Tensor vs. vector theories

the functional renormalization group perspective

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Tensor vs. vector theories

Same universality class at large N

- vector models are branched polymers
- possibility of large-N in tensors: melons [Gurau 1011 ff.]
- "melons are branched polymers", same universality class

Field theory: RG flow in phase space

- how do they compare?
- what's the phase structure of field theory with tensorial interactions?
- quantum/random geometry interpretation: emergence of a continuum?

Caveat: two kinds of "tensor field theory"!
here: propagating tensor degrees of freedom (no background spacetime)

Non-Gaussian fixed point in tensor theory?

Various methods and claims in the literature:

- [Brezin/Zinn-Justin '92] RG for matrix model integrating out $N+1 \mapsto N$
- [Eichhorn/Koslowski 1309,1408] FRG setup and methods for matrix model
- [E/K 1710] NGFP with several relevant directions in $r=3$ sextic melonic TM
- [E/K/Lumma/Pereira 1811] NGFP in $r=4$ full sextic and [E/L/P/S 1912] octic TM

Propagating tensor fields in $\phi^{2n}_{d_{\mathrm{G}},r}$ truncation:

- [Benedetti/BenGeloun/Oriti 1411] FRG setup, non-autonomous equation $\phi^4_{1,3}$ due to compactness, phase transition only when hidden compactness scale sent to infinity
- [Benedetti/Lahoche 1508] $\phi^4_{1,6}$ with gauge constraint: Wilson-Fisher like FP
- [BenGeloun/Martini/Oriti 1508, 1601] $\phi^4_{1,r}$ on $\mathbb{R}$: Wilson-Fisher like FP
- extending truncation: $\phi^6_{1,4}$ [BGK1606], necklaces [CLO1703], tetrahedron [BGKOP1805] ...
- [Lahoche/OuSamary 1608/1803/1809/1812/1904/1908] Ward identities, eff.-vertex exp.
- [Carrozza/Lahoche 1612] $\phi^{2n}_{3,3}$ on $SU(2)$ with gauge constraint, cyclic-melonic potential approximation, stable (?) Wilson-Fisher type FP up to $n=6$

Crucial: convergence with higher truncations!

Improve the local potential approximation of ∞ many cyclic-melonic couplings!

Outline

1. FRG for $\mathrm{O}(N)$ scalar field theory
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   - The cyclic-melonic potential approximation
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Functional renormalization group

Theory at scale k given by generating function with k-dep. IR-regulator $\mathcal{R}_k$

$$e^{W_k[J]} = \int D\phi \, e^{-S[\phi]-(\phi,\mathcal{R}_k\phi)+(J,\phi)}$$

Scale-dependent effective action via Legendre transform w.r.t. $\varphi = \frac{\delta W_k[J]}{\delta J}$

$$\Gamma_k[\varphi] = \sup_J \{(J,\varphi) - W_k[J]\} - (\varphi, \mathcal{R}_k \varphi)$$

Renormalization group flow determined by functional equation [Wetterich'93, Morris'94]

$$\boxed{k\partial_k \Gamma_k[\varphi] = \frac{1}{2} \mathrm{Tr} \frac{k\partial_k \mathcal{R}_k}{\Gamma_k^{(2)}[\varphi] + \mathcal{R}_k}}$$

Flow interpolates between microscopic theory $k \to \infty$ and full quantum effective action $\Gamma = \lim_{k\to 0} \Gamma_k$

$\mathrm{O}(N)$ scalar field theory

$\mathrm{O}(N)$-symmetric scalar field theory in d dimensions in the local potential approximation (LPA, $Z_k = cons.$) parametrized by potential $U = U_k$,

$$\Gamma_k = \int \mathrm{d}^d x \left(\frac{1}{2} Z_k \partial\phi^a \partial\phi_a + U(\phi^a\phi_a) \right)$$

Projection of the flow on constant average field $\rho = \frac{1}{2}\phi^a\phi_a$:

$$k\partial_k U_k = \frac{1}{2}\mathrm{Tr}_q \frac{(N-1)\, k\partial_k \mathcal{R}_k(q)}{Z_k q^2 + \mathcal{R}_k(q) + U'} + \frac{k\partial_k \mathcal{R}_k(q)}{Z_k q^2 + \mathcal{R}_k(q) + U' + 2\rho U''}$$

$$\overset{\text{optimized } \mathcal{R}_k}{=} c_d Z_k k^{d+1} \left(\frac{N-1}{Z_k k^2 + U'} + \frac{1}{Z_k k^2 + U' + 2\rho\, U''} \right)$$

Flow equations

Rescaling $u = U/c_d Z_k k^d$ and $\rho = \frac{1}{2} Z_k k^{2-d} \phi^a \phi_a$ by their canonical dimension:

$$\boxed{k\partial_k u + du - (d-2)\rho u' = \frac{N-1}{1+u'} + \frac{1}{1+u'+2\rho\, u''}}$$

Flow equations for couplings $\tilde{\lambda}_n$ in $u(\rho) = \sum \frac{\tilde{\lambda}_n}{n!}\rho^n$ from Taylor exp. at $\rho = 0$:

$$k\partial_k \tilde{\lambda}_n + d\tilde{\lambda}_n - n(d-2)\tilde{\lambda}_n = (N-1)\beta^n(\tilde{\lambda}_i) + \bar{\beta}^n(\tilde{\lambda}_i)$$

Infinite tower of coupled algebraic equations of order $n+1$

- $\beta^n(\tilde{\lambda}_i)$ Taylor coeff. of $\frac{1}{1+u'}$ of the form $\frac{1}{(1+\tilde{\mu})^{n+1}} Pol^{(n)}(\tilde{\mu}, \tilde{\lambda}_2, ..., \tilde{\lambda}_{n+1})$
- $\bar{\beta}^n(\tilde{\lambda}_i) = \beta^n((2i-1)\tilde{\lambda}_i)$ Taylor coeff. of $\frac{1}{1+u'+2\rho u''}$

$\rightarrow$ approximate solutions by truncation at a give order n, but most do not converge with n!

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Large-N solutions

In the large-N limit, the u'' term vanishes:

$$k\partial_k u + du - (d-2)\rho u' = \frac{1}{1+u'}$$

Flow equations for vacuum expansion $u(\rho) = \sum_{n\geq 2} \frac{g_n}{n!}(\rho - \kappa)^n$ decouple:

$$\partial_t \kappa + (d-2)\kappa = 1$$
$$\partial_t g_n + dg_n + (d-2)ng_n = \beta^n(0, g_2, ..., g_n, g_{n+1} = 0)$$

$\rightarrow$ exact recursive fixed point solution ($\partial_t g_n = 0$)

$$\kappa^* = \frac{1}{d-2}, \quad g_2^* = \frac{4-d}{2}, \quad g_3^* = \frac{3}{4}\frac{(d-4)^3}{d-6}, \ldots$$

the Wilson-Fisher fixed point

- non-vanishing vacuum (κ local minimum) for $2 < d < 4$
- converges to the Gaussian fixed point $u = 0$ for $d \to 4$
- scaling exponents $\theta_i = d - 2i$, only $\theta_1 > 0$

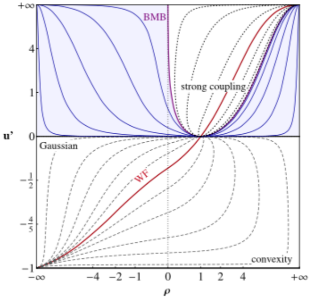
Exact solutions

Possible to solve the flow for u' exactly with method of characteristics:
[Busiello/DeCesare/Rabuffo '81]...[Litim/Tetradis 9501]

$$k\partial_k u' + 2u' - (d-2)\rho u'' = \frac{u''}{(1+u')^2}$$

has implicit 1-parameter fixed-point solutions

$$\rho = \frac{1}{d-2}\,{}_2F_1\left(2, \frac{2-d}{2}, \frac{4-d}{2}, -u'\right) + c(u')^{\frac{d-2}{2}}$$



[Litim/Marchais/Mati 1702]

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FRG Setup for dynamic tensor fields

Tensor field $\phi : G^{\times r} \to \mathbb{C}/\mathbb{R}$ on compact G with explicit curvature scale a:
[straightforward, first done in [Benedetti/BenGeloun/Oriti 1411]

$$(\phi, \phi') = \int_{G^{\times r}} \mathrm{d}\boldsymbol{g}\, \bar{\phi}(\boldsymbol{g})\phi'(\boldsymbol{g}) \quad , \quad \int_G \mathrm{d}g = a$$

Tensorial interactions: unitary transformations $U^c : L^2(G) \to L^2(G)$:

$$\phi(\boldsymbol{g}) \mapsto \left(\bigotimes_{c=1}^{r} U^c\, \phi \right)(\boldsymbol{g}) = \int \mathrm{d}h_1 ... \mathrm{d}h_r \prod_{c=1}^{r} U^c(g_c, h_c)\phi(h_1, ..., h_r)$$

Theory space labelled by (bipartite) $r+1$ edge-coloured graphs b:

$$\Gamma_k[\bar{\phi}, \phi] = (\phi, \mathcal{K}_k \phi) + \sum_{b\in B} \lambda_{b,k} \mathrm{Tr}_b[\phi, \bar{\phi}] \quad , \quad \mathcal{K}_k = (-1)^{d_{\mathrm{G}}} Z_k \Delta + \mu_k$$

Supertrace for complex field

Hessian is 2×2 matrix with respect to ϕ and $\bar{\phi}$ [overlooked in literature so far],

$$\left(\Gamma_k^{(2)} + \mathcal{R}_k \mathbb{I}_2\right) = \begin{pmatrix} P_{\mathrm{R}} + F & F_{12} \\ F_{21} & P_{\mathrm{R}} + F \end{pmatrix}$$

with effective propagator $P_{\mathrm{R}} = (-1)^{d_{\mathrm{G}}} Z_k \Delta + \mu_k + \mathcal{R}_k$ and

$$F = \frac{\delta^2 \Gamma_k[\phi, \bar{\phi}]}{\delta\phi(\boldsymbol{g})\delta\bar{\phi}(\boldsymbol{h})}, \quad F_{12} = \frac{\delta^2 \Gamma_k[\phi, \bar{\phi}]}{\delta\phi(\boldsymbol{g})\delta\phi(\boldsymbol{h})} \quad , \quad F_{21} + \frac{\delta^2 \Gamma_k[\phi, \bar{\phi}]}{\delta\bar{\phi}(\boldsymbol{g})\delta\bar{\phi}(\boldsymbol{h})}$$

if invertible (possible in momentum space):

$$k\partial_k \Gamma_k = \frac{1}{2} \overline{\mathrm{Tr}} \frac{k\partial_k \mathcal{R}_k \mathbb{I}_2}{\Gamma_k^{(2)} + \mathcal{R}_k \mathbb{I}_2} = \frac{1}{2} \mathrm{Tr}_{\hat{G}^r} \frac{2(P_{\mathrm{R}} + F)\, k\partial_k \mathcal{R}_k}{(P_{\mathrm{R}} + F)^2 - F_{12} F_{21}}$$

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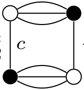
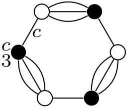
Cyclic-melonic interactions

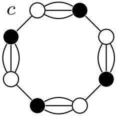
Interactions parametrized by a polynomial function: [Carrozza/Lahoche 1612]

$$\sum_{b \in B_{\text{cyc-mel}}} \lambda_{b,k} \mathrm{Tr}_b[\phi, \bar{\phi}] = \sum_{c=1}^{r} \mathrm{Tr}_G V_k^c(\bar{\phi} \cdot_{\hat{c}} \phi) \quad , \quad V_k^c(z) = \sum_{n=2}^{\infty} \frac{\lambda_{n,k}^c}{n!} z^n$$

Diagrammatically:

$\mathrm{Tr}_G V_k(\bar{\phi} \cdot_{\hat{c}} \phi) = \lambda_2^c$ 

$+\lambda_3^c$ 

$+\lambda_4^c$ 

$+ \dots$

Hessian: sum over choices of 2 vertices – multi-edge pair: melonic contribution

$$F[\phi, \bar{\phi}](\boldsymbol{g}, \boldsymbol{h}) = \sum_{c=1}^{r} \sum_{n=2}^{\infty} \frac{n}{n!} \lambda_n^c \Bigg[\sum_{p=1}^{n-2} (\bar{\phi} \cdot_{\hat{c}} \phi)^p (g^c, h^c) (\bar{\phi} \cdot_c \phi)^{n-p-1} (\hat{\boldsymbol{g}}_c, \hat{\boldsymbol{h}}_c)$$
$$+ \delta(g_c, h_c) (\bar{\phi} \cdot_c \phi)^{n-1} (\hat{\boldsymbol{g}}_c, \hat{\boldsymbol{h}}_c) + \prod_{b \neq c} \delta(g_b, h_b) \, (\bar{\phi} \cdot_{\hat{c}} \phi)^{n-1} (g_c, h_c) \Bigg].$$

Constant-field projection

Projection to constant field

$$\rho := (\phi, \phi) = a^r \, \bar{\phi}\phi \, .$$

allows to express everything in terms of potentials V_k^c:

$$F[\bar{\phi}, \phi](\boldsymbol{g}, \boldsymbol{h}) = a^{-r} \sum_{c=1}^{r} \left[\left(\prod_{b \neq c} a\delta(g_b, h_b) + a\delta(g_c, h_c) - 1 \right) V_k^{c\prime}(\rho) + \rho V_k^{c\prime\prime}(\rho) \right],$$

Combinatorial non-locality captured by operator

$$\mathcal{O}^c(\boldsymbol{g}, \boldsymbol{g}') := \prod_{b \neq c} a\delta(g_b, h_b) + a\delta(g_c, h_c) - 1$$

In momentum (representation) space

$$\mathcal{O}^c_{\boldsymbol{j}} := \delta_{0 j_c} + (1 - \delta_{0 j_c}) \prod_{b \neq c} \delta_{0 j_b}$$

Projected FRG equation

FRG equation for $u(\rho) := \mu_k \rho + \sum_{c=1}^{r} V_k^c(\rho)$ similar to $\mathrm{O}(N=2)$ theory

$$k\partial_k u(\rho) = \frac{1}{2}\mathrm{Tr}_{\hat{G}^r} \frac{k\partial_k \mathcal{R}_k}{P_{\mathrm{R}} + \sum_c \mathcal{O}_{\boldsymbol{j}}^c V_k^{c\prime}(\rho)} + \frac{k\partial_k \mathcal{R}_k}{P_{\mathrm{R}} + \sum_c \mathcal{O}_{\boldsymbol{j}}^c V_k^{c\prime}(\rho) + 2\rho \left(\prod_c \delta_{0 j_c}\right) \sum_c V_k^{c\prime\prime}(\rho)}$$

(Same analysis for real ϕ yields only second term)

To evaluate trace choose

- $G = U(1)$
- kinetic term $\frac{1}{a^2} C_{\boldsymbol{j}} = \frac{1}{a^{2\zeta}} \sum_{c=1}^{r} |j_c|^{2\zeta}$
- optimized regulator $\mathcal{R}_k = Z_k \left(k^{2\zeta} - a^{-2\zeta} C_{\boldsymbol{j}}\right) \theta\left(k^{2\zeta} - a^{-2\zeta} C_{\boldsymbol{j}}\right)$

Relation to Tensor models: $N_k := ak$

- trace $\mathrm{Tr}_{\hat{G}^r}$ becomes $\prod_{c=1}^{r} \sum_{j_c \in \mathbb{Z}} \theta(N_k^{2\zeta} - \sum |j_c|^{2\zeta})$
- scaling in $\mathcal{R}_k$ fixed naturally by field propagation (no freedom [Eichhorn et. al.])
- crucial difference: $\mathcal{K} = Z_k \Delta + \mu$, not just $\mathcal{K} = Z_k$

Full equation

Identifying $V_k^c = V_k/r$ allows to calculate full FRG equation for potential

$$U_k = \mu_k \rho + V_k(\rho) = \mu_k \rho + \sum_{n=2}^{\infty} \frac{\lambda_n}{n!} \rho^n$$

Non-autonomous FRG equation for real/complex ϕ ($N_\phi = 1, 2$) at arbitrary r:
[as expected from Benedetti/BenGeloun/Oriti 1411]

$$\begin{aligned} k\partial_k U_k(\rho) = \left(\zeta - \frac{\eta_k}{2}\right) k^{2\zeta} Z_k \Bigg(& \frac{1}{k^{2\zeta} Z_k + U_k'(\rho) + 2\rho U_k''(\rho)} \\ & + \frac{N_\phi - 1 + r N_\phi I_0^{(1)}(N_k)}{k^{2\zeta} Z_k + U_k'(\rho)} + N_\phi \sum_{s=2}^{r} \binom{r}{s} \frac{I_0^{(s)}(N_k)}{k^{2\zeta} Z_k + M_k^{(s)}(\rho)} \Bigg) \\ & + \frac{\eta_k}{2} k^{2\zeta} Z_k \frac{1}{N_k^{2\zeta}} \left(\frac{r N_\phi I_{2\zeta}^{(1)}(N_k)}{k^{2\zeta} Z_k + U_k'(\rho)} + N_\phi \sum_{s=2}^{r} \binom{r}{s} \frac{s I_{2\zeta}^{(s)}(N_k)}{k^{2\zeta} Z_k + M_k^{(s)}(\rho)} \right) \end{aligned}$$

Different relative weight between mass and couplings at order s: effective mass

$$M_k^{(s)}(\rho) := \mu_k + \frac{r-s}{r} V_k'(\rho)$$

Threshold functions

$$I_\gamma^{(s)}(N_k) = \left(\prod_{c=1}^{r} \sum_{j_c \in \mathbb{Z}\setminus\{0\}}\right) |j_c|^\gamma \, \theta\left(N_k^{2\zeta} - \sum_c |j_c|^{2\zeta}\right) .$$

- "box" scheme: approximation by a sum over the hypercube
- "integral" scheme: approximation, sufficient in the $N_k \to \infty$ limit,
- "simplex" scheme: exact trace for $\zeta = 1/2$

$$I_0^{(s)}(N_k) = \begin{cases} (2N_k)^s & \text{box} \\ v_s^{(\zeta)} N_k^s & \text{integral, } v_s^{(\zeta)} := \frac{(2\Gamma(1+1/2\zeta))^s}{\Gamma(1+s/2\zeta)} \\ \frac{2^s}{s!}\frac{\Gamma(N_k+1)}{\Gamma(N_k+1-s)} & \text{simplex} \end{cases}$$

$$I_{2\zeta}^{(s)}(N_k) = \begin{cases} 2H_{N_k}^{(-2\zeta)}(2N_k)^{s-1} & \text{box} \\ \frac{v_s^{(\zeta)}}{s+2\zeta} N_k^{s+2\zeta} & \text{integral} \\ \frac{2^s}{(s+1)!}\prod_{i=0}^{s}(N_k+i) & \text{simplex, } \zeta = 1/2 \\ \frac{2^s}{(s+2)!}(2N_k+s)\prod_{i=0}^{s}(N_k+i) & \text{simplex, } \zeta = 1 \end{cases}$$

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Non-autonomous beta equations

Full FRG equation is not tractable → flow equation for couplings:

$$\frac{k\partial_k \lambda_n}{Z_k k^{2\zeta}} = \left(\zeta - \frac{\eta_k}{2}\right) \bar{\beta}^n(\mu_k, \lambda_i) + N_\phi \sum_{l=1}^{n} \left(\zeta F_r^l(N_k) - \frac{\eta_k}{2} G_r^l(N_k)\right) \beta_l^n(\mu_k, \lambda_i)$$

with expansion of Taylor coefficients due to effective mass:

$$\beta^n\left(\mu_k, \frac{r-s}{r}\lambda_i\right) = \sum_{l=1}^{n} \left(\frac{r-s}{r}\right)^l \beta_l^n(\mu_k, \lambda_i)$$

Non-autonomous part at order l:

$$F_r^l(N_k) := \frac{N_\phi - 1}{N_\phi} + 2rN_k + \frac{1}{r^l} \sum_{s=2}^{r} \binom{r}{s} (r-s)^l \, I_0^{(s)}(N_k)$$

$$G_r^l(N_k) := F_r^l(N_k) - rN_k^{-2\zeta} I_{2\zeta}^{(1)}(N_k) - \frac{1}{r^l} \sum_{s=2}^{r} \binom{r}{s} (r-s)^l s \, N_k^{-2\zeta} I_{2\zeta}^{(s)}(N_k)$$

Rescaling and dimension

Rescaling: at scale N_k where a given order $N_k^{d_{\text{eff}}}$ is dominant:

$$\lambda_n = Z_k^n k^{d_{\text{eff}}-(d_{\text{eff}}-2\zeta)n} a^{(1-n)d_{\text{eff}}} \tilde{\lambda}_n$$

[Possibility of consistent rescaling in k and a first explored in BenGeloun/Martini/Oriti 1601]

$$k\partial_k \tilde{\lambda}_n + d_{\text{eff}}\tilde{\lambda}_n - n(d_{\text{eff}} - 2\zeta + \eta_k)\tilde{\lambda}_n$$
$$= \left(\zeta - \frac{\eta_k}{2}\right) \frac{1}{N_k^{d_{\text{eff}}}} \bar{\beta}^n(\tilde{\mu}, \tilde{\lambda}_i) + N_\phi \sum_{l=1}^{n} \left(\zeta \frac{F_r^l(N_k)}{N_k^{d_{\text{eff}}}} - \frac{\eta_k}{2} \frac{G_r^l(N_k)}{N_k^{d_{\text{eff}}}}\right) \beta_l^n(\tilde{\mu}, \tilde{\lambda}_i)$$

Small-N_k limit: $d_{\text{eff}} = 0$

Confirms the argument in [Benedetti/BenGeloun/Oriti 1411] based on [Benedetti 1403]

Large-N_k limit: $d_{\text{eff}} = r - 1$

- stems from melonic single-$\delta_{j_c,0}$ contributions

Consistent with renormalization: eff. dimension $d_r = d_{\text{G}}(r-1)$

- vertex weights $d_r - n(d_r - 2\zeta)$
- divergence degree $\omega^{\text{s.d.}} = d_r - \frac{d_r - 2\zeta}{2} N - d_{\text{G}}\left(\delta_{\text{Gurau}} + K_\partial - 1\right)$
  [Carrozza/Oriti/Rivasseau 1303], [OusmaneSamary/Vignes-Tourneret 1211], [BenGeloun 1306]...

Dimensional flow

Continuous rescaling: [improves ideas in Benedetti/BenGeloun/Oriti 1411]

$$\lambda_n = Z_k^n k^{2\zeta n} F_r^1(ak)^{1-n}\tilde{\lambda}_n.$$

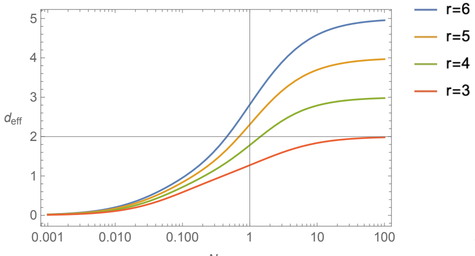
effective dimension

$$d_{\mathrm{eff}}(k) := \frac{\partial \log F_r^1(N_k)}{\partial \log N_k}$$

full FRG equations

$$k\partial_k\tilde{\lambda}_n = -\, d_{\mathrm{eff}}(k)\tilde{\lambda}_n + n(d_{\mathrm{eff}}(k) - 2\zeta + \eta_k)\tilde{\lambda}_n$$
$$+ \left(\zeta - \frac{\eta_k}{2}\right)\frac{\bar{\beta}^n(\tilde{\lambda}_i)}{F_r^1(N_k)} + N_\phi \sum_{l=1}^{n}\left(\zeta\frac{F_r^l(N_k)}{F_r^1(N_k)} - \frac{\eta_k}{2}\frac{G_r^l(N_k)}{F_r^1(N_k)}\right)\beta_l^n(\tilde{\lambda}_i)$$

- dimension changes continuously through scales
- d_{eff} is NOT the spectral dim. of the generated geometry!
- tensorial theory behaves like a d_{eff}-dim. local field theory



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No phase transition in TGFT with fixed curvature a

In given regime, rederive equation for potential from beta equations:

Small-N_k: equivalence to O(N_ϕ) theory in $d = 0$

Small-N_k limit cyclic-melonic potential approximation gives exactly

$$\frac{1}{\zeta} k\partial_k u(\rho) + 2\rho\, u'(\rho) = \frac{N_\phi - 1}{1 + u'(\rho)} + \frac{1}{1 + u'(\rho) + 2\rho\, u''(\rho)}$$

Consequence for theory with fixed curvature scale a: small-N_k is small-k limit

- UV fixed points do not persist to IR
- no phase transition for $d_{\text{eff}} < 2$ (Mermin-Wagner theorem)
- **universal symmetry restoration** due to compactness of G [Benedetti 1403]
- cyclic-melonic potential approximation not valid in IR, BUT
- result independent of specific interactions (r-fold zero mode always there)

Way out: $a \to \infty$ limit

Then large-N_k equations valid at any k! Only melonic (single-δ) contributions

$$k\partial_k\bar{\mu} - (-2\zeta + \eta_k)\,\bar{\mu} = r\left(1 - \frac{\eta_k}{d_r + 2\zeta}\right)\frac{-\bar{\lambda}_2}{(1+\bar{\mu})^2}$$

$$k\partial_k\bar{\lambda}_n + d_r\bar{\lambda}_n - (d_r - 2\zeta + \eta_k)n\bar{\lambda}_n = \left(1 - \frac{\eta_k}{d_r + 2\zeta}\right)\beta^n(\bar{\mu}, \bar{\lambda}_i)$$

independent of N_ϕ (upon rescaling constant $c_{d_r} = \zeta v_{d_r}^{(\zeta)} N_\phi$)

[agrees with and generalizes equations in BenGeloun/Martini/Oriti 1601]

Equivalence to O(N) theory in d_r dimensions for $N \to \infty$

up to the relative factor r between mass and couplings

$$\left(k\partial_k + d_r - (d_r - 2\zeta + \eta_k)\,\rho\partial_\rho\right)\left(\frac{1}{r}\bar{\mu}\rho + \bar{V}(\rho)\right) = \frac{1 - \frac{\eta_k}{d_r+2\zeta}}{1 + \bar{\mu} + \bar{V}_k'(\rho)}$$

Flow of anomalous dimension

There is one other difference: no argument that $\eta_k = -k\partial_k \log Z_k$ vanishes...

$$\eta_k = \frac{d_r \bar{\lambda}_2}{-2\zeta(1+\bar{\mu})^2 + \bar{\lambda}_2}\,.$$

"$-$" sign in tensorial flow of η_k is unusual!

Understanding large-a tensorial theory in the cyclic-melonic LPA($'$)

- first expectation: similar result to large-N_k O(N) theory in d_r dim.
- what's the effect of the r factor (LPA)?
- what's the effect of the specific flow of η_k (LPA$'$)?

Outline

Critical dimension

At origin of phase space ("Gaussian fixed point", GFP):
Scaling exponents given by scaling dimension $d_r - n(d_r - 2\zeta)$ (vertex weights)

$$\rightarrow \text{ critical dimension } \quad d_r = d_{\text{crit}} = 4\zeta$$

UV perspective:

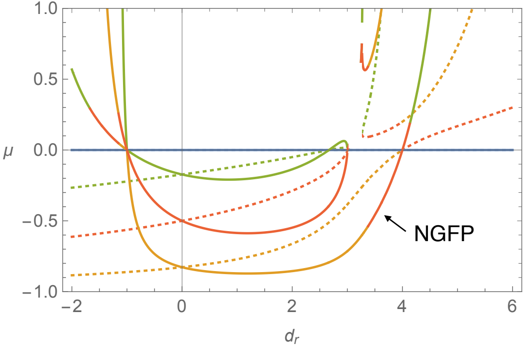
- just-renormalizable quartic theory at $d_{\text{crit}} = 4\zeta$
- for $d_r > d_{\text{crit}}$ only one UV-relevant (renormalizable) direction
- $\rightarrow$ quantum triviality

IR perspective (phase transitions):

- for $d_r > d_{\text{crit}}$ IR-attractive surface of codimension one
- splits the phase space in two regions
- $\rightarrow$ phase transition, can be described by mean field theory

Wilson-Fisher fixed point in the LPA

Wilson-Fisher fixed point: non-Gaussian fixed point with one UV-relevant direction, continuously connected in d_r to the GFP at $d_r = d_{\rm crit}$



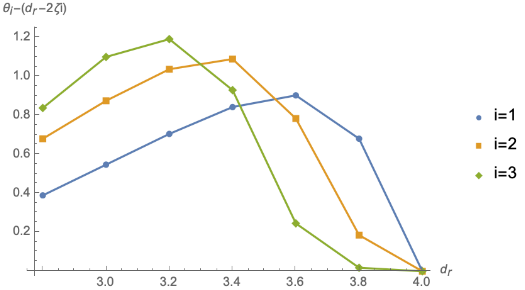
- no decoupling of equations due to factor r
- only quantitative modification due to factor r
- phase transition for $a \to \infty$ in LPA

Stability of the NGFP, $r = d_r + 1 = 4$ ($\zeta = 1$):

n	$10\tilde{\mu}$	$10^2\bar{\lambda}_2$	$10^3\bar{\lambda}_3$	$10^4\bar{\lambda}_4$	$10^5\bar{\lambda}_5$	$10^6\bar{\lambda}_6$	$10^7\bar{\lambda}_7$	$10^8\bar{\lambda}_8$	$10^9\bar{\lambda}_9$	$10^{10}\bar{\lambda}_{10}$
6	-7.1817	2.8522	3.5074	3.7706	1.3424	-6.2297				
7	-7.1720	2.8680	3.5233	3.7406	1.0193	-8.3707	-17.591			
8	-7.1740	2.8647	3.5200	3.7469	1.0866	-7.9239	-13.910	41.128		
9	-7.1751	2.8630	3.5182	3.7503	1.1232	-7.6812	-11.912	63.425	304.07	
10	-7.1750	2.8631	3.5184	3.7501	1.1205	-7.6994	-12.062	61.750	281.24	-358.82
11	-7.1749	2.8633	3.5186	3.7497	1.1167	-7.7245	-12.268	59.449	249.87	-851.88
12	-7.1749	2.8633	3.5186	3.7497	1.1166	-7.7252	-12.274	59.384	248.98	-865.87

Quantitative difference to $\mathrm{O}(N)$ theory

Scaling exponents $\theta_i = d_r - 2\zeta i + \delta\theta_i(r,\zeta)$



Stability of scaling exponents, $r = d_r + 1 = 4$ ($\zeta = 1$):

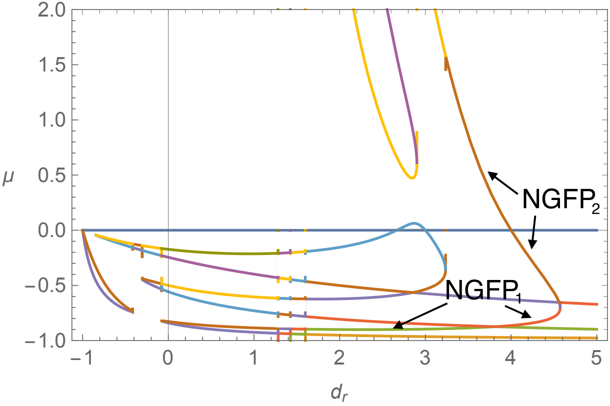
n	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}
6	0.44448	-1.9006	-6.1670	-11.553	-16.454	-28.527				
7	0.45290	-1.8256	-4.7984	-9.8777	-13.603	-21.312	-34.652			
8	0.45314	-1.8669	-4.1832	-8.2540	-12.239	-17.179	-26.712	-41.022		
9	0.45218	-1.8834	-4.0306	-7.0618	-11.165	-14.647	-21.814	-32.301	-47.464	
10	0.45205	-1.8787	-4.0690	-6.3878	-10.063	-13.168	-18.442	-26.782	-38.014	-53.954
11	0.45214	-1.8757	-4.1043	-6.1630	-9.0992	-12.228	-16.073	-22.864	-31.940	-43.840
12	0.45217	-1.8761	-4.1011	-6.1886	-8.4649	-11.474	-14.452	-19.951	-27.551	-37.247

Outline

1. FRG for $\mathrm{O}(N)$ scalar field theory
   - FRG equations
   - Wilson-Fisher fixed point
2. FRG in the cyclic-melonic potential approximation
   - FRG for tensorial interactions
   - The cyclic-melonic potential approximation
3. Rescaling and dimension
   - Beta equations and rescaling
   - Equivalence to $\mathrm{O}(N)$ field theory
4. Wilson-Fisher fixed point
   - Existence in the LPA
   - Indications in the LPA'

A peculiar anomalous dimension

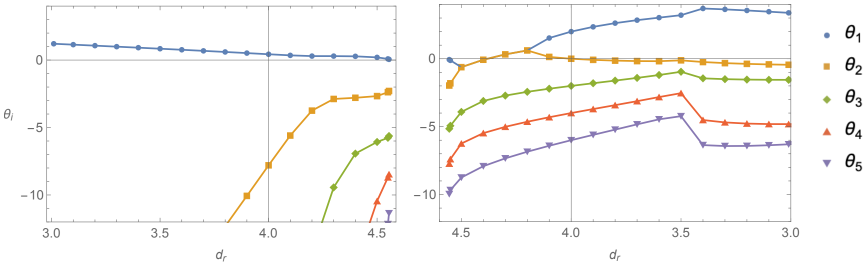
Tensor-specific flow of anomalous dimension η changes the phase structure:
mass parameter as function of dimension in n=4 truncation ($\zeta = 1$):



- solution with two branches $\rightarrow$ **two non-Gaussian fixed points**
- agrees with ϵ expansion around $d_r = d_{\text{crit}}$ at $n = 2$ in [Benedetti/Lahoche 1508]

Scaling exponents

Scaling exponents of lower branch / upper branch ($\zeta = 1$, $n = 10$ truncation)

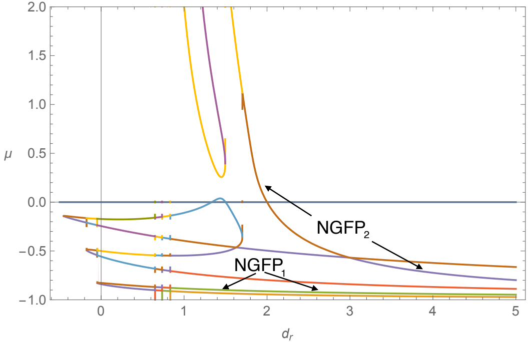


- NGFP_1 is of Wilson-Fisher type, occurs for $d_r < d_\bullet$ (branching point)
- hint for Wilson-Fisher FP at $d_r = d_{\text{crit}}$ [Benedetti/Lahoche 1508], [BenGeloun/Martini/Oriti 1601] – but convergence unclear...
- NGFP_2 exists only for $d_r >\approx 3.5$ – no second NGFP for $d_r < d_{\text{crit}}$!
- asymptotic-safe NGFP close to $d_r > d_{\text{crit}}$? But no integer rank r...

Asymptotic safety?

For $\zeta < 1$ the branching point is shifted to higher dimension:

- branching point for $\zeta = 1/2$ around $d_\bullet \approx 100$
- NGFP_2 with two relevant directions at finite rank
- converges



scaling exponents at the NGFP_2 for $d_r = 3 > d_{\text{crit}}$ with $\zeta = 1/2$:

n	$\theta_{1/2}$	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}
6	$0.24545 \pm 1.33237i$	-2.3665	-4.8104	-9.1162	-16.366				
7	$0.24662 \pm 1.33276i$	-2.4041	-4.3120	-7.2845	-12.247	-20.146			
8	$0.24671 \pm 1.33264i$	-2.4115	-4.2725	-6.3724	-9.9634	-15.513	-23.995		
9	$0.24663 \pm 1.33264i$	-2.4074	-4.3130	-6.0972	-8.6451	-12.814	-18.883	-27.893	
10	$0.24664 \pm 1.33265i$	-2.4073	-4.3116	-6.1298	-8.0245	-11.124	-15.803	-22.338	-31.831
11	$0.24664 \pm 1.33265i$	-2.4076	-4.3078	-6.1566	-7.9189	-10.141	-13.775	-18.904	-25.863
12	$0.24664 \pm 1.33265i$	-2.4076	-4.3083	-6.1486	-7.9787	-9.7579	-12.459	-16.564	-22.099

Summary

Insights:

- tensor models and field theory related via $N_k = a \cdot k$
- field theory allows to find interesting phase structure for $a \to \infty$
- tensorial field theory behaves like local field theory in $d = d_r = d_{\mathrm{G}}(r-1)$
- equivalence to large-N $\mathrm{O}(N)$ scalar field theory in the cyclic-melonic LPA
- tensor-specific flow of η deforms the Wilson-Fisher FP curve:
  - hint for Wilson Fisher FP beyond critical dimension
  - room for asymptotic safety

Outlook

Cyclic-melonic approximation is only a first step:

- analytic methods necessary for more precise statements about NGFPs
- extend ∞ truncation: all melonic – multi-traces – ...
- non-compact groups, realistic models of quantum gravity
- insights from relation to Dyson-Schwinger equations?

Thanks for your attention!