Double scaling limit of the $O(N)^3$-tensor model

joint work with V. Bonzom & A. Tanasa
after arXiv:2109.07238

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Introduction

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- Generalization of matrix models to $d > 2$ dimensions
- Large N expansion governed by the degree $\omega \geq 0$. Graphs $\mathcal{G}$ such that $\omega(\mathcal{G}) = 0$ are know as *melons*
- For matrix model, the double-scaling limit consists in taking $N \to +\infty, \lambda \to \lambda_c$ holding some ratio of the two fixed. It corresponds to a continuum limit.
- Tensor model have similar mechanism which has been studied for various model with a single (tetrahedral) interaction:
  1. *Bonzom & al., 2014* (arxiv:1404.7517) for the colored tensor model
  2. *Gurau & al., 2015* (arxiv:1505.00586) for the multi-orientable tensor model
  3. *Benedetti & al., 2020* (arxiv:2003.02100) for the $U(N)^2 \times O(D)$ multi-matrix model

Outline

1. The model
   - The quartic $O(N)^3$-tensor model
   - A useful combinatorial tool : the schemes
2. Finiteness of the number of schemes
3. Identification of the dominant schemes
   - Relevant singularities
   - Structure of the dominant schemes
4. Conclusion

The model

Introduced by S. Carrozza and A. Tanasa in *Carrozza & Tanasa, 2015* (1512.06718)

- The tensor ϕ_{abc} is invariant under the action of $O(N)^3$:

$$\phi_{abc} \to \phi'_{a'b'c'} = \sum_{a,b,c=1}^{N} O^1_{a'a} O^2_{b'b} O^3_{c'c} \phi_{abc} \qquad O^i \in O(N)$$

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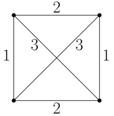
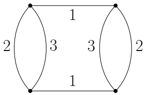
- Two different quartic invariants:

$$I_t(\phi) = \sum_{a,a',b,b',c,c'} \phi_{abc}\phi_{ab'c'}\phi_{a'bc'}\phi_{a'b'c} =$$



$$I_{p,1}(\phi) = \sum_{a,a',b,b',c,c'} \phi_{abc}\phi_{a'bc}\ \phi_{ab'c'}\phi_{a'b'c'} =$$



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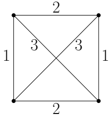

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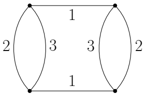
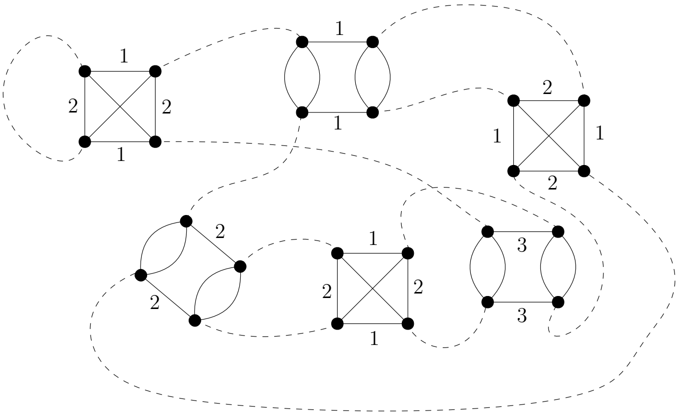


- The action reads

$$S_N(\phi) = -\frac{N^2}{2}\phi^2 + N^{5/2}\frac{\lambda_1}{4} I_t(\phi) + N^2\frac{\lambda_2}{4}\Big(I_{p,1}(\phi) + I_{p,2}(\phi) + I_{p,3}(\phi)\Big) \quad (1)$$

An example of graph of the model



The large N limit expansion

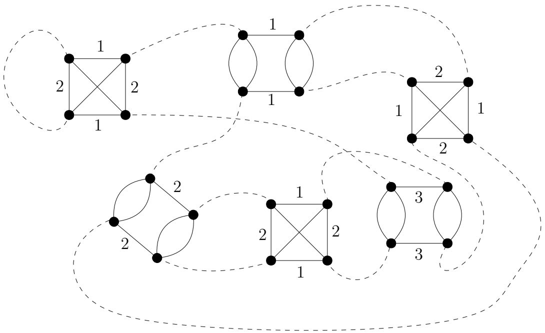
The free energy admits a large N expansion

$$F_N(\lambda_1, \lambda_2) = \ln Z_N(\lambda_1, \lambda_2) = \sum_{\bar{\mathcal{G}} \in \bar{\mathbb{G}}} N^{3-\omega(\bar{\mathcal{G}})} \mathcal{A}(\bar{\mathcal{G}}). \tag{2}$$

where

$$\omega(\bar{\mathcal{G}}) = 3 + \frac{3}{2} n_t(\bar{\mathcal{G}}) + 2 n_p(\bar{\mathcal{G}}) - F(\bar{\mathcal{G}}) \tag{3}$$

An example of graph (again)



$$n_t = 3$$
$$n_p = 3$$
$$F_1 = 1$$
$$F_2 = 3$$
$$F_3 = 1$$
$$\Rightarrow \omega = \frac{17}{2}$$

Melons and melonic move

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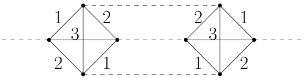
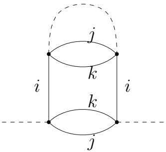
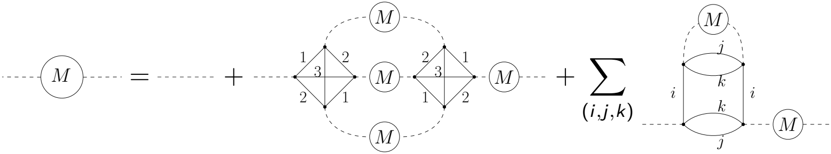


Type II:





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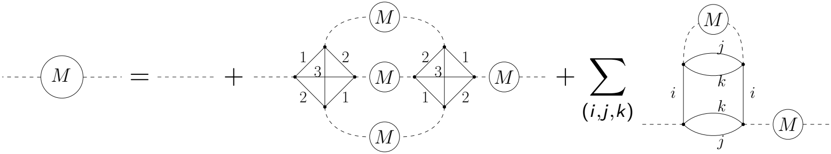

(4)

After change of variables $(t, \mu) = (\lambda_1^2, \frac{3\lambda_2}{\lambda_1^2})$ the above equation reads

$$M(t, \mu) = 1 + tM(t, \mu)^4 + t\mu M(t, \mu)^2 \tag{5}$$

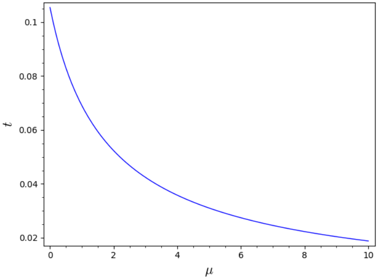

Figure: The critical points of $M(t, \mu)$

What are schemes ?

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- A scheme (of degree ω) is a "blueprint" that tells us how to obtain graphs of the same degree ω.
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  - *Chapuy & al., 2007)* (arxiv:0712.3649) for maps
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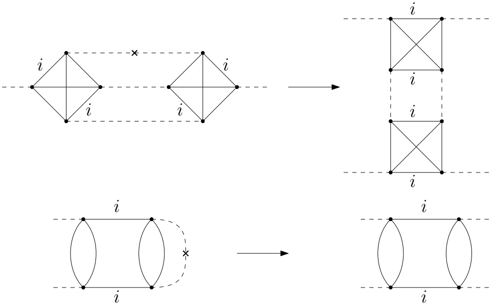
We have already seen one such move earlier with melonic move !

Dipoles

Definition

A dipole is a 4-point graph obtained by cutting an edge in an elementary melon.



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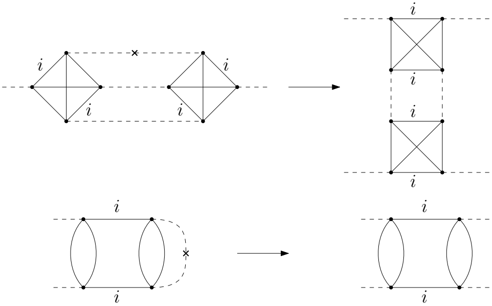

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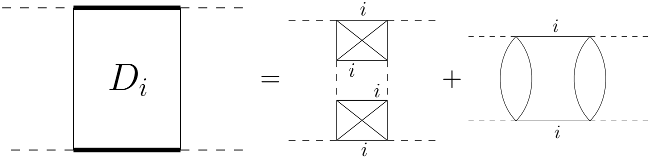
Dipoles



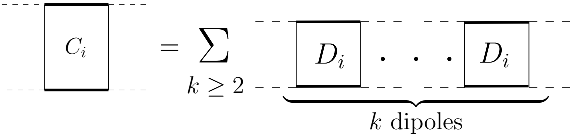
(6)

$$U(t,\mu) = \left(\lambda_1^2 M(t,\mu)^2 + \lambda_2\right) M(t,\mu)^2$$
$$= tM(t,\mu)^4 + \frac{1}{3} t\mu M(t,\mu)^2 \underset{(5)}{=} M(t,\mu) - \frac{2}{3} t\mu M(t,\mu)^2 - 1 \qquad (7)$$

Chains

Definition

Chains are the 4-point functions obtained by connecting an arbitrary number of dipoles by matching one side of a dipole to another side of a distinct dipole.



(8)

$$C_i(t,\mu) = U(t,\mu)^2 \sum_{k \geq 0} U(t,\mu)^k = \frac{U(t,\mu)^2}{1 - U(t,\mu)} \tag{9}$$

$$B(t,\mu) = \frac{\left(3U(t,\mu)\right)^2}{1 - 3U(t,\mu)} - 3\frac{U(t,\mu)^2}{1 - U(t,\mu)} = \frac{6U(t,\mu)^2}{(1 - 3U(t,\mu))(1 - U(t,\mu))} \tag{10}$$

Schemes

Definition

The scheme $\mathcal{S}$ of a 2-point graph $\mathcal{G}$ is obtained by

1. Removing all melonic 2-point subgraphs in $\mathcal{G}$
2. Replacing all maximal (broken and not broken) chains with chain-vertices of the same type, and all dipoles with dipole-vertex of the same color.

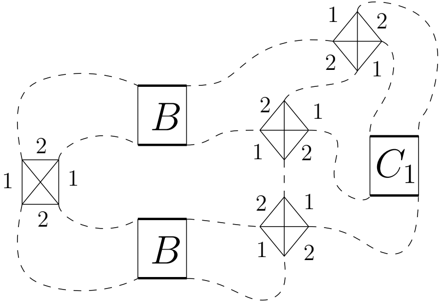


Figure: An example of scheme

Schemes

- Is it possible to find more moves to repackage even more graphs together ?

Schemes

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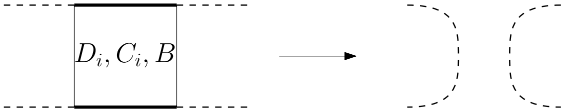
Theorem

The set of schemes of a given degree is finite in the quartic $O(N)^3$-invariant tensor model.

Overview of the proof

1. Study what happens when removing a dipole/chain in a scheme
2. Show that there can only be finitely many other faces

Chain removal



Two different cases:

1. Removing the chain disconnects the scheme

$$\Delta\omega = 0$$

2. Removing the chain doesn't disconnect the scheme:

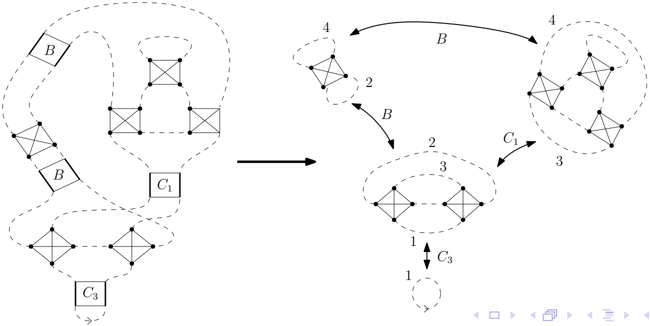
$$-3 \leq \Delta\omega \leq -1$$

One last definition..

Definition

For a scheme $\mathcal{S}$, its *skeleton graph* $\mathcal{I}(\mathcal{S})$ is such that

- The vertex set of $\mathcal{I}(\mathcal{S})$ is the set of connected components obtained by removing all chain-vertices and dipole-vertices of $\mathcal{S}$.
- There is an edge between two vertices in $\mathcal{I}(\mathcal{S})$ if the two corresponding connected components are connected by a chain-vertex or a dipole-vertex. This edge is labeled by the type of chain- or dipole-vertex.



Properties of the skeleton graph

Lemma

For any scheme $\mathcal{S}$, define by convention $\omega(\mathcal{I}(\mathcal{S})) = \omega(\mathcal{S})$. We have the following properties.

1. *If $\bar{\mathcal{G}}^{(r)}$ for $r \in \{1, \ldots, p\}$ has vanishing degree, then the corresponding vertex in $\mathcal{I}(\mathcal{S})$ has valency at least equal to 3.*

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2. *Let $\mathcal{T} \subset \mathcal{I}(\mathcal{S})$ be a spanning tree. Let q be the number of edges of $\mathcal{I}(\mathcal{S})$ which are not in $\mathcal{T}$. Then, $\omega(\mathcal{T}) \leq \omega(\mathcal{S}) - q$.*

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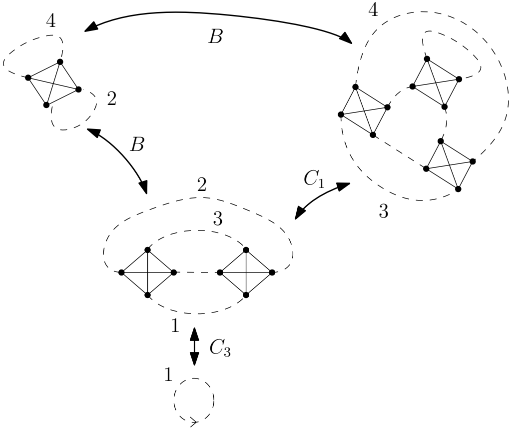
3. *$\omega(\mathcal{T}) = \omega(\mathcal{G}^{(0)}) + \sum_{r=1}^{p} \omega(\bar{\mathcal{G}}^{(r)})$, in other words, if $\mathcal{I}(\mathcal{S})$ is a tree, then the degree of $\mathcal{S}$ is the sum of the degrees of its components obtained by removing all chain- and dipole-vertices.*

There are finitely many chains in a scheme

$$N(\mathcal{S}) \leq 7\omega(\mathcal{S}) - 1 \quad (11)$$



What about scheme with no chains ?

Through each edge of color 0 passes exactly one face of each color.

$$\sum_{p\geq 1} pF_i^{(p)}(\mathcal{G}) = E_0(\mathcal{G}) = 2n(\mathcal{G}) \tag{12}$$

Using equation of the degree we get

$$\sum_{p\geq 5}(p-4)F^{(p)}(\mathcal{G}) = 4(\omega - 3) + 3F^{(1)}(\mathcal{G}) + 2F^{(2)}(\mathcal{G}) + F^{(3)}(\mathcal{G}) \tag{13}$$

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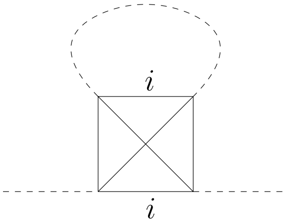
- We only have to show that there are finitely many "short faces"

Faces of length 1 and 2

1. Faces of length 1 are tadpoles

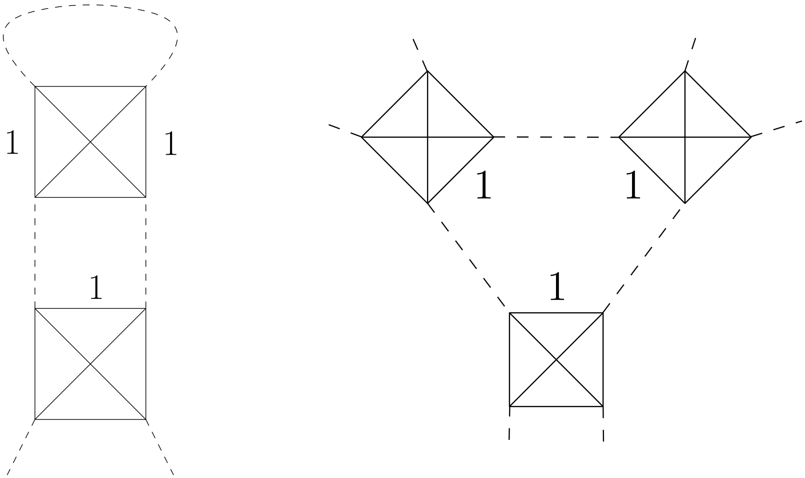


Removing the tadpole gives

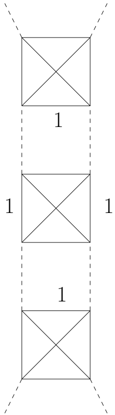
$$\Delta\omega = -\frac{1}{2} \tag{14}$$

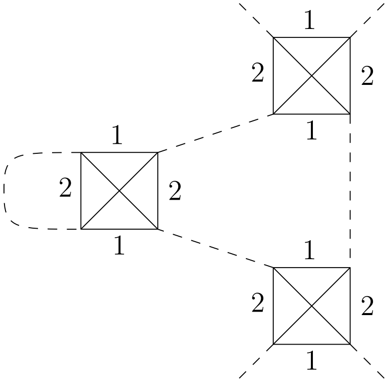
2. Faces of length 2 are dipoles, which have already been shown to be bounded.

Faces of length 3



Self-intersecting faces of length 4





Non-self intersecting faces of length 4

- Too many cases to handle..

Non-self intersecting faces of length 4

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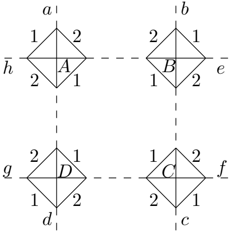
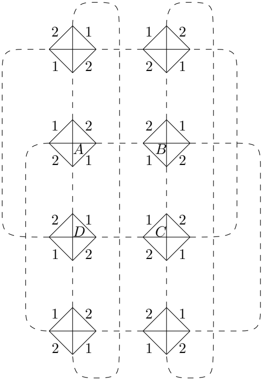
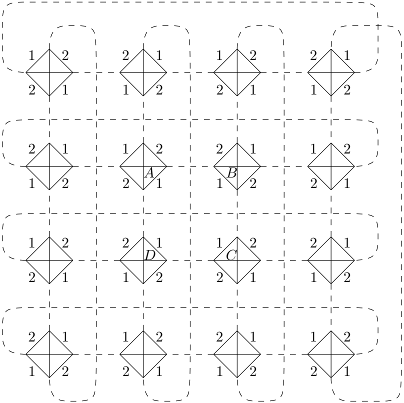
- Instead, we can show that all bubbles contributing to a face of length 4 are at bounded distance of a bubble which is in finite quantity in the scheme.



Two possible schemes





We have shown that

1. There are finitely many chains in a scheme of fixed degree
2. There are finitely many scheme with no chains.

Therefore..

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1. There are finitely many chains in a scheme of fixed degree
2. There are finitely many scheme with no chains.

Therefore..

Theorem

The set of schemes of a given degree is finite in the quartic $O(N)^3$-invariant tensor model.

Note however that this proof told us very little on the structure of the said schemes..

Which singularities are relevant ?

- Critical points of $M(t, \mu)$

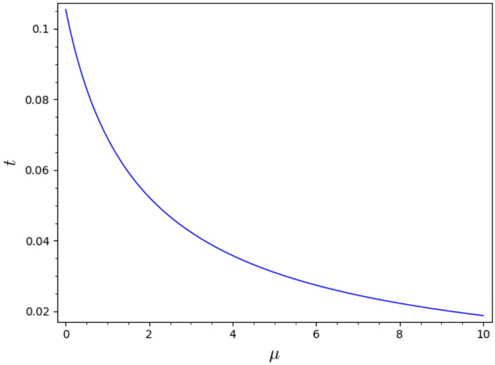

Figure: The critical points of $M(t, \mu)$

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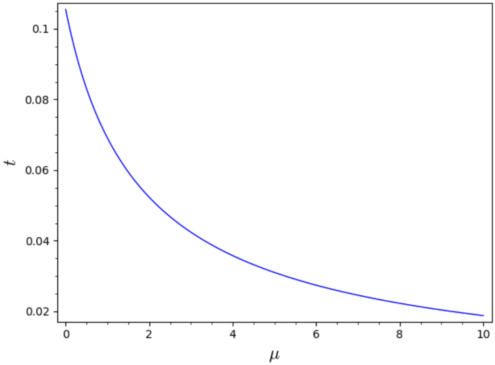

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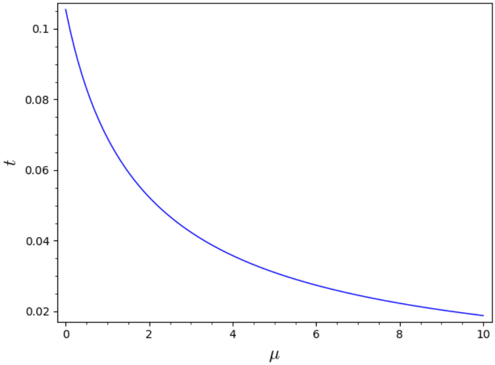

Figure: The critical points of $M(t,\mu)$

- Points such that $U(t,\mu) = 1$, critical for any chains
- Points such that $U(t,\mu) = \frac{1}{3}$ for broken chains.

Critical points for broken chains

We can use the equation of $M(t, \mu)$ get an expression for t

$$t = \frac{M(t,\mu) - 1}{M(t,\mu)^4 + \mu M(t,\mu)^2} \tag{15}$$

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$$\begin{aligned} U(t,\mu) = \frac{1}{3} &\iff M(t,\mu) - \frac{4}{3} - \frac{2}{3}\frac{(M(t,\mu)-1)M(t,\mu)^2\mu}{M(t,\mu)^4 + \mu M(t,\mu)^2} = 0 \\ &\iff -3M(t,\mu)^3 + 4tM(t,\mu)^2 - \mu M(t,\mu) + 2\mu = 0 \end{aligned} \tag{16}$$

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Which is exactly the equation defining critical points of $M(t,\mu)$!

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Therefore, the skeleton graph of a dominant scheme must be a tree whose edges corresponds to broken chains and nodes to graphs without melons, dipoles or chains, which carry the degree. Moreover, to maximize the number of edges in the tree, each vertex must have a degree and valency as small as possible.

Structure of the dominant schemes

Theorem

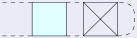
The dominant schemes of degree $\omega > 0$ *are given bijectively by rooted plane binary trees with* $4\omega - 1$ *edges, with the following correspondence*

- *The root of the tree corresponds to the two external legs of the 2-point function.*
- *Edges of the tree correspond to broken chains.*

Structure of the dominant schemes

Theorem

The dominant schemes of degree $\omega > 0$ are given bijectively by rooted plane binary trees with $4\omega - 1$ edges, with the following correspondence

- *The root of the tree corresponds to the two external legs of the 2-point function.*
- *Edges of the tree correspond to broken chains.*
- *The leaves are tadpoles:* 

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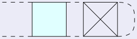
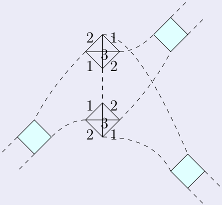
- *The leaves are tadpoles:* 
- *There are two types of internal nodes,*



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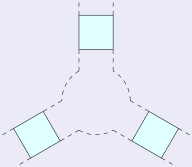

Generating function of dominant scheme

The generating function associated to one dominant schemes is

$$
\begin{aligned}
G^{\omega}_{\mathcal{T}}(t,\mu) &= (3t^{\frac{1}{2}})^{2\omega}(1+6t)^{2\omega-1}B(t,\mu)^{4\omega-1} \\
&= (3t^{\frac{1}{2}})^{2\omega}(1+6t)^{2\omega-1}\frac{6^{4\omega-1}U^{8\omega-2}}{((1-U)(1-3U))^{4\omega-1}}
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Summing over the different trees and taking into account melonic insertions at the root gives

$$G_{\mathrm{dom}}^{\omega}(t,\mu) = M(t,\mu)\sum_{\substack{\mathcal{T} \\ 2\omega \text{ leaves}}} G_{\mathcal{T}}^{\omega}(t,\mu)$$
$$= \mathrm{Cat}_{2\omega-1}\, M(t,\mu) G_{\mathcal{T}}^{\omega}(t,\mu) \quad (18)$$

Double scaling parameter

Near critical point

$$G^{\omega}_{dom}(t,\mu) \underset{t\to t_c(\mu)}{\sim} N^{3-\omega} M_c(\mu)\,\mathrm{Cat}_{2\omega-1}\,9^{\omega}\,t_c^{\omega}\,(1+6t_c)^{2\omega-1} \times \left(\frac{1}{\left(1-\frac{4}{3}t_c(\mu)\mu M_c(\mu)\right) K(\mu)\sqrt{1-\frac{t}{t_c(\mu)}}} \right)^{4\omega-1} \quad (19)$$

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Therefore we define the following double scaling parameter

$$\kappa(\mu)^{-1} = \frac{1}{3}\frac{1}{t_c(\mu)^{\frac{1}{2}}\,(1+6t_c(\mu))}\left(\left(1-\frac{4}{3}t_c(\mu)\mu M_c(\mu)\right)K(\mu)\right)^2\left(1-\frac{t}{t_c(\mu)}\right)N^{\frac{1}{2}} \quad (20)$$

Final computation

We now have

$$G^{\omega}_{dom}(\mu) = M_c(\mu) \frac{N^{\frac{11}{4}}}{\kappa(\mu)^{\frac{1}{2}}} \sqrt{3} \frac{t_c(\mu)^{\frac{1}{4}}}{(1+6t_c(\mu))^{\frac{1}{2}}} \, \mathrm{Cat}_{2\omega-1} \, \kappa(\mu)^{2\omega} \tag{21}$$

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And therefore..

$$\begin{aligned} G_2^{DS}(\mu) &= N^{-3}\sum_{\omega\in\mathbb{N}/2} G^{\omega}_{dom}(\mu) \\ &= M_c(\mu)\left(1 + N^{-\frac{1}{4}}\sqrt{3}\frac{t_c(\mu)^{\frac{1}{4}}}{(1+6t_c(\mu))^{\frac{1}{2}}}\frac{1-\sqrt{1-4\kappa(\mu)}}{2\kappa(\mu)^{\frac{1}{2}}}\right) \end{aligned} \tag{22}$$

which is convergent for $\kappa(\mu) \leq \frac{1}{4}$.

Conclusion

Recap

- Showed that there were finitely many schemes of any degree ω
- Defined the double scaling limit of the quartic $O(N)^3$ tensor models with both tetrahedral and pillow interaction and computed the 2-point function.

Remarks

- Despite not knowing all schemes of given degree ω, we could still identify dominant schemes !
- Similar method apply to other models, such as the $U(N)^2 \times O(D)$ multi-matrix model

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Questions

- Could we use similar method for higher-order interactions (e.g. sextic wheel/prismatic interaction) ?
- Is there a model where broken chains are not dominating in the double scaling limit ?

Thank you for your attention !