trace of projector

- Re-write the Kronecker delta

$$\delta(\sigma \cdot u = u) \longleftrightarrow \langle u| \sigma |u\rangle \,.$$

- Burnside's lemma becomes

$$\begin{aligned} \left|S_k \backslash U_{m,k}\right| &= \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{u \in U_{m,k}} \langle u| \sigma |u\rangle \\ &= \sum_{u \in U_{m,k}} \langle u| P_{V_0}^{S_k} |u\rangle \\ &= \mathrm{Tr}_{V_{m,k}}(P_{V_0}^{S_k}). \end{aligned}$$

- 2-colored directed graphs with m, n edges on k vertices

$$[(a_1^-, a_1^+), \ldots, (a_m^-, a_m^+)], [(b_1^-, b_1^+), \ldots, (b_n^-, b_n^+)],$$
$$a_i^\pm, b_i^\pm \in \{1, \ldots, k\}$$

- $U_{m,n,k}$ defines permutation representation

$$V_{m,n,k} = \mathrm{Span}(\, |u\rangle \mid u \in U_{m,n,k})$$

- $V_{m,n,k}$ is a bosonic oscillator space

$$|u\rangle = A^\dagger_{a_1^- a_1^+} \ldots A^\dagger_{a_m^- a_m^+} B^\dagger_{b_1^- b_1^+} \ldots B^\dagger_{b_n^- b_n^+} |0\rangle$$

- Number of orbits is equal to dimension of subspace of invariants

$$\left| S_k \setminus U_{m,n,k} \right| = \mathrm{Tr}_{V_{m,n,k}}(P^{S_k}_{V_0})$$

- Replace k with D
- It is the same vector space of matrix polynomials

$$V_{m,D} \cong W_{m,D} = \mathrm{Sym}^m(V_D \otimes V_D)$$
$$V_{m,n,D} \cong W_{m,n,D} = \mathrm{Sym}^m(V_D \otimes V_D) \otimes \mathrm{Sym}^n(V_D \otimes V_D)$$

- The action of S_D on W is the same as S_k on V

$$\mathrm{Dim}(m, D) = \mathrm{Tr}_{W_{m,D}}(P_{V_0}^{S_D}) = \mathrm{Tr}_{V_{m,D}}(P_{V_0}^{S_D}) = \left| S_D \setminus U_{m,D} \right|$$
$$\mathrm{Dim}(m, n, D) = \mathrm{Tr}_{W_{m,n,D}}(P_{V_0}^{S_D}) = \mathrm{Tr}_{V_{m,n,D}}(P_{V_0}^{S_D}) = \left| S_D \setminus U_{m,n,D} \right|.$$

- $\text{Dim}(m, D) = \left|S_D \setminus U_{m,D}\right|$ stabilizes for $D \geq 2m$, because the most number of vertices that can be occupied by m edges is $2m$.

- $\text{Dim}(m, n, D) = \left|S_D \setminus U_{m,n,D}\right|$ stabilizes because $m + n$ edges can occupy no more than $2m + 2n$ vertices

- We set out to understand a basis at finite D.
- For $D < 2m + 2n$, a basis is labeled by graphs with $m + n$ edges on D vertices.

$$\mathrm{Dim}(m, n, D) = \left| S_D \setminus U_{m,n,D} \right|.$$

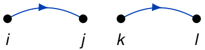
Summary: Observables

- Burnside's lemma using permutation representations

$$\mathrm{Dim}(m, n, D) = \left| S_D \setminus U_{m,n,D} \right|$$

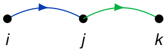
- Orbits are graphs with $m + n$ edges, D vertices
- Explicit bijection to observables

$$\sum_{i,j,k,l} M_{ij} M_{kl} \leftrightarrow$$



,

$$\sum_{i,j,k} M_{ij} N_{jk} \leftrightarrow$$



Expectation Values of Observables

When the VEV is non-zero, Wick's theorem takes the form

$$\langle T_{i_1 j_1} \dots T_{i_r j_r} \rangle = \sum_{p \in P_r^{1,2}} \prod_{c \in p} \prod_{(a,b) \in p} \langle T_{i_a j_a} T_{i_b j_b} \rangle_c \langle T_{i_c j_c} \rangle,$$

where T can be either M or N.

At degree $m = 2, n = 1$ we have

$$\begin{aligned}\langle M_{i_1 j_1} M_{i_2 j_2} N_{i_3 j_3} \rangle = \; & \langle M_{i_1 j_1} M_{i_2 j_2} \rangle_c \langle N_{i_3 j_3} \rangle + \langle M_{i_1 j_1} N_{i_3 j_3} \rangle_c \langle M_{i_2 j_2} \rangle \\ & + \langle M_{i_2 j_2} N_{i_3 j_3} \rangle_c \langle M_{i_1 j_1} \rangle + \langle M_{i_1 j_1} \rangle \langle M_{i_2 j_2} \rangle \langle N_{i_3 j_3} \rangle\end{aligned}$$

- $V_D \otimes V_D$ decomposes into irreducible representations of S_D

$$V_D \otimes V_D \longrightarrow 2\,V_0 \oplus 3\,V_H \oplus V_2 \oplus V_3$$

- Use Clebsch-Gordan coefficients

$$S_a^{V_A;\alpha} = \sum_{i,j} C_{a,ij}^{V_A;\alpha} M_{ij}, \quad a = 1, \ldots, \dim(V_A)$$

- They transform irreducibly under S_D

$$\sum_{i,j} C_{a,ij}^{V_A;\alpha} M_{\sigma^{-1}(i)\sigma^{-1}(j)} = D^{V_A}(\sigma)_a{}^b S_b^{V_A;\alpha}.$$

- Correlators are simple in the representation basis

$$\langle S_a^{V_A;\alpha} S_b^{V_B;\beta} \rangle_c = \left(\Lambda_A^{MM} \right)^{-1}_{\alpha\beta} \delta_{ab} \delta(V_A, V_B).$$

- We have symmetric coupling matrices

$$\frac{2 \cdot 3}{2} + \frac{3 \cdot 4}{2} + 1 + 1 = 11$$

- Invert the change of basis

$$M_{ij} = \sum_{V_A,a,\alpha} C^{V_A;\alpha}_{a,ij} S^{V_A;\alpha}_a$$

- The two point function becomes

$$\begin{aligned}\langle M_{ij} M_{kl} \rangle_c &= \sum_{V_A,a,\alpha} \sum_{V_B,b,\beta} C^{V_A;\alpha}_{a,ij} \langle S^{V_A;\alpha}_a S^{V_B;\beta}_b \rangle_c C^{V_B;\beta}_{b,kl} \\ &= \sum_{V_A,a,\alpha,\beta} C^{V_A;\alpha}_{a,ij} \left(\Lambda^{MM}_A \right)^{-1}_{\alpha\beta} C^{V_A;\beta}_{a,kl}\end{aligned}$$

- Repeat for N

$$R_a^{V_A;\alpha} = \sum_{i,j} C_{a,ij}^{V_A;\alpha} N_{ij}$$

- The correlators are simple in this basis

$$\langle R_a^{V_A;\alpha} R_b^{V_B;\beta} \rangle_c = \left(\Lambda_A^{NN}\right)^{-1}_{\alpha\beta} \delta_{ab} \delta(V_A, V_B),$$

$$\langle S_a^{V_A;\alpha} R_b^{V_B;\beta} \rangle_c = \left(\Lambda_A^{MN}\right)^{-1}_{\alpha\beta} \delta_{ab} \delta(V_A, V_B).$$

- $\left(\Lambda_A^{MN}\right)_{\alpha\beta}$ is not a symmetric matrix.

$$2^2 + 3^2 + 1 + 1 = 15$$

- Inverting the change of basis

$$\langle N_{ij} N_{kl} \rangle_c = \sum_{V_A, a, \alpha, \beta} C_{a,ij}^{V_A;\alpha} \left(\Lambda_A^{NN}\right)^{-1}_{\alpha\beta} C_{a,kl}^{V_A;\beta}$$

$$\langle M_{ij} N_{kl} \rangle_c = \sum_{V_A, a, \alpha, \beta} C_{a,ij}^{V_A;\alpha} \left(\Lambda_A^{MN}\right)^{-1}_{\alpha\beta} C_{a,kl}^{V_A;\beta}$$

- In general

$$\langle T_{ij} T_{kl} \rangle_c = \sum_{V_A, a, \alpha, \beta} C^{V_A;\alpha}_{a,ij} \left(\Lambda^{TT}_A \right)^{-1}_{\alpha\beta} C^{V_A;\beta}_{a,kl}$$

- We need to compute the tensors

$$\sum_a C^{V_A;\alpha}_{a,ij} C^{V_A;\beta}_{a,kl}$$

- They can all be written in terms of the same tensor

$$F_{ij} = \delta_{ij} - \frac{1}{D}$$

- Also true about VEV

- We had an orthonormal basis for $V_0 \oplus V_H$

$$|E_0\rangle = \sum_{i=1}^{D} C_{0i} |i\rangle = \frac{1}{\sqrt{D}} \sum_{i=1}^{D} |i\rangle$$

$$|E_a\rangle = \sum_{i=1}^{D} C_{ai} |i\rangle \,, \;\; a = 1, \ldots, D-1$$

- F_{ij} are matrix elements of P_{V_H}

$$F_{ij} = \sum_{a=1}^{D-1} \langle i|E_a\rangle \langle E_a|j\rangle = \delta_{ij} - \frac{1}{D}.$$

- Derived from

$$\langle i|j\rangle = \delta_{ij} = \langle i| \left(|E_0\rangle \langle E_0| + \sum_{a=1}^{D-1} |E_a\rangle \langle E_a| \right) |j\rangle$$

- There are 15 combinations, here are a select few

$$\sum_a C^{V_0;2}_{a,ij} C^{V_0;2}_{a,kl} = \frac{F_{ij}F_{kl}}{D-1}$$

$$\sum_a C^{V_H;1}_{a,ij} C^{V_H;1}_{a,kl} = \frac{F_{jl}}{D}$$

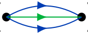
$$\sum_a C^{V_H;3}_{a,ij} C^{V_H;3}_{a,kl} = \sum_{p,q} F_{ip}F_{jp}F_{pq}F_{qk}F_{ql}$$

$$\sum_a C^{V_3;1}_{a,ij} C^{V_3;1}_{a,kl} = \frac{1}{2}\left(F_{ik}F_{jl} - F_{il}F_{jk}\right)$$

- Expectation values are weighted sums of products of F
- Some contributions to $\langle M_{i_1 j_1} M_{i_2 j_2} N_{i_3 j_3} \rangle$ are proportional to

$$F_{i_1 j_1} F_{i_2 j_2} F_{i_3 j_3},$$
$$\sum_{p,q=1}^{D} F_{i_1 p} F_{j_1, p} F_{pq} F_{i_2 q} F_{j_2 q} F_{i_3 j_3}.$$

- Observables have all external indices summed.

$\langle$  $\rangle$ $= \sum_{i,j} \langle M_{ij} M_{ij} N_{ij} \rangle$

- The F contributions become

$$\sum_{i,j=1}^{D} F_{ij} F_{ij} F_{ij},$$

$$\sum_{i,j,p,q=1}^{D} F_{ip} F_{jp} F_{pq} F_{iq} F_{jq} F_{ij}.$$

The general strategy is to decompose the sums into parts

$$\sum_{i,j=1}^{D} = \sum_{i=j} + \sum_{i\neq j},$$

$$\sum_{i,j,k=1}^{D} = \sum_{i=j=k} + \sum_{i=j\neq k} + \sum_{i=k\neq j} + \sum_{j=k\neq i} + \sum_{i\neq j\neq k}.$$

Because F_{ij} is almost like a Kronecker delta,

$$F_{ij} = \begin{cases} 1 - \frac{1}{D} & i = j \\ \;\;- \frac{1}{D} & i \neq j \end{cases}$$

it is constant on each part

Example 1

$$\sum_{i,j}(F_{ij})^A = \sum_{i=j}(F_{ij})^A + \sum_{i\neq j}(F_{ij})^A$$
$$= D\left[1-\frac{1}{D}\right]^A + D(D-1)\left[\frac{1}{(-D)}\right]^A$$
$$= \left[D(1-D)^A + D(D-1)\right]\frac{1}{(-D)^A}.$$

Example 2

$$\sum_{i_1,i_2,i_3=1}^{D} F_{i_1 i_2}^{A_{12}} F_{i_1 i_3}^{A_{13}} F_{i_2 i_3}^{A_{23}} = \left(\sum_{123} + \sum_{12|3} + \sum_{13|2} + \sum_{23|1} + \sum_{1|2|3}\right) F_{i_1 i_2}^{A_{12}} F_{i_1 i_3}^{A_{13}} F_{i_2 i_3}^{A_{23}}$$

$$= \frac{D!}{(D-1)!}\left[1-\frac{1}{D}\right]^{A_{12}+A_{13}+A_{23}} + \frac{D!}{(D-2)!}\left[1-\frac{1}{D}\right]^{A_{12}}\left[\frac{1}{-D}\right]^{A_{13}+A_{23}}$$

$$+ \frac{D!}{(D-2)!}\left[1-\frac{1}{D}\right]^{A_{13}}\left[\frac{1}{-D}\right]^{A_{12}+A_{23}} + \frac{D!}{(D-2)!}\left[1-\frac{1}{D}\right]^{A_{23}}\left[\frac{1}{-D}\right]^{A_{12}+A_{13}}$$

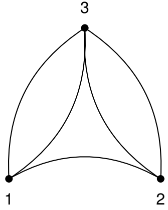
$$+ \frac{D!}{(D-3)!}\left[\frac{1}{-D}\right]^{A_{12}+A_{13}+A_{23}}$$

$$= \left[\frac{1}{-D}\right]^{A_{12}+A_{13}+A_{23}} \Big(\frac{D!(1-D)^{A_{12}+A_{13}+A_{23}}}{(D-1)!}$$

$$+ \frac{D!(1-D)^{A_{12}}}{(D-2)!} + \frac{D!(1-D)^{A_{13}}}{(D-2)!} + \frac{D!(1-D)^{A_{23}}}{(D-2)!} + \frac{D!}{(D-3)!}\Big)$$

- The procedure can be organized in terms of undirected graphs
- For example,

$$\sum_{i_1,i_2,i_3=1}^{D} F^1_{i_1 i_2} F^2_{i_1 i_3} F^2_{i_2 i_3} \longleftrightarrow$$



- Sum over p indices gives graph on p vertices
- Each factor F_{i_a,i_b} is an edge from a to b

- General products with p indices summed

$$\sum_{i_1,\ldots,i_p} \prod_{1 \le a < b \le p} F_{i_a i_b}^{A_{ab}}$$

- Parameterized by matrix A_{ab}
- Interpret A_{ab} as an adjacency matrix

- Decomposition

$$\sum_{i_1,i_2,i_3=1}^{D} = \sum_{i_1=i_2=i_3} + \sum_{i_1=i_2\neq i_3} + \sum_{i_1=i_3\neq i_2} + \sum_{i_2=i_3\neq i_1} + \sum_{i_1\neq i_2\neq i_3}$$

- Corresponds to set partitions

$$i_1 = i_2 = i_3 \leftrightarrow \{123\}, \quad i_1 = i_2 \neq i_3 \leftrightarrow \{12\},\{3\},$$
$$i_1 \neq i_2 \neq i_3 \leftrightarrow \{1\},\{2\},\{3\}$$

- Notation for sums

$$\sum_{i_1,i_2,i_3=1}^{D} = \sum_{\{123\}} + \sum_{\{12\},\{3\}} + \sum_{\{13\},\{2\}} + \sum_{\{23\},\{1\}} + \sum_{\{1\},\{2\},\{3\}}$$

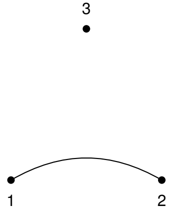
- Number of terms in sum with b blocks is

$$\frac{D!}{(D-b)!}$$

- Compute each sum using graphs operations

$$\sum_{i_1=i_2\neq i_3} F^1_{i_1 i_2} F^2_{i_1 i_3} F^2_{i_2 i_3} \leftrightarrow \sum_{\{12\},\{3\}}^{D} F^1_{i_1 i_2} F^2_{i_1 i_3} F^2_{i_2 i_3}$$

- Sum is associated with a new graph

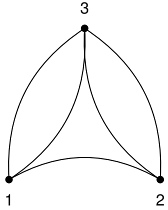


$$= \frac{D!}{(D-2)!}(1-D)^1 \frac{1}{(-D)^5}$$

- All indices equal

$$\sum_{i_1=i_2=i_3}^{D} F^1_{i_1 i_2} F^2_{i_1 i_3} F^2_{i_2 i_3} \leftrightarrow \sum_{\{123\}}^{D} F^1_{i_1 i_2} F^2_{i_1 i_3} F^2_{i_2 i_3}$$

- Associated graph contributes

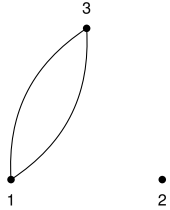


$$= \frac{D!}{(D-1)!}(1-D)^5 \frac{1}{(-D)^5}$$

- i_1 and i_3 equal

$$\sum_{i_1=i_3\neq i_2}^{D} F^1_{i_1 i_2} F^2_{i_1 i_3} F^2_{i_2 i_3} \leftrightarrow \sum_{\{13\},\{2\}}^{D} F^1_{i_1 i_2} F^2_{i_1 i_3} F^2_{i_2 i_3}$$

- Associated graph contributes

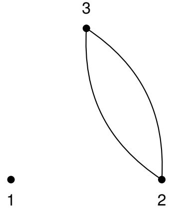


$$= \frac{D!}{(D-2)!}(1-D)^2 \frac{1}{(-D)^5}$$

- i_2 and i_3 equal

$$\sum_{i_2=i_3\neq i_1}^{D} F^1_{i_1 i_2} F^2_{i_1 i_3} F^2_{i_2 i_3} \leftrightarrow \sum_{\{23\},\{1\}}^{D} F^1_{i_1 i_2} F^2_{i_1 i_3} F^2_{i_2 i_3}$$

- Associated graph contributes

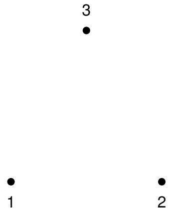


$$= \frac{D!}{(D-2)!}(1-D)^2 \frac{1}{(-D)^5}$$

- All different

$$\sum_{i_1 \neq i_2 \neq i_3}^{D} F^1_{i_1 i_2} F^2_{i_1 i_3} F^2_{i_2 i_3} \leftrightarrow \sum_{\{1\},\{2\},\{3\}}^{D} F^1_{i_1 i_2} F^2_{i_1 i_3} F^2_{i_2 i_3}$$

- Associated graph contributes



$$= \frac{D!}{(D-3)!}(1-D)^0 \frac{1}{(-D)^5}$$

Summary: Expectation Values

- Expectation values are computed using Wick's theorem
- They ultimately boil down to products of F with all indices summed
- We organized the sums using undirected graphs
- Procedure manifestly independent of D

Summary

- We proved that observables correspond to directed colored graphs
- The one-point and two-point function are simple in a representation basis
- Computing expectation values is largely about the interplay between the graph and representation basis
- The interplay can be described using undirected uncolored graphs.

Outlook

- Theory
  - Extend the permutation invariant program to tensor models
  - Correlators in the matrix models
  - Interesting parameter limits
  - Double cosets
- Applied
  - Apply 2-matrix model to data
  - Improve algorithm

Thank you for listening!