From HARMONIC OSCILLATORS *to* QUANTUM GRAVITY

🙂 🙂 **John R . Klauder** 🙂 🙂

CANONICAL QUANTIZATION – 1

Classical physics

$$p\,,q \in \mathbb{R}\,,\ \{q,p\} = 1\,,\quad H(p,q) = [p^2 + q^2]/2$$

$$A = \int [p(t)\,\dot{q}(t) - H(p(t),q(t))]\,dt$$

Favored variables

$$p \to P\ (=P^\dagger)\quad,\quad q \to Q\ (=Q^\dagger)\qquad;\qquad [Q,P] = i\hbar I$$

Schrödinger′s representation

$$Q \to x \in \mathbb{R}\quad,\quad P \to -\,i\hbar(\partial/\partial\,x)$$

Schrödinger′s equation

$$i\hbar\ \partial\,\psi(x,t)/\partial\,t\ =\ H(-i\hbar\,\partial/\partial\,x,x)\ \psi(x,t)$$

$$\int_{-\infty}^{\infty} |\,\psi(x,t)\,|^2\,dx < \infty$$

CANONICAL QUANTIZATION – 2

Classical physics

$$-\infty < p < \infty \quad , \quad \underline{0 < q < \infty} \quad , \quad H(p,q) = [p^2 + q^2]/2$$

$$A = \int [p(t)\,\dot{q}(t) - H(p(t), q(t))]\,dt$$

Favored variables

$$p \to \underline{P\ (\neq P^\dagger)} \quad , \quad q \to Q\ (= Q^\dagger) \qquad ; \qquad [Q, P] = i\hbar I$$

Hamiltonian operator(s)

$$H_0 = [PP^\dagger + Q^2]/2 \neq [P^\dagger P + Q^2]/2 = H_1$$

Hamiltonian spektra

$$E_0 = \hbar[(0,2,4,6,...) + 1/2], \quad E_1 = \hbar[(1,3,5,7,...) + 1/2]$$

$$E = \hbar[(0,2,3,5,7,8,10,...) + 1/2]$$

CANONICAL QUANTIZATION FAILS

AFFINE QUANTIZATION – 1

Classical variables

$$-\infty < p < \infty \;\;,\; 0 < q < \infty\;, \qquad H' = [(pq)^2/q^2 + q^2]/2$$

$$A = \int \{p(t)q(t)[\dot{q}(t)/q(t)] - H'(p(t)q(t), q(t))\}\, dt$$

Favored variables

$$p \to P\;(\neq P^\dagger) \quad , \quad q \to Q\;(=Q^\dagger) > 0$$

$$pq \to (P^\dagger Q + QP)/2 \equiv D\;(=D^\dagger) \quad , \quad [Q, D] = i\hbar Q$$

Schrödinger′s representation

$$Q \to x \in I\!R^+ \quad , \quad D \to -i\hbar[(\partial/\partial x)x + x(\partial/\partial x)]/2$$

Schrödinger′s equation

$$i\hbar\, \partial\psi(x,t)/\partial t \;=\; H'(-i\hbar[(\partial/\partial x)x + x(\partial/\partial x)]/2,\, x)\; \psi(x,t)$$

$$D\,x^{-1/2} = 0 \quad , \qquad \int_0^\infty |\psi(x,t)|^2\, dx < \infty$$

AFFINE QUANTIZATION – 2

$$half - harmonic\ oscillator \quad (m = 1)$$

$$H(p,q) = (p^2 + \omega^2 q^2)/2 \ , \qquad 0 < q < \infty$$

$$H'(pq,q) = ((pq)^2/q^2 + \omega^2 q^2)/2$$

$$\mathscr{H}'(D,Q) = (DQ^{-2}D + \omega^2 Q^2)/2 \quad , \qquad Q \to x > 0$$

$$= [-\hbar^2(x\,\partial/\partial x + 1/2)x^{-2}(x\,\partial/\partial x + 1/2) + \omega^2 x^2]/2$$

$$= [-\hbar^2\partial^2/\partial x^2 + (3/4)\hbar^2/x^2 + \omega^2 x^2]/2$$

Special Result : **The eigenvalues are equally spaced**! 🙂

$$E_n = 2(n+1)\hbar\omega \ , \ n = 0,1,2,...$$

Laure Gouba , *arXiv* : 2005.08696 🙂

FAVORED COORDINATES – 1

Dirac: "Cartesian coordinates should lead to $\mathscr{H}(p,q) = H(p,q)$

$$|p,q\rangle = e^{-iqP/\hbar}\, e^{ipQ/\hbar}\,|\omega\rangle \quad , \qquad \langle\omega|(Q + iP/\omega)|\omega\rangle = 0$$

$$H(p,q) = \langle p,q|\mathscr{H}(P,Q)|p,q\rangle \ ,$$

$$= \langle\omega|\mathscr{H}(P+p, Q+q)|\omega\rangle = \mathscr{H}(p,q) + \mathscr{O}(\hbar;p,q)$$

$$2\hbar[\,\|d|p,q\rangle\|^2 - |\langle p,q|d|p,q\rangle|^2] = \omega^{-1}dp^2 + \omega\, dq^2$$

0

$$|p;q\rangle = e^{ipQ/\hbar}\, e^{-i\ln(q)D/\hbar}\,|b\rangle \quad , \qquad \langle b|[(Q - \mathbb{1}) + iD/b]|b\rangle = 0$$

$$H'(pq,q) = \langle p;q|\mathscr{H}'(D,Q)|p;q\rangle \ ,$$

$$= \langle b|\mathscr{H}'(D + pqQ, qQ)|b\rangle = \mathscr{H}'(pq,q) + \mathscr{O}'(\hbar;p,q)$$

$$2\hbar[\,\|d|p;q\rangle\|^2 - |\langle p;q|d|p;q\rangle|^2] = b^{-1}q^2dp^2 + bq^{-2}dq^2$$

$-2/b$

🙂 **CQ → flat** , 🙂 **AQ → constant negative curvature**

AFFINE QUANTIZATION – 3

Action for Schrödinger′s equation

$$A_Q = \int \langle\psi(t)\,|\,[i\hbar(\partial/\partial t) - \mathscr{H}(P,Q)]\,|\,\psi(t)\rangle\; dt$$

$$A'_Q = \int \langle\psi(t)\,|\,[i\hbar(\partial/\partial t) - \mathscr{H}'(D,Q)]\,|\,\psi(t)\rangle\; dt$$

Canonical coherent states

$$|\,p,q\rangle = e^{-iqP/\hbar}\,e^{ipQ/\hbar}\,|\,\omega\rangle \qquad\qquad |\,p;q\rangle = e^{ipQ/\hbar}e^{-i\ln(q)D/\hbar}\,|\,b\rangle$$

Action for enhanced classical equations

$$A_C = \int \langle p(t),q(t)\,|\,[i\hbar(\partial/\partial t) - \mathscr{H}(P,Q)]\,|\,p(t),q(t)\rangle\; dt$$

$$A'_C = \int \langle p(t);q(t)\,|\,[i\hbar(\partial/\partial t) - \mathscr{H}'(D,Q)]\,|\,p(t);q(t)\rangle\; dt$$

$$A_c = \int \{p(t)\dot{q}(t) - H(p(t),q(t))\}\; dt\;, \qquad\qquad A'_c = \int \{-q(t)\dot{p}(t) - H'(p(t)q(t),q(t)\}\; dt,$$

Both operator pairs lead to similar classical stories, and with hbar > 0.

🙂 *Bonus topic – 1 (no charge)* 🙂

Classical covariant scalar field **NR!**

$$\kappa(x) \equiv \pi(x)\varphi(x) \quad , \quad \{\varphi(x), \kappa(y)\} = \delta^s(x-y)\,\varphi(x) \quad , \quad s \geq 4$$

$$H'(\kappa, \varphi) = \int \{[\kappa(x)\varphi(x)^{-2}\kappa(x) + (\overrightarrow{\nabla}\varphi)(x)^2 + m_o^2\varphi(x)^2]/2 + g_o\varphi(x)^4\}\, d^s x$$

Affine quantization

$$\varphi(x) \rightarrow \hat{\varphi}(x) \quad , \quad \kappa(x) \rightarrow \hat{\kappa}(x) \quad , \quad [\hat{\varphi}(x), \hat{\kappa}(y)] = i\hbar\,\delta^s(x-y)\,\hat{\varphi}(x)$$

Schrödinger′s representation

$$\hat{\varphi}(x) \rightarrow \varphi(x) \quad , \quad \hat{\kappa}(x) \rightarrow -i\hbar[\varphi(x)(\delta/\delta\varphi(x)) + (\delta/\delta\varphi(x))\varphi(x)]/2$$

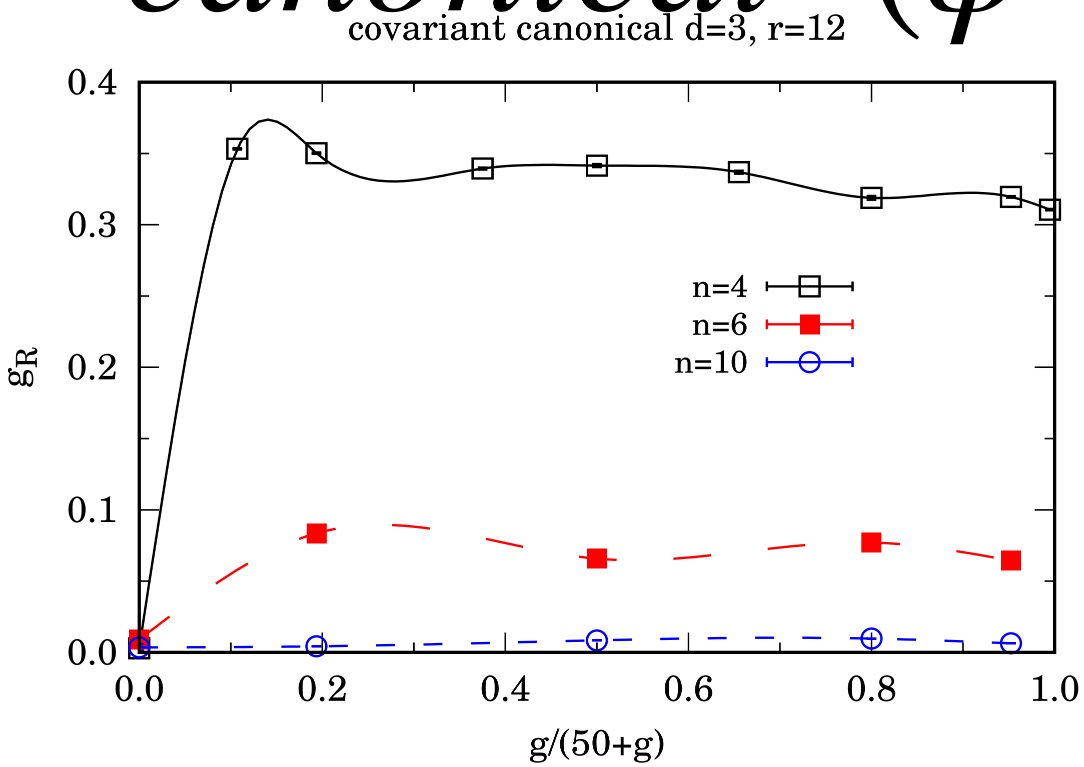
Schrödinger′s equation

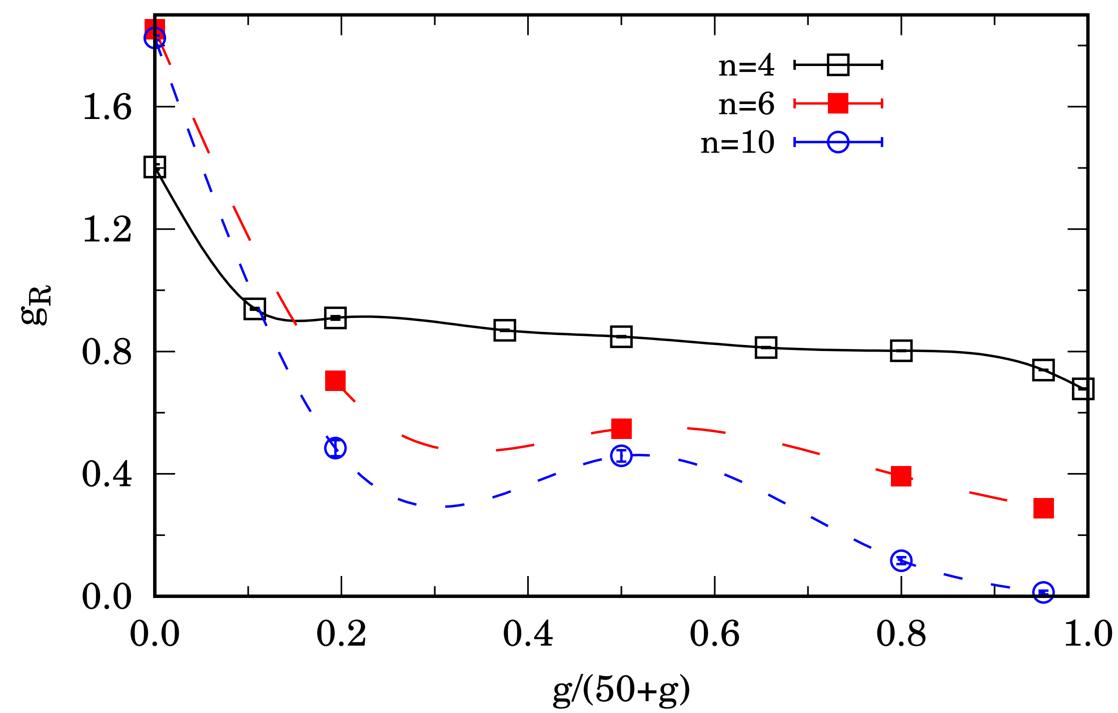
$$i\hbar\,\partial\Psi(\varphi, t)/\partial t = \int \{[\hat{\kappa}(x)\varphi(x)^{-2}\hat{\kappa}(x) + (\overrightarrow{\nabla}\varphi(x))^2 + m_o^2\varphi(x)^2]/2 + g_o\varphi(x)^4\}\, d^{!}x\; \Psi(\varphi, t)$$

$$\hat{\kappa}(x)\,\varphi(x)^{-1/2} = 0$$ 🙂 $$\int |\Psi(\varphi)|^2\,\mathscr{D}\varphi < \infty$$

Bonus topic – 2 (no charge)

$$canonical\ (\varphi^{12})_3 \quad affine\ (\varphi^{12})_3$$





Riccardo Fantoni; arXiv : 2011.09862

$$canonical(\varphi^4)_4 \downarrow,\ affine\ (\varphi^4)_4 \uparrow$$

R. Fantoni, J. R. Klauder; arXiv : 2012.09991

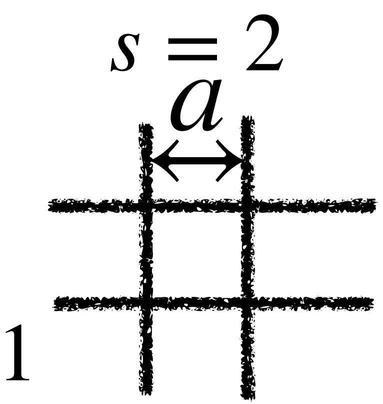
🙂 *Bonus topic – 3 (no charge)* 🙂

An Ultralocal (= NO gradients) Model $s \geq 1 \quad NR!$

$$i\hbar\,\partial\Psi(\varphi,t)/\partial t = [\,\int\{[\hat{\kappa}(x)\varphi(x)^{-2}\hat{\kappa}(x) + m_o^2\varphi(x)^2]/2 + g_o\varphi(x)^4\}\,d^sx\,]\,\Psi(\varphi,t)$$

$$\hat{\kappa}(x)\,\varphi(x)^{-1/2} = 0\ , \qquad \Psi(\varphi) = e^{-W(\varphi)}\,\Pi_x\,|\,\varphi(x)\,|^{-1/2}$$

Regularization and continuum limit



$$\varphi(x) \to \varphi_{\mathbf{k}} \equiv \varphi(\mathbf{k}a)\,, \quad a > 0\,, \ \mathbf{k} \in \{0, \pm 1, \pm 2, \pm 3, \cdots\}^s$$

$$\Psi_r(\varphi) = e^{-W_r(\varphi)}\,\Pi_{\mathbf{k}}(ba^s)^{1/2}\,|\,\varphi_{\mathbf{k}}\,|^{-(1-2ba^s)/2}\ , \ \int|\,\Psi_r(\varphi)\,|^2\,\Pi_{\mathbf{k}}d\varphi_{\mathbf{k}} = 1$$

$$C(f) = \lim_{a\to 0}\,\Pi_{\mathbf{k}}\int e^{if_{\mathbf{k}}\varphi_{\mathbf{k}}/\hbar}\,(ba^s)\,e^{-2W_r(\varphi_{\mathbf{k}})}\,|\,\varphi_{\mathbf{k}}\,|^{-(1-2ba^s)}\,d\varphi_{\mathbf{k}}$$

$$C(f) = \lim_{a\to 0}\,\Pi_{\mathbf{k}}\int\{1 - (ba^s)\int[1 - e^{if_{\mathbf{k}}\varphi_{\mathbf{k}}/\hbar}]\,e^{-2W_r(\varphi_{\mathbf{k}})}\,|\,\varphi_{\mathbf{k}}\,|^{-(1-2ba^s)}\,d\varphi_{\mathbf{k}}\,\}$$

$$C(f) = \exp\{-b\int d^sx\int[1 - e^{if(x)\lambda/\hbar}]\,e^{-2w(\lambda)}\,d\lambda/|\,\lambda\,|\,\}$$ 🙂

AFFINE GRAVITY – 1

Classical variables $\quad a, b, c, \ldots = 1, 2, 3$

🙂 *metric* $\quad g_{ab}(x) = g_{ba}(x)$, 🙂 *momentum* $\quad \pi^{cd}(x) = \pi^{dc}(x)$

$$ds(x)^2 = g_{ab}(x)\,dx^a\,dx^b > 0 \;,\quad g(x) \equiv \det[g_{ab}(x)] > 0 \;,\quad \{g\} \equiv \{g_{ab}(x)\} > 0$$

$$g_{ac}(x)\,g^{bc}(x) \equiv \delta^b_a \;, \qquad \pi^{ac}(x)\,g_{bc}(x) \equiv \pi^a_b(x)$$ 🙂 🙂

Classical Hamiltonian

$$H'(\pi, g) = \int \{g^{-1/2}[\pi^a_b\pi^b_a - (1/2)\pi^a_a\pi^b_b] + g^{1/2}\,{}^{(3)}\!R\}\,d^3x$$

Poisson brackets

$$\{\pi^a_b(x), \pi^c_d(x')\} = (1/2)\,\delta^3(x, x')\,[\delta^a_d\pi^c_b(x) - \delta^c_b\pi^a_d(x)]$$

$$\{g_{ab}(x), \pi^c_d(x')\} = (1/2)\,\delta^3(x, x')\,[\delta^c_a g_{bd}(x) + \delta^c_b g_{ad}(x)]$$

$$\{g_{ab}(x), g_{cd}(x')\} = 0 \qquad \{g_{ab}(x)\} > 0$$

FAVORED COORDINATES – 2

$$|p,q\rangle = e^{-iqP/\hbar}\, e^{ipQ/\hbar}\,|\omega\rangle\ , \qquad \langle\omega|(Q+iP/\omega)|\omega\rangle = 0$$

$$2\hbar[\,\|\,d\,|p,q\rangle\,\|^2 - |\langle p,q\,|\,d\,|p,q\rangle|^2] = \omega^{-1}dp^2 + \omega\,dq^2$$

$$|p;q\rangle = e^{ipQ/\hbar}e^{-i\ln(q)D/\hbar}\,|b\rangle\ , \qquad \langle b|[(Q-1\!\!1)+iD/b]|b\rangle = 0$$

$$2\hbar[\,\|\,d\,|p;q\rangle\,\|^2 - |\langle p;q\,|\,d\,|p;q\rangle|^2] = b^{-1}q^2dp^2 + b\,q^{-2}dq^2$$

$$|\pi;g\rangle = e^{(i/\hbar)\int \pi^{ab}(x)\,\hat{g}_{ab}(x)\,d^3x}\, e^{-(i/\hbar)\int \eta^a_b(x)\,\hat{\pi}^b_a(x)\,d^3x}\,|\beta\rangle\ , \qquad \{g(x)\} = e^{\{\eta(x)\}} > 0$$

$$C\hbar[\,\|\,d\,|\pi;g\rangle\,\|^2 - |\langle\pi;g\,|\,d\,|\pi;g\rangle|^2]\ ; \qquad \langle\beta|[(\hat{g}_{ab}(x)-\delta_{ab}1\!\!1)+i\hat{\pi}^c_d(x)/\beta(x)\hbar]|\beta\rangle = 0$$

$$= \int[(\beta(x)\hbar)^{-1}g_{ab}g_{cd}\,d\pi^{bc}d\pi^{da} + (\beta(x)\hbar)\,g^{ab}g^{cd}\,dg_{bc}dg_{da}]\,d^3x$$

affine gravity : constant negative curvature

AFFINE GRAVITY – 2

Quantum gravity coherent states

$$\langle \pi''; g'' | \pi'; g' \rangle = \exp\Big\{ -2 \int \beta(x)\, d^3x$$

$$\times \ln\Big\{ \frac{\det\{\frac{1}{2}[g''^{ab}(x) + g'^{ab}(x)] + i\frac{1}{2\hbar}\beta(x)^{-1}[\pi''^{ab}(x) - \pi'^{ab}(x)]\}}{\det[g''^{ab}(x)]^{1/2}\ \det[g'^{ab}(x)]^{1/2}} \Big\} \Big\}$$

$$H'(\pi^a_b(x), g_{cd}(x)) = \langle \pi; g | \mathcal{H}'(\hat{\pi}^a_b(x), \hat{g}_{cd}(x)) | \pi; g \rangle$$

$$= \langle \beta | \mathcal{H}'(\hat{\pi}^a_b(x) + \pi^{aj}(x)[e^{\eta(x)/2}]^e_j \hat{g}_{ef}(x)[e^{\eta(x)/2}]^f_b, [e^{\eta(x)/2}]^e_c \hat{g}_{ef}(x)[e^{\eta(x)/2}]^f_d) | \beta \rangle$$

$$= \mathcal{H}'(\pi^{aj}(x) g_{jb}(x), g_{cd}(x)) + \mathcal{O}'(\hbar; \pi, g)$$

Basic quantum commutations

$$[\hat{\pi}^a_b(x), \hat{\pi}^c_d(x')] = i\hbar(1/2)\,\delta^3(x,x')\,[\delta^a_d \hat{\pi}^c_b(x) - \delta^c_b \hat{\pi}^a_d(x)]$$

$$[\hat{g}_{ab}(x), \hat{\pi}^c_d(x')] = i\hbar(1/2)\,\delta^3(x,x')\,[\delta^c_a \hat{g}_{bd}(x) + \delta^c_b \hat{g}_{ad}(x)]$$

$$[\hat{g}_{ab}(x), \hat{g}_{cd}(x')] = 0 \quad , \qquad \{\hat{g}_{ab}(x)\} > 0$$

AFFINE GRAVITY – 3

Schrödinger′s representation

$$\hat{g}_{ab}(x) = g_{ab}(x)$$

$$\hat{\pi}^a_b(x) = -\,i\hbar(1/2)[g_{bc}(x)(\delta/\delta g_{ac}(x)) + (\delta/\delta g_{ac}(x))g_{bc}(x)]$$

Schrödinger′s equation

$$i\hbar\,\partial\Psi(\{g\},t)/\partial t = \Big[\int [\hat{\pi}^a_b(x)g(x)^{-1/2}\hat{\pi}^b_a(x) - \tfrac{1}{2}\hat{\pi}^a_a(x)g(x)^{-1/2}\hat{\pi}^b_b(x) + g(x)^{1/2}\,{}^{(3)}\!R(x)]\,d^3x\Big]\,\Psi(\{g\},t)$$

Functional properties

$$\hat{\pi}^a_b(x)\,g(x)^{-1/2} = 0\ , \qquad \Psi(\{g\}) = Y(\{g\})\,\Pi_x\,g(x)^{-1/2} \qquad \text{formal}$$

$$\Psi_r(\{g\}) = Y_r(\{g\})\Pi_{\mathbf{k}}\,[\Sigma_{\mathbf{l}}J_{\mathbf{k},\mathbf{l}}\,g_{\mathbf{l}}]^{-(1-ba^3)/2} \qquad \text{regularized}$$

SMOOTH METRICS & CONSTRAINTS

$$\{[\hat{\pi}^a_b(x)\hat{\pi}^b_a(x) - \tfrac{1}{2}\hat{\pi}^a_a(x)\hat{\pi}^b_b(x)] + g(x)\,{}^{(3)}\!R(x)\,\}\Phi(\{g\}) = 0 \qquad , \quad \{g(\,\cdot\,)\} \in C^2$$

AFFINE QUANTUM GRAVITY

- *Canonical quantization takes favored classical variables and promotes them to basic operators . Schrödinger's representation and equation quantizes via CQ, for a selected set of problems .*

- *Affine quantization takes favored classical variables and promotes them to basic operators . Schrödinger's representation and equation quantizes via AQ, for a selected set of problems .*

- *There is also SQ, which is spin quantization .* •••

Quantization involves SQ , CQ , and AQ, all together!

Dictated by : Constant (+ , 0, –) Numerical Curvatures .

From H O(S) to Q G

Affine quantization is exactly like canonical quantization .
The difference lies in the choice of BASIC OPERATORS .

Affine quantization does NOT compete with canonical quantization . It JOINS with canonical quantization to make ENHANCED QUANTIZATION .

🙂 Thank You 🙂

spin , favored gravity variables , REF

FAVORED COORDINATES – 3

Spin variable properties , $SU(2)$, $SO(3)$

$$[S_2, S_3] = i\hbar\, S_1 \quad , \quad S_1^2 + S_2^2 + S_3^2 = \hbar^2\, s(s+1) I_{2s+1} \quad , \quad s \in (1/2)\{1,2,3,\cdots\}$$

$$S_3\,|\,s,m\rangle = m\hbar\,|\,s,m\rangle \ , \ m \in \{-s,\cdots,s-1,s\} \ , \qquad (S_1 + iS_2)\,|\,s,s\rangle = 0$$

Spin coherent states

$$|\,\theta,\varphi\rangle \equiv e^{-i\varphi S_3/\hbar} e^{-i\theta S_2/\hbar}\,|\,s,s\rangle \quad , \quad -\pi < \varphi \le \pi \ , \ -\pi/2 \le \theta \le \pi/2$$

latitude *longitude*

$$-\pi(s\hbar)^{1/2} < q \le \pi(s\hbar)^{1/2} \quad , \quad -(s\hbar)^{1/2} \le p \le (s\hbar)^{1/2}$$

$$|\,p,q\rangle \equiv e^{-i(q/(s\hbar)^{1/2})S_3/\hbar}\, e^{-i\cos^{-1}(p/(s\hbar)^{1/2})S_2/\hbar}\,|\,s,s\rangle$$

$$d\sigma^2 = 2\hbar[\,\|\,d\,|\,\theta,\varphi\rangle\,\|^2 - |\,\langle\theta,\varphi\,|\,d\,|\,\theta,\varphi\rangle\,|^2\,]$$
$$= (s\hbar)[d\theta^2 + \cos(\theta)^2\, d\varphi^2]$$
$$= (1 - p^2/s\hbar)^{-1} dp^2 + (1 - p^2/s\hbar)\, dq^2$$

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$Loop\ quantum\ gravity$

arXiv : 1910.11139

JHEPGC J. R. Klauder papers