Instability of complex CFTs with operators in the principal series

Dario Benedetti

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Motivation

Conformal Field Theories (CFTs) typically appear as fixed points of the renormalization group, and are important for both high-energy and statistical phisics

Conformal invariance $\Rightarrow$ tight constraints on correlators
$\Rightarrow$ all the n-point functions are in principle determined by the CFT data:

- Scaling dimensions: $O_i'(x') = \Omega(x)^{-\Delta_i} O_i(x)$
  $\Rightarrow \langle O_i(x) O_j(y) \rangle = \delta_{ij}/|x-y|^{2\Delta_i}$
- OPE coefficients: $O_i(x) O_j(y) = \sum_k c_{ijk} P(x, \partial_y) O_k(y)$
  $\Rightarrow$ fixes higher n-point functions

Unitarity (reflection positivity in Euclidean case) imposes additional constraints:
$\Delta_i, c_{ijk} \in \mathbb{R}$, and unitarity bounds (e.g. $\Delta_i \geq (d-2)/2$ for scalar operators)

However, in statistical physics there is no reason to have reflection positivity
$\Rightarrow$ **complex CFT data** are allowed

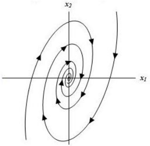
Complex CFTs could be of theoretical interest [Gorbenko, Rychkov, Zan - 2018]

Complex scaling dimensions

Complex scaling dimensions appear in various ways:

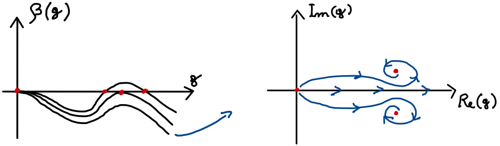
- Real fixed points with diagonalizable but non-symmetric stability matrix

  $\Rightarrow$ **Focus** or **spiral point**



  (e.g. in systems with long-range disorder [Weinrib, Halperin 1982])

- At complex fixed points appearing after a merger of real fixed points (e.g. fate of Banks-Zaks fixed point at $N_f < N_f^{\mathrm{crit}}(N_c)$ [Gies, Jaeckel 2005; Kaplan et al. 2009])



Scaling dimensions in the "principal series"

In the large-N limit of tensor models in d dimensions, a special case of complex scaling dimensions is commonly found, namely

$$\Delta = \frac{d}{2} + \mathrm{i}\, r\,, \quad r \in \mathbb{R}$$

also labelling the principal series representations of the Euclidean conformal group $SO(d+1,1)$

Such type of dimensions appeared before in other contexts, always in some large-N limit, e.g.:

- non-supersymmetric orbifolds of $\mathcal{N}=4$ super Yang-Mills [Dymarsky, Klebanov, Roiban 2005]
- gauge theories with matter in the Veneziano limit [Kaplan et al. 2009]
- fishnet models [Kazakov et al. 2017-2019]

Typical mechanism:
in the OPE $\phi \times \phi$, operator $\mathcal{O}(x)$ ($\sim \mathrm{Tr}(\phi^2)$) whose dimension Δ merges with that of its "shadow operator" $\widetilde{\Delta} = d - \Delta$ ($\Rightarrow$ at $\Delta = d/2$) and then moves into the complex plane

Spontaneous breaking of conformal symmetry?

Conjecture [Kim, Klebanov, Tarnopolsky, Zhao - 2019]

If the assumption of conformal invariance in a large N theory leads to a single-trace operator with a complex scaling dimension of the form $d/2 + if$,
then in the true low-temperature phase this operator acquires a VEV

Actually two statements at once:

- Implicit: the conformal vacuum is unstable (AdS/CFT argument)
- Explicit: there exists a stable vacuum with spontaneous breaking of conformal invariance ($\langle \mathcal{O}(x) \rangle = 0$ in a CFT)

They provide a very neat $d = 1$ example, in the melonic limit:
two flavors SYK, or SYK-like tensor model, for which both statements can be checked explicitly

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two flavors SYK, or SYK-like tensor model, for which both statements can be checked explicitly

$\Rightarrow$ can it be proved in some generality?

The AdS/CFT picture

AdS/CFT dictionary:

Scalar operator with dimension Δ in CFT$_d$ $\Leftrightarrow$ scalar field with mass $m^2 = \Delta(\Delta - d)$ in AdS$_{d+1}$

$$\Downarrow$$

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}$$

$$\Downarrow$$

$$\Delta = \frac{d}{2} + \mathrm{i}\, r \quad \Leftrightarrow \quad m^2 < \underbrace{-\frac{d^2}{4}}_{\text{BF bound}}$$

$\Rightarrow$ Tachyonic/thermodynamic instability (BF = Breitenlohner-Freedman)

(notice: no instability for $-\frac{d^2}{4} \le m^2 < 0$, thanks to AdS curvature)

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$\Rightarrow$ First goal: prove instability from the CFT side, without referring to AdS/CFT

A standard example of instability

Consider the **effective potential** of a (Euclidean) scalar field theory in flat space:

$$W[J] = \log \int [d\varphi] e^{-S[\varphi]+J\cdot\varphi} \xrightarrow[\text{Legendre tr.}]{} \Gamma[\phi] \xrightarrow[\phi=\text{const.}]{} V(\phi)$$

Free energy: $F = \Gamma[\phi_0]$, with ϕ_0 solution of $\delta\Gamma/\delta\phi = 0$ ("on shell")

If $V(\phi) = m^2\phi^2 + O(\phi^3)$, then:

- for $m^2 > 0$, the $\phi_0 = 0$ configuration is stable (local minimum of F);
- for $m^2 < 0$, the $\phi_0 = 0$ configuration is unstable (local maximum of F).

Notice: on AdS, the constant configuration is not a normalizable mode
$\Rightarrow \phi(-\nabla^2)\phi$ contributes a positive term $\Rightarrow$ instability bound is shifted to $m^2 < 0$

Aim

Claim

Consider a Euclidean quantum field theory whose Schwinger-Dyson equations admit a conformal solution. If the OPE of two fundmental scalar fields includes a contribution from one primary operator $\mathcal{O}_{h_\star}$ of dimension $h_\star = \frac{d}{2} + \mathrm{i}\, r_\star$, with non-vanishing $r_\star \in \mathbb{R}$, then the conformal solution is unstable.

Unlike usual SSB, we are not solving for the VEV of the field ϕ ($= 0$ in a CFT), but for the two-point function

And we want to show that the conformal solution is unstable

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And we want to show that the conformal solution is unstable

$\Rightarrow$ For our purpose we will need the 2PI effective action $\Gamma[G]$

2PI formalism

Notation: $\phi_a(x) = \phi(X)$ with $X = (x, a)$; $\int_X = \sum_a \int \mathrm{d}^d x$, $\delta(X - X') = \delta_{aa'}\delta(x - x')$, etc.

Introduce a bilocal source:

$$\mathbf{W}[\mathcal{J}] = \ln Z[\mathcal{J}] = \ln \int [d\phi] \exp\left\{ -S[\phi] + \frac{1}{2}\int_{X,Y} \phi(X)\mathcal{J}(X,Y)\phi(Y) \right\} .$$

The 2PI effective action is defined by the Legendre transform:

$$\begin{aligned}\mathbf{\Gamma}[G] &= \left(-\mathbf{W}[\mathcal{J}] + \frac{1}{2}\mathrm{Tr}[\mathcal{J}G] \right)\Big|_{\frac{\delta \mathbf{W}}{\delta \mathcal{J}} = \frac{1}{2}G} \\ &= \frac{1}{2}\mathrm{Tr}[C^{-1}G] + \frac{1}{2}\mathrm{Tr}[\ln G^{-1}] + \mathbf{\Gamma}_2[G]\end{aligned}$$

$\mathbf{\Gamma}_2[G]$: sum of 2PI diagrams constructed from the vertices of $S[\phi]$, but with G as propagator.

The field equations of $\mathbf{\Gamma}[G]$ are the Schwinger-Dyson equations:

$$\frac{\delta \mathbf{\Gamma}}{\delta G(X_1, X_2)}\Big|_{G = G_\star} = 0 \quad \Rightarrow \quad G^{-1}(X, X') = C^{-1}(X, X') - \Sigma(X, X')$$

with the self energy given by $\Sigma[G] = -2\,\delta\mathbf{\Gamma}_2/\delta G$

First Hypothesis

Hypothesis 1

Let a Euclidean quantum field theory of N real scalar fields in $\mathbb{R}^d$ be given, and assume that the Schwinger-Dyson equations for the two-point functions, for some choice of renormalized couplings corresponding to a fixed point of the renormalization group, admit a conformal solution

$$G_\star(X_1, X_2) \sim \delta_{a_1 a_2} |x_1 - x_2|^{-2\Delta_1} ,$$

where $\Delta_i \in \mathbb{R}$ is the scaling dimension of ϕ_{a_i}; moreover, also the four-point functions (and possibly all the other n-point functions, the ones with even n being related to functional derivatives of $\mathbf{\Gamma}[G]$ with respect to G, evaluated at $G_\star$) are conformal.

On-shell effective action = free energy : $\mathbf{F} = \mathbf{\Gamma}[G_\star]$

Stability test: introduce fluctuations $\delta G = G - G_\star$, expand $\mathbf{\Gamma}[G]$ as

$$\mathbf{\Gamma}[G] - \mathbf{F} \simeq \frac{1}{2} \int_{X_1 \dots X_4} \delta G(X_1, X_2) \frac{\delta^2 \mathbf{\Gamma}}{\delta G(X_1, X_2) \delta G(X_3, X_4)} \Big|_{G=G_\star} \delta G(X_3, X_4)$$

and check whether there are perturbations giving a negative contribution.

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and check whether there are perturbations giving a negative contribution.

$\Rightarrow$ We need to control the space of fluctuations and the structure of the Hessian

Hessian of $\mathbf{\Gamma}[G]$ and Bethe-Salpeter kernel

We write the Hessian of the 2PI effective action as

$$\boxed{\left.\frac{\delta^2\mathbf{\Gamma}[G]}{\delta G(X_1,X_2)\delta G(X_3,X_4)}\right|_{G=G_\star} = \frac{1}{2}\int_{Y_1,Y_2} G_\star^{-1}(X_1,Y_1)G_\star^{-1}(X_2,Y_2)\left(\mathbb{I}-K\right)(Y_1,Y_2,X_3,X_4)}$$

where $\mathbb{I}$ is the identity operator

$$\mathbb{I}(X_1,X_2,X_3,X_4) = \frac{1}{2}\left(\delta(X_1-X_3)\delta(X_2-X_4)+\delta(X_1-X_4)\delta(X_2-X_3)\right)$$

and K is the Bethe-Salpeter kernel defined by

$$K(X_1,X_2,X_3,X_4) = -2\int_{Y_1,Y_2} G_\star(X_1,Y_1)G_\star(X_2,Y_2)\left.\frac{\delta^2\mathbf{\Gamma}_2[G]}{\delta G(Y_1,Y_2)\delta G(X_3,X_4)}\right|_{G=G_\star}$$

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$=$ 

The vector space of perturbations

[Dobrev et al. "Harmonic analysis on the n-dimensional Lorentz group and its applications to conformal quantum field theory" 1977]

$\delta G(X_1, X_2) \in \mathcal{V}$, the space of smooth symmetric functions which are square integrable with respect to inner product

$$(f_1, f_2) = \frac{1}{2} \int_{X_1 \dots X_4} \overline{f_1(X_1, X_2)} \left(G_\star^{-1}(X_1, X_3) G_\star^{-1}(X_2, X_4) + G_\star^{-1}(X_1, X_4) G_\star^{-1}(X_2, X_3) \right) f_2(X_3, X_4)$$

and satisfy the asymptotic boundary conditions[1]

$$f_i(X_1, X_2) \sim |x_1|^{-2\Delta_1} \quad \text{for} \quad |x_1| \to \infty$$
$$f_i(X_1, X_2) \sim |x_2|^{-2\Delta_2} \quad \text{for} \quad |x_2| \to \infty$$

Shadow space: $\tilde{\mathcal{V}} = \mathcal{V}_{\Delta_i \to \widetilde{\Delta}_i}$

Notice: $G_\star^{-1} G_\star^{-1} : \mathcal{V} \to \tilde{\mathcal{V}}$

[1] $\mathcal{V}$ is the union of Kronecker products of two type I (scalar) complementary series representations, satisfying $|\mathrm{Re}(\Delta_1 - \frac{d}{2})| + |\mathrm{Re}(\Delta_2 - \frac{d}{2})| \leq \frac{d}{2}$

A basis of bilocal functions [Dobrev et al. 1977]

$f \in \mathcal{V}$ has the representation

$$f(X_1, X_2) = \frac{1}{2} \sum_{J \in \mathbb{N}_0} \int \mathrm{d}^d z \int_{\mathcal{P}} \frac{\mathrm{d}h}{2\pi\,\mathrm{i}} \rho(h, J) \sum_{\sigma} V_{\tilde{h};\sigma}^{\mu_1 \cdots \mu_J}(X_1, X_2; z) F_{h;\sigma}^{\mu_1 \cdots \mu_J}(z)$$

where J is the spin, and

$$\mathcal{P} = \left\{ h \;\middle|\; h = \frac{d}{2} + \mathrm{i}\, r,\ r \in \mathbb{R} \right\} : \quad \text{“principal series”}$$

$$\rho(h, J) = \frac{\Gamma(\frac{d}{2} + J)}{2(2\pi)^{d/2} J!} \frac{\Gamma(\tilde{h} - 1)\Gamma(h - 1)}{\Gamma(\frac{d}{2} - h)\Gamma(\frac{d}{2} - \tilde{h})} (h + J - 1)(\tilde{h} + J - 1) : \quad \text{“Plancherel weight”}$$

The functions

$$V_{h;\sigma}^{\mu_1 \cdots \mu_J}(X_1, X_2; x_3) = \mathcal{N}_{h,J}^{\Delta_1, \Delta_2} \langle \phi_{\Delta_1}(x_1) \phi_{\Delta_2}(x_2) \mathcal{O}_h^{\mu_1 \cdots \mu_J}(x_3) \rangle_{\mathrm{cs}}\, E_{a_1 a_2}^{\sigma, J}$$

form a complete and orthonormal basis (in the continuous sense)
and $F_{h;\sigma}^{\mu_1 \cdots \mu_J}(z)$ is the projection of $f(X_1, X_2)$ on the basis

Analogy to Fourier decomposition: $V \leftrightarrow$ plane waves, $F \leftrightarrow$ Fourier transform of f
Group theory analogy: $V \sim$ Clebsch-Gordan coefficients

Eigenbasis of the Bethe-Salpeter kernel

Hypothesis of conformal invariance $\Rightarrow K$ transforms in the $\Delta_1 \times \Delta_2 \times \widetilde{\Delta}_3 \times \widetilde{\Delta}_4$ rep.

Moreover, if the kernel is real, as we will assume, then it can be shown to be also self-adjoint (wrt to inner product on $\mathcal{V}$), and thus diagonalizable
$\Rightarrow$ we can choose $E^{\sigma,J}_{a_1 a_2}$ s.t.

$$\boxed{\int_{X_3,X_4} K(X_1,X_2,X_3,X_4)\, V^{\mu_1\cdots\mu_J}_{h;\sigma}(X_3,X_4;z) = k_\sigma(h,J)\, V^{\mu_1\cdots\mu_J}_{h;\sigma}(X_1,X_2;z)}$$

$$\Downarrow$$

$$\begin{aligned}\mathbf{\Gamma}[G] - \mathbf{F} &\simeq \frac{1}{4}\int_{X_1\ldots X_6} \delta G(X_1,X_2) G_\star^{-1}(X_1,X_5) G_\star^{-1}(X_2,X_6) \\ &\qquad\qquad \times (\mathbb{I} - K)(X_5,X_6,X_3,X_4)\delta G(X_3,X_4) \\ &= \frac{1}{8}\sum_{J\in\mathbb{N}_0}\int_{\mathcal{P}} \frac{\mathrm{d}h}{2\pi\,\mathrm{i}}\,\rho(h,J)\sum_\sigma (1 - k_\sigma(h,J))\int \mathrm{d}^d z F^{\mu_1\cdots\mu_J}_{\tilde{h};\sigma}(z) F^{\mu_1\cdots\mu_J}_{h;\sigma}(z)\end{aligned}$$

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\end{aligned}$$

Now we need to introduce the hypothesis of existence of a primary operator $\mathcal{O}_{h_\star}$ of dimension $h_\star \in \mathcal{P}$

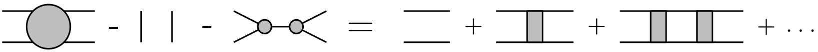
4-point function and Bethe-Salpeter kernel

The Hessian is the inverse of the four-point function, connected and 1PI in the s-channel:

$$\int_{Y_1,Y_2} \frac{\delta^2\mathbf{\Gamma}[G]}{\delta G(X_1,X_2)\delta G(Y_1,Y_2)}\bigg|_{G=G_\star} \mathcal{F}_s(Y_1,Y_2,X_3,X_4) = \mathbb{I}(X_1,X_2,X_3,X_4)$$

with

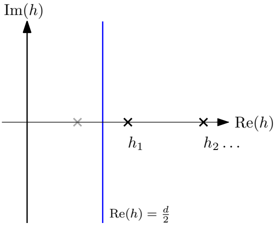
$$\begin{aligned}\mathcal{F}_s(X_1,X_2,X_3,X_4) \equiv & \langle\phi(X_1)\phi(X_2)\phi(X_3)\phi(X_4)\rangle - G_\star(X_1,X_2)G_\star(X_3,X_4) \\ & - \int_{Y_1,Y_2} \langle\phi(X_1)\phi(X_2)\phi(Y_1)\rangle G_\star^{-1}(Y_1,Y_2)\langle\phi(Y_2)\phi(X_3)\phi(X_4)\rangle\end{aligned}$$



OPE spectrum

$$\mathcal{F}_s(X_1,X_2,X_3,X_4) = \sum_{J\in\mathbb{N}_0}\int_{\mathcal{P}_+}\frac{\mathrm{d}h}{2\pi\,\mathrm{i}}\sum_{\sigma}\frac{2\,\rho(h,J)}{1-k_\sigma(h,J)} \times \int \mathrm{d}^d z\, V^{\mu_1\cdots\mu_J}_{h;\sigma}(X_1,X_2;z)V^{\mu_1\cdots\mu_J}_{\tilde{h};\sigma}(X_3,X_4;z)$$

$$= \sum_{J\in\mathbb{N}_0}\int_{\mathcal{P}}\frac{\mathrm{d}h}{2\pi\,\mathrm{i}}\sum_{\sigma}\frac{2\,\hat{\rho}_{\Delta_i}(h,J)}{1-k_\sigma(h,J)}\,\mathcal{G}^{\Delta_i}_{h,J}(x_i)E^{\sigma,J}_{a_1a_2}E^{\sigma,J}_{a_3a_4}$$



poles at solutions of $k_\sigma(h,J)=1$

$$\Rightarrow \quad = \sum_J\sum_n \underbrace{c^2_{h_n(J),J}}_{\text{OPE coeff.}}\underbrace{\mathcal{G}^{\Delta_i}_{h_n(J),J}(x_i)}_{\text{Conformal blocks}}E^{\sigma,J}_{a_1a_2}E^{\sigma,J}_{a_3a_4}$$

Second Hypothesis

Solutions of $k_\sigma(h,J)=1 \Rightarrow$ spectrum of primary operators in the OPE of $\phi\times\phi$

$$\Downarrow$$

Hypothesis 2

Let $K(X_1,X_2,X_3,X_4)$ be the Bethe-Salpeter kernel of the conformal field theory of Hypothesis 1, and assume that it is real, and hence diagonalizable, with eigenvalue $k_\sigma(h,J)$, which for each J and σ is real on $h\in\mathcal{P}$, and analytically continued to a meromorphic function in the half-plane $\mathrm{Re}(h)\geq d/2$.
Moreover, let the equation $k_\sigma(h,J)=1$ admit, for some fixed J and σ, a simple root of the form $h=h_\star\equiv\frac{d}{2}+\mathrm{i}\,r_\star$, with $r_\star\in\mathbb{R}$ and different from zero.

Putting the pieces back together

By Hypothesis 1, we have obtained:

$$\begin{aligned}\mathbf{\Gamma}[G]-\mathbf{F} \simeq \frac{1}{4}\int_{X_1\ldots X_6} \delta G(X_1,X_2)G_\star^{-1}(X_1,X_5)G_\star^{-1}(X_2,X_6) \\ \times\left(\mathbb{I}-K\right)(X_5,X_6,X_3,X_4)\delta G(X_3,X_4) \\ =\frac{1}{8}\sum_{J\in\mathbb{N}_0}\int_{\mathcal{P}}\frac{\mathrm{d}h}{2\pi\,\mathrm{i}}\,\rho(h,J)\sum_\sigma\left(1-k_\sigma(h,J)\right)\int \mathrm{d}^d z F^{\mu_1\cdots\mu_J}_{\tilde{h};\sigma}(z)F^{\mu_1\cdots\mu_J}_{h;\sigma}(z)\end{aligned}$$

where $\rho(h,J)$ and the z-integrand are positive functions.

By Hypothesis 2, $(1-k_\sigma(h,J))$ must change sign on the integration contour around the simple root $h_\star\in\mathcal{P}$

$$\Downarrow$$

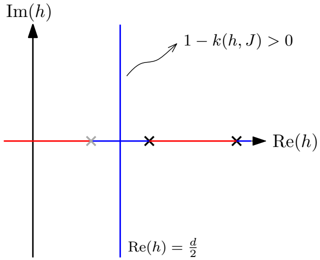
Theorem

Given Hypothesis 1 and 2, there exist perturbations $\delta G(X_1,X_2)\in\mathcal{V}$ such that the second variation of the 2PI effective action $\mathbf{\Gamma}[G]$ around the solution $G_\star(X_1,X_2)$ is negative. Therefore, the conformal solution $G_\star(X_1,X_2)$ is unstable.

Generalizations to complex and/or Grassmann fields, and to $d=1$, are possible

Pictorial explanation

Illustration in the complex h plane of some hypothetical solutions of $k(h, J) = 1$:



Black crosses: physical solutions
Gray crosses: their shadow
Blue intervals: $1 - k(h, J) > 0$
Red intervals: $1 - k(h, J) < 0$

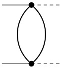
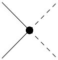
Example 1: long-range $O(N)^3$ model

[Giombi, Klebanov, Tarnopolsky 2017; DB, Gurau, Harribey 2019]

$$\mathbf{\Gamma}[G] = N^3 \left(\frac{1}{2}\mathrm{Tr}[(-\partial^2)^\zeta G] + \frac{1}{2}\mathrm{Tr}[\ln G^{-1}] + \frac{m^{2\zeta}}{2}\int_x G(x,x) + \frac{\lambda_2}{4}\int_x G(x,x)^2 - \frac{\lambda^2}{8}\int_{x,y} G(x,y)^4 \right)$$

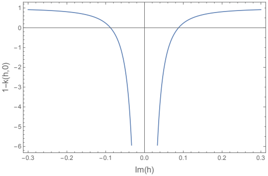
$$\Rightarrow \quad \text{SDE} \quad \Rightarrow \quad G_\star(x,y) \sim |x-y|^{-d/2}$$

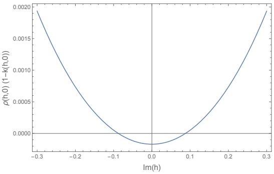
$$K(x_1,x_2,x_3,x_4) = G_\star(x_1,x_3)G_\star(x_2,x_4)\left(3\lambda^2 G_\star(x_3,x_4)^2 - \lambda_2\delta(x_3-x_4)\right)$$

$= 3\lambda^2$  $- \lambda_2$ 

$$\Rightarrow \quad k(h,J) = \frac{3g^2}{(4\pi)^d}\frac{\Gamma(-\frac{d}{4}+\frac{h+J}{2})\Gamma(\frac{d}{4}-\frac{h-J}{2})}{\Gamma(\frac{3d}{4}-\frac{h-J}{2})\Gamma(\frac{d}{4}+\frac{h+J}{2})}$$

At $d=3$ and $g=1$:





Example 2: Two-flavor SYK-like model [Kim, Klebanov, Tarnopolsky, Zhao - 2019]

$2N^3$ Majorana fermions ψ_i^{abc}, with action:

$$S[\psi] = \int \mathrm{d}\tau \sum_{i=1,2} \left(\frac{1}{2} \psi_i^{\mathbf{a}} \partial_\tau \psi_i^{\mathbf{a}} + \frac{\lambda}{4} \hat{\delta}^t_{\mathbf{abcd}} \psi_i^{\mathbf{a}} \psi_i^{\mathbf{b}} \psi_i^{\mathbf{c}} \psi_i^{\mathbf{d}} \right)$$
$$+ \int \mathrm{d}\tau \frac{\lambda\alpha}{2} \hat{\delta}^t_{\mathbf{abcd}} \left(\psi_1^{\mathbf{a}} \psi_1^{\mathbf{b}} \psi_2^{\mathbf{c}} \psi_2^{\mathbf{d}} + \psi_1^{\mathbf{a}} \psi_2^{\mathbf{b}} \psi_1^{\mathbf{c}} \psi_2^{\mathbf{d}} + \psi_1^{\mathbf{a}} \psi_2^{\mathbf{b}} \psi_2^{\mathbf{c}} \psi_1^{\mathbf{d}} \right) ,$$

Symmetry group $\mathcal{G} \supset \mathbb{Z}_2 \times \mathbb{Z}_2 \Rightarrow G_{12}(\tau) = \langle \psi_1^{\mathbf{a}}(\tau) \psi_2^{\mathbf{a}}(0) \rangle = 0$

$$\Downarrow$$

Conformal solution: $G_{12} = G_{21} = 0$,

$$G_{11} = G_{22} = G_\star(\tau) = \left(\frac{1}{4\pi(1+3\alpha^2)} \right)^{\frac{1}{4}} \frac{\mathrm{sgn}(\tau)}{|\lambda\tau|^{1/2}}$$

Fluctuations: $(\delta G_{11}, \delta G_{22}, \delta G_{12}, \delta G_{21})$

Bethe-Salpeter kernel: $K = \begin{pmatrix} 1+\alpha^2 & 2\alpha^2 & 0 & 0 \\ 2\alpha^2 & 1+\alpha^2 & 0 & 0 \\ 0 & 0 & 2\alpha & 2\alpha^2 \\ 0 & 0 & 2\alpha^2 & 2\alpha \end{pmatrix} \frac{K_c(\tau_1,\tau_2;\tau_3,\tau_4)}{1+3\alpha^2}$

Example 2: Two-flavor SYK-like model [Kim, Klebanov, Tarnopolsky, Zhao - 2019]

The matrix structure is diagonalized by the following eigenvectors:

$$E^1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad E^2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad E^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad E^4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

The kernel K_c is diagonalized as usual by (two) three-point conformal structures

The interesting eigenvalue is $k_4(h) = -\frac{3\alpha(1-\alpha)}{1+3\alpha^2} \frac{\tan(\frac{\pi}{2}(h+\frac{1}{2}))}{h-1/2}$

For $\alpha < 0$, the equation $k_4(h) = 1$ admits the solutions $h = \frac{1}{2} \pm \mathrm{i}\, f(\alpha)$, where

$$f \tanh(\pi f/2) = -\frac{3\alpha(1-\alpha)}{1+3\alpha^2}$$

$\Rightarrow$ instability in the $(\delta G_{12}, \delta G_{21})$ sector
$\Rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$ breaks down to diagonal subgroup $\mathbb{Z}_2$

• Existence of a stable symmetry-breaking solution shown numerically by Kim et al.

• Similar results in $SU(N)^2 \times O(N) \times U(1)^2$ model
(complex scaling dimension $\Rightarrow$ breaking of $U(1)^2$ to diagonal subgroup)

Example 3: Fishnet model

[Gurdogan,Kazakov (2015); Grabner,Gromov,Kazakov,Korchemsky (2017); Kazakov,Olivucci (2018)]

A non-melonic model which however has a similar structure

- Two (matrix) complex scalar fields in the adjoint of $SU(N)$, with action

$$S_{\text{fishnet}} = \frac{N_c}{(4\pi)^{\frac{d}{2}}} \int_x \Big(\text{Tr}[\phi_1^\dagger \; (-\partial^2)^{d/2} \, \phi_1 + \phi_2^\dagger \; (-\partial^2)^{d/2} \, \phi_2 + \xi^2 \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2] \\ + \alpha_1^2 \sum_{i=1}^{2} \text{Tr}(\phi_i \phi_i) \, \text{Tr}(\phi_i^\dagger \phi_i^\dagger) - \alpha_2^2 \, \text{Tr}(\phi_1 \phi_2) \text{Tr}(\phi_2^\dagger \phi_1^\dagger) \\ - \alpha_2^2 \text{Tr}(\phi_1 \phi_2^\dagger) \text{Tr}(\phi_2 \phi_1^\dagger) \Big)$$

  Notice: $U(1)^2$ symmetry

- First line (lack of hermitian conjugate of single-trace vertex) gives in the large-N limit a very rigid structure of diagrams (fishnets)

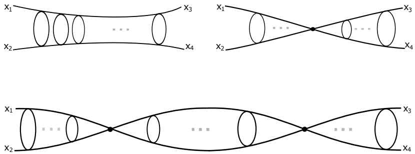
- No wave function renormalization in $d < 4$ because long-range; but also in $d = 4$, because of planar fishnet structure (no melonic two-point function) $\Rightarrow$ trivial solution for G

Example 3: Fishnet model

[Gurdogan,Kazakov (2015); Grabner,Gromov,Kazakov,Korchemsky (2017); Kazakov,Olivucci (2018)]

- Double-trace terms are needed for renormalization

- They are renormalized by a special case of fishnets, those with cycle of length two edges, i.e. ladders!



$\Rightarrow$ same renormalization structure as pillow and double-trace in $O(N)^3$ model

- Spectrum of bilinears is found in the same way from the Bethe-Salpeter equation, with similar **complex scaling dimension** in $\mathcal{P}$ appearing for real ξ^2

- But trivial solution of SDE $G(x,y) = C(x,y) \Rightarrow \mathbf{\Gamma}_2[G] = 0$?
How can the theorem apply?

Example 3: Fishnet model

[Gurdogan,Kazakov (2015); Grabner,Gromov,Kazakov,Korchemsky (2017); Kazakov,Olivucci (2018)]

Actually, the vanishing of the self-energy relies on the assumption of unbroken $U(1)^2$ symmetry

Source terms:

$$S_{\text{symm.}}[\phi,\mathcal{J}] = N\int \mathrm{d}^d x \mathrm{d}^d y \sum_{i=1,2} \mathcal{J}_{\bar{i}i}(x,y)\mathrm{tr}[\phi_i^\dagger(x)\phi_i(y)]$$

$$S_{\text{break.}}[\phi,\mathcal{J}] = N\int \mathrm{d}^d x \mathrm{d}^d y \sum_{i=1,2} \Big(\mathcal{J}_{ii}(x,y)\mathrm{tr}[\phi_i(x)\phi_i(y)] + \mathcal{J}_{\bar{i}\,\bar{i}}(x,y)\mathrm{tr}[\phi_i^\dagger(x)\phi_i^\dagger(y)]\Big)$$

Breaking term reduces $U(1)^2$ symmetry to $\mathbb{Z}_2{}^2$

Legendre transform $\Rightarrow$ new diagrammatic rules with non-vanishing $G_{ii}(x,y)$ and $G_{\bar{i}\,\bar{i}}(x,y) \Rightarrow$ non-trivial $\mathbf{\Gamma}_2[G]$

Diagrams necessarily have an even number of "symmetry breaking" propagators, hence

$$\frac{\delta\mathbf{\Gamma}_2}{\delta G_{\bar{i}i}}\Big|_{G_{ii}=G_{\bar{i}\,\bar{i}}=0} = \frac{\delta\mathbf{\Gamma}_2}{\delta G_{ii}}\Big|_{G_{ii}=G_{\bar{i}\,\bar{i}}=0} = \frac{\delta\mathbf{\Gamma}_2}{\delta G_{\bar{i}\,\bar{i}}}\Big|_{G_{ii}=G_{\bar{i}\,\bar{i}}=0} = 0\,,$$

$$\Rightarrow G^\star_{\bar{i}i}(x,y) = C(x,y)\,, \quad G^\star_{ii} = G^\star_{\bar{i}\,\bar{i}} = 0$$

However, $K_{i\,i\,\bar{i}\,\bar{i}}(x_1,x_2,x_3,x_4) \neq 0$, and at large-$N$ limit, only two 2PI planar diagrams with exactly one G_{ii} and one $G_{\bar{i}\,\bar{i}}$ leading to the same kernel as in $O(N)^3$ model, having a complex scaling dimension in $\mathcal{P}$

$\Rightarrow$ The fishnet model has an instability associated to the perturbations δG_{ii} and $\delta G_{\bar{i}\,\bar{i}}$

Summary and outlook

- A proof of the Breitenlohner-Freedman instability directly on the CFT side

  i.e. CFTs with a primary operator of dimension $h = d/2 + \mathrm{i}\, r$ are unstable

- Several melonic examples, as well as fishnet model

- It should be stressed that sometimes instability can be avoided (e.g. at imaginary coupling)

- The large-N limit is not needed for the proof, but probably it is needed for finding an operator dimension with real part exactly equal to $d/2$
  (open question)

- Conjecture: "Under the same assumptions, in the true vacuum of the theory, the operator $\mathcal{O}_{h_\star}$ acquires a non-trivial vacuum expectation value: $\langle \mathcal{O}_{h_\star} \rangle \neq 0$." [Kim, Klebanov, Tarnopolsky, Zhao - 2019]
  Probably needs further assumptions on the 2PI effective action

- Similar technique for a derivation of AdS/CFT from $O(N)$ model
  [de Mello Koch, Jevicki, Suzuki,Yoon 2018; Aharony, Chester, Urbach 2020]
  $\Rightarrow$ understand the relation between our construction and the proof of the Breitenlohner-Freedman bound in AdS$_{d+1}$?