

On mutimatrix models motivated by noncommutative geometry:

a gauge theory perspective

Tensor Journal Club

Carlos Pérez-Sánchez

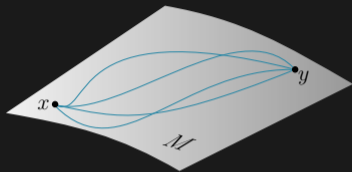
Institute for Theoretical Physics
University of Heidelberg

24. Nov 2021

I. MOTIVATING NONCOMMUTATIVE GEOMETRY

Heuristics:

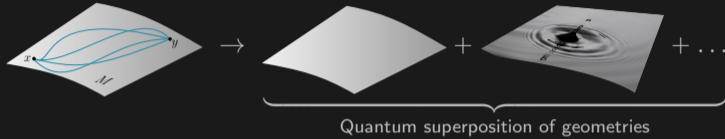
- Path integrals on M



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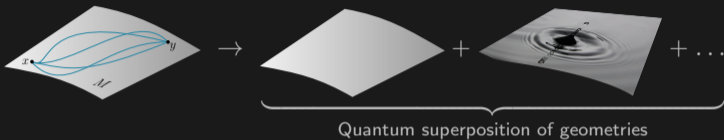
- Path integrals on $M \rightarrow$ Path integral of spacetime (Quantum Gravity)



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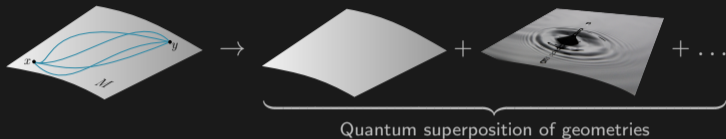
- Quantum Gravity \rightarrow Random Geometry

$$\mathcal{Z}_{\text{QG}} = \int\!\!\int_{\substack{\text{topologies} \\ \text{geometries}}} e^{\frac{i}{\hbar} S_{\text{EH}}[g]} \mathcal{D}[g] \rightarrow \mathcal{Z} = \int\!\!\int_{\substack{\text{topologies} \\ \text{geometries}}} e^{-\frac{1}{\hbar} S_{\text{EH}}[g]} \mathcal{D}[g]$$

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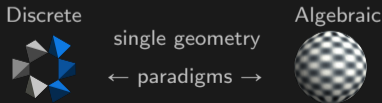
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- Quantum Gravity \rightarrow Random Geometry

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- To access \mathcal{Z} , models for 'quantum space' are proposed, e.g.



(from Wikipedia)

The Spectral Standard Model

$$\begin{aligned}
 & -\frac{1}{2}\partial_\nu g_\mu^a \partial_\nu g_\mu^a - g_s f^{abc} \partial_\mu g_\nu^a g_\mu^b g_\nu^c - \frac{1}{4}g_s^2 f^{abc} f^{ade} g_\mu^b g_\nu^c g_\mu^d g_\nu^e + \\
 & \frac{1}{2}ig_s^2 (\bar{q}_i^\mu \gamma^\mu q_j^\mu) g_\mu^a + \bar{G}^a \partial^2 G^a + g_s f^{abc} \partial_\mu \bar{G}^a G^b g_\mu^c - \partial_\nu W_\mu^+ \partial_\nu W_\mu^- - \\
 & M^2 W_\mu^+ W_\mu^- - \frac{1}{2}\partial_\nu Z_\mu^0 \partial_\nu Z_\mu^0 - \frac{1}{2c_w^2} M^2 Z_\mu^0 Z_\mu^0 - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \\
 & \frac{1}{2}\partial_\mu H \partial_\mu H - \frac{1}{2}m_h^2 H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \\
 & \frac{1}{2c_w} M \phi^0 \phi^0 - \beta_h \left[\frac{2M^2}{g^2} + \frac{2M}{g} H + \frac{1}{2}(H^2 + \phi^0 \phi^0 + 2\phi^+ \phi^-) \right] + \frac{2M^4}{g^2} \alpha_h - \\
 & ig_{c_w} [\partial_\nu Z_\mu^0 (W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) - Z_\nu^0 (W_\mu^+ \partial_\nu W_\mu^- - W_\mu^- \partial_\nu W_\mu^+) + \\
 & Z_\mu^0 (W_\nu^+ \partial_\nu W_\mu^- - W_\nu^- \partial_\nu W_\mu^+)] - ig_{s_w} [\partial_\nu A_\mu (W_\mu^+ W_\nu^- - \\
 & W_\nu^+ W_\mu^-) - A_\nu (W_\mu^+ \partial_\nu W_\mu^- - W_\mu^- \partial_\nu W_\mu^+) + A_\mu (W_\nu^+ \partial_\nu W_\mu^- - \\
 & W_\nu^- \partial_\nu W_\mu^+)] - \frac{1}{2}g^2 W_\mu^+ W_\nu^+ W_\mu^- W_\nu^- + \frac{1}{2}g^2 W_\nu^+ W_\mu^+ W_\nu^- W_\mu^- + \\
 & g^2 c_w^2 (Z_\mu^0 W_\nu^+ Z_\nu^0 W_\mu^- - Z_\nu^0 Z_\mu^0 W_\nu^+ W_\mu^-) + g^2 s_w^2 (A_\mu W_\nu^+ A_\nu W_\mu^- - \\
 & A_\nu A_\mu W_\nu^+ W_\mu^-) + g^2 s_w c_w [A_\mu Z_\nu^0 (W_\nu^+ W_\mu^- - W_\mu^- W_\nu^+) - \\
 & 2A_\mu Z_\mu^0 W_\nu^+ W_\nu^-] - g\alpha [H^3 + H\phi^0 \phi^0 + 2H\phi^+ \phi^-] - \frac{1}{8}g^2 \alpha_h [H^4 + \\
 & (\phi^0)^4 + 4(\phi^+ \phi^-)^2 + 4(\phi^0)^2 \phi^+ \phi^- + 4H^2 \phi^+ \phi^- + 2(\phi^0)^2 H^2] - \\
 & gM W_\mu^+ W_\mu^- H - \frac{1}{2}g \frac{M}{c_w} Z_\mu^0 Z_\mu^0 H - \frac{1}{2}ig [W_\mu^+ (\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) - \\
 & W_\mu^- (\phi^0 \partial_\mu \phi^+ - \phi^+ \partial_\mu \phi^0)] + \frac{1}{2}g [W_\mu^+ (H \partial_\mu \phi^- - \phi^- \partial_\mu H) - \\
 & W_\mu^- (H \partial_\mu \phi^+ - \phi^+ \partial_\mu H)] + \frac{1}{2}g \frac{1}{c_w} (Z_\mu^0 (H \partial_\mu \phi^0 - \phi^0 \partial_\mu H) - \\
 & ig \frac{s_w^2}{c_w} M Z_\mu^0 (W_\mu^+ \phi^- - W_\mu^- \phi^+) + ig_{s_w} M A_\mu (W_\mu^+ \phi^- - W_\mu^- \phi^+) - \\
 & ig \frac{1-2c_w^2}{2c_w} Z_\mu^0 (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) + ig_{s_w} A_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) - \\
 & \frac{1}{4}g^2 W_\mu^+ W_\mu^- [H^2 + (\phi^0)^2 + 2\phi^+ \phi^-] - \frac{1}{4}g^2 \frac{1}{c_w^2} Z_\mu^0 Z_\mu^0 [H^2 + (\phi^0)^2 + \\
 & 2(2s_w^2 - 1)^2 \phi^+ \phi^-] - \frac{1}{2}g^2 \frac{s_w^2}{c_w} Z_\mu^0 \phi^0 (W_\mu^+ \phi^- + W_\mu^- \phi^+) -
 \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{2}ig^2 \frac{s_w^2}{c_w} Z_\mu^0 H (W_\mu^+ \phi^- - W_\mu^- \phi^+) + \frac{1}{2}g^2 s_w A_\mu \phi^0 (W_\mu^+ \phi^- + \\
 & W_\mu^- \phi^+) + \frac{1}{2}ig^2 s_w A_\mu H (W_\mu^+ \phi^- - W_\mu^- \phi^+) - g^2 \frac{s_w}{c_w} (2c_w^2 - \\
 & 1) Z_\mu^0 A_\mu \phi^+ \phi^- - g^1 s_w^2 A_\mu A_\mu \phi^+ \phi^- - \bar{e}^\lambda (\gamma \partial + m_e^\lambda) e^\lambda - \bar{\nu}^\lambda \gamma \partial \nu^\lambda - \\
 & \bar{u}_j^\lambda (\gamma \partial + m_u^\lambda) u_j^\lambda - \bar{d}_j^\lambda (\gamma \partial + m_d^\lambda) d_j^\lambda + ig_{s_w} A_\mu [-\bar{e}^\lambda \gamma^\mu e^\lambda + \\
 & \frac{2}{3}(\bar{u}_j^\lambda \gamma^\mu u_j^\lambda) - \frac{1}{3}(\bar{d}_j^\lambda \gamma^\mu d_j^\lambda)] + \frac{ig}{4c_w} Z_\mu^0 [(\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + \\
 & (\bar{e}^\lambda \gamma^\mu (4s_w^2 - 1 - \gamma^5) e^\lambda) + (\bar{u}_j^\lambda \gamma^\mu (\frac{4}{3}s_w^2 - 1 - \gamma^5) u_j^\lambda) + (\bar{d}_j^\lambda \gamma^\mu (1 - \\
 & \frac{8}{3}s_w^2 - \gamma^5) d_j^\lambda)] + \frac{ig}{2\sqrt{2}} W_\mu^+ [(\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) e^\lambda) + (\bar{u}_j^\lambda \gamma^\mu (1 + \\
 & \gamma^5) C_{\lambda\kappa}^\dagger d_j^\kappa)] + \frac{ig}{2\sqrt{2}} W_\mu^- [(\bar{e}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger \gamma^\mu (1 + \\
 & \gamma^5) u_j^\lambda)] + \frac{ig}{2\sqrt{2}} \frac{m_\lambda^2}{M} [-\phi^+ (\bar{\nu}^\lambda (1 - \gamma^5) e^\lambda) + \phi^- (\bar{e}^\lambda (1 + \gamma^5) \nu^\lambda)] - \\
 & \frac{g}{2} \frac{m_\lambda^2}{M} [H (\bar{e}^\lambda e^\lambda) + i\phi^0 (\bar{e}^\lambda \gamma^5 e^\lambda)] + \frac{ig}{2M\sqrt{2}} \phi^+ [-m_a^2 (\bar{u}_j^\lambda C_{\lambda\kappa}^\dagger (1 - \\
 & \gamma^5) d_j^\kappa) + m_u^2 (\bar{u}_j^\lambda C_{\lambda\kappa}^\dagger (1 + \gamma^5) d_j^\kappa) + \frac{ig}{2M\sqrt{2}} \phi^- [m_d^2 (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger (1 + \\
 & \gamma^5) u_j^\kappa) - m_u^2 (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger (1 - \gamma^5) u_j^\kappa) - \frac{g}{2} \frac{m_h^2}{M} H (\bar{u}_j^\lambda u_j^\lambda) - \\
 & \frac{g}{2} \frac{m_d^2}{M} H (\bar{d}_j^\lambda d_j^\lambda) + \frac{ig}{2} \frac{m_\lambda^2}{M} \phi^0 (\bar{u}_j^\lambda \gamma^5 u_j^\lambda) - \frac{ig}{2} \frac{m_\lambda^2}{M} \phi^0 (\bar{d}_j^\lambda \gamma^5 d_j^\lambda)
 \end{aligned}$$

Num. of generations and $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \rightsquigarrow$

NCG

\rightsquigarrow Classical Lagrangian of the Standard Model

[Chamseddine-Connes-Marcolli *ATMP* 2007 (Euclidean); J. Barrett *J. Math. Phys.* 2007 (Lorentzian)]

go to sketch of proof \triangleright

Commutative spectral triples

A spin manifold M yields (A_M, H_M, D_M)

- $A_M = C^\infty(M)$ is a comm. $*$ -algebra
- $H_M := L^2(M, \mathbb{S})$ a representation of A_M
- $D_M = -i\gamma^\mu(\partial_\mu + \omega_\mu)$ is self-adjoint
- for each $a \in A_M$, $[D_M, a]$ is bounded, and in fact $[D_M, x^\mu] = -ic(dx^\mu) = -i\gamma^\mu$, being c Cliff. mult.
- D_M has compact resolvent

[go to more on spin geometry >](#)

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A *spectral triple* (A, H, D) consists of

- a $*$ -algebra A
- a representation H of A
- a self-adjoint operator D on H with compact resolvent and such that $[D, a]$ is bounded for each $a \in A$

The 'commutative case' motivates

$$\Omega_D^1(A) := \left\{ \sum_{(\text{finite})} b[D, a] \mid a, b \in A \right\}$$

go to more on spin geometry \triangleright

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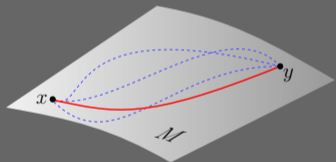
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RECONSTRUCTION THEOREM: [A, Connes, JNCG '13] (quite roughly formulated)
Commutative spectral triples +some more axioms are Riemannian manifolds.

The idea is to replace the metric in (M, g) by D_M

Connes' geodesic distance

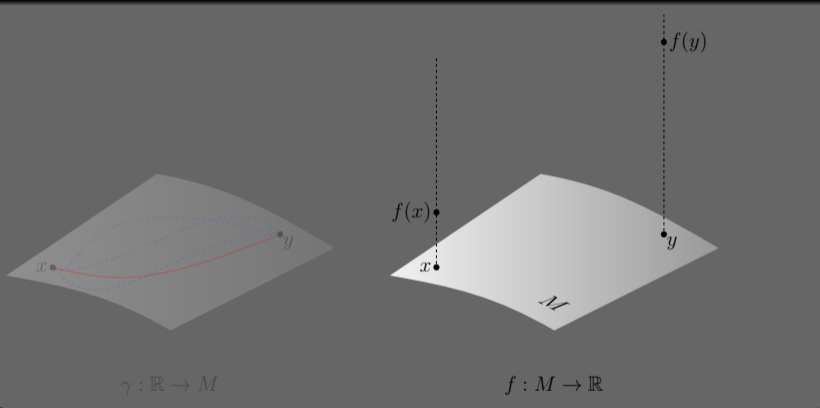


$$\gamma : \mathbb{R} \rightarrow M$$

$$\inf_{\gamma \text{ as above}} \left\{ \int_{\gamma} ds \right\} = d(x, y)$$

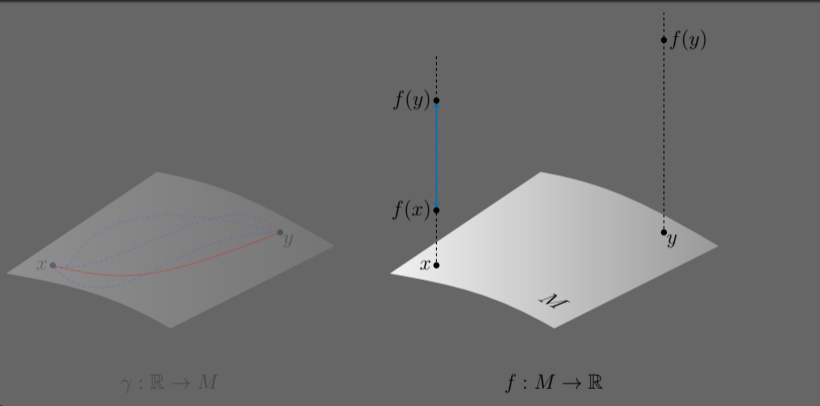
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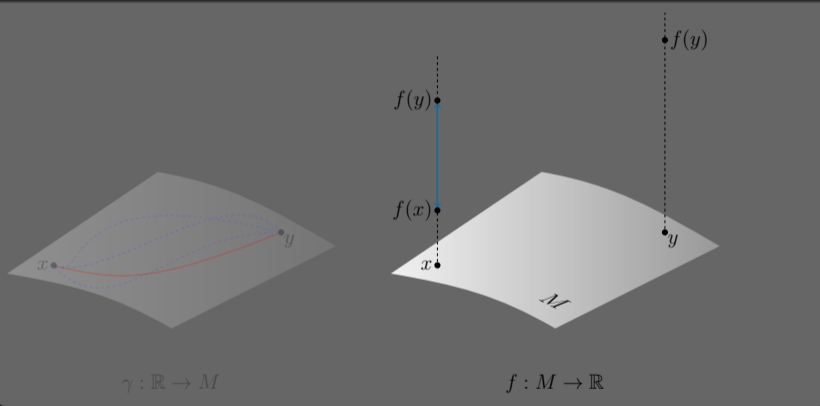
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$$|f(x) - f(y)|$$

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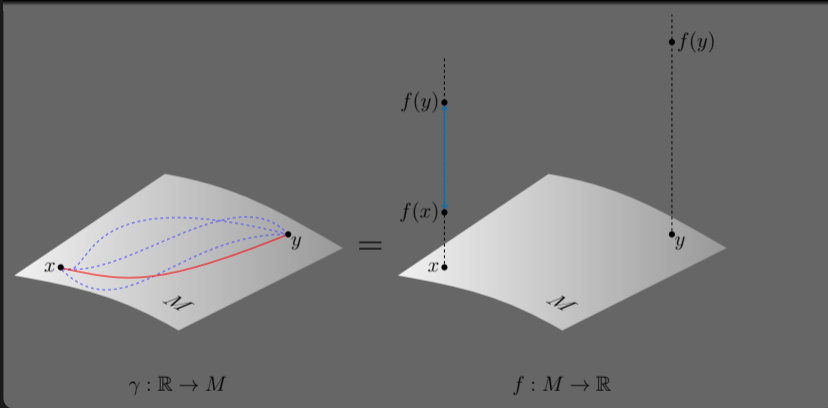
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$$\sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| : \|D_M f - f D_M\| \leq 1 \}$$

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$$\inf_{\gamma \text{ as above}} \left\{ \int_{\gamma} ds \right\} = d(x, y) = \sup_{f \in C^{\infty}(M)} \left\{ |f(x) - f(y)| : \|D_M f - f D_M\| \leq 1 \right\}$$

go to examples ▽

NCG toolkit in high energy physics

- On a spectral triple (A, H, D) the (bosonic) classical action is given by

$$S(D) = \text{Tr}_H f(D/\Lambda) \quad [\text{Chamseddine-Connes } CMP '97]$$

for a bump function f around the origin and Λ a cut off scale. It's computed with heat kernel expansion [P. Gilkey, *J. Diff. Geom.* '75]

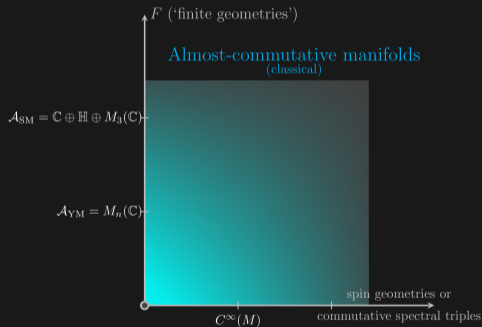
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- Realistic, classical models come from *almost-commutative manifolds* $M \times F$, where F is a finite-dim. spectral triple $(C^\infty(A_F), H_M \otimes H_F, D_M \otimes 1_F + \gamma_5 \otimes D_F)$



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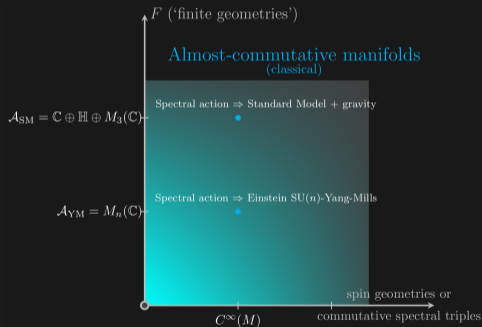
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- applications require (A, H, D) to have a *reality* $J : H \rightarrow H$ antiunitary ^{+some axioms}, implementing a right A -action on H ,

$$\psi a = J a^* J^{-1} \psi \quad \psi \in H, a \in A$$



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- let's sketch *connections*: if S^G is a G -invariant functional on M

$$S^G \rightsquigarrow S^{\text{Maps}(M, G)}$$

$$d \rightsquigarrow d + \mathbb{A} \quad \mathbb{A} \in \Omega^1(M) \otimes \mathfrak{g}$$

$$\mathbb{A}' = u \mathbb{A} u^{-1} + u du^{-1} \quad u \in \text{Maps}(M, G)$$

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- given (A, H, D) and a Morita equivalence $A \simeq_M B$ (i.e. $\text{End}_A(E) \cong B$) yields new $(B, E \otimes_A H, D')$. For $A = B$, in fact a tower

$$\{(A, H, D_\omega)\}_{\omega \in \Omega_D^1(A)}$$

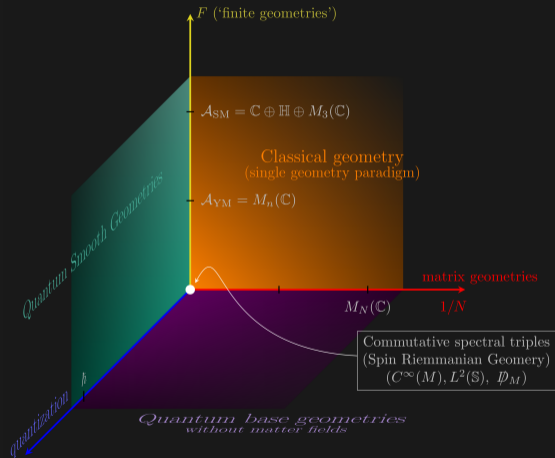
with $D_\omega = D + \omega$. In presence of J

$$D_\omega = D + \omega \pm J \omega J^{-1}$$

$$D_\omega \mapsto \text{Ad}(u) D_\omega \text{Ad}(u)^* = D_{\omega_u}$$

$$\omega \mapsto \omega_u = u \omega u^* + u [D, u^*] \quad u \in \mathcal{U}(A)$$

Organization



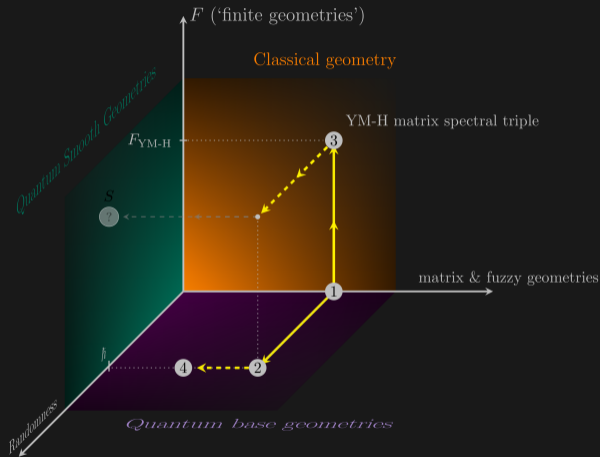
AIM: Make sense of

$$\mathcal{Z} = \int_{\text{DIRAC}} e^{-\frac{1}{\hbar} \text{Tr}_H f(D)} dD$$

- Plane $(\hbar, 1/N, 0)$ of 'base geometries'
- Plane $(\hbar, 0, F) = \lim_{N \rightarrow \infty} (\hbar, 1/N, F)$
- Plane $(0, 1/N, F) = \lim_{\hbar \rightarrow 0} (\hbar, 1/N, F)$
of classical geometries

[CP 2105.01025]

Organization



- 1 Matrix Geometries [J. Barrett, *J. Math. Phys.* 2015]
- 2 Dirac ensembles [J. Barrett, L. Glaser, *J. Phys. A* 2016] and how to compute the spectral action [CP 1912.13288]
- 3 Gauge matrix spectral triples [CP 2105.01025]
- 4 Functional Renormalization [CP 2007.10914] and [CP 2111.02858]

A *fuzzy geometry* (of signature (p, q) (thus of dim. $p + q$ and KO-dim $q - p$) consists of

- $A = M_N(\mathbb{C})$
- $H = \mathbb{S} \otimes M_N(\mathbb{C})$, being \mathbb{S} a $\mathcal{Cl}(p, q)$ -module
... (axioms for D omitted, go to axioms ∇) ...

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... (axioms for D omitted, go to axioms ∇) ...
- Gamma-matrices conventions:
 - $(\gamma^\mu)^2 = +1$,
 $\mu = 1, \dots, p, \gamma^\mu$ Hermitian
 - $(\gamma^\mu)^2 = -1$,
 $\mu = 1 + p, \dots, q + p, \gamma^\mu$ anti-Hermitian
 - $\Gamma^I := \gamma^{\mu_1} \dots \gamma^{\mu_r}$ for $\mu_i = 1, \dots, p + q$,
 $I = (\mu_1, \dots, \mu_r)$

- Characterization of D in even dimensions:

$$D = \sum_J \Gamma_{\text{s.a.}}^J \otimes \{H_J, \cdot\} + \sum_J \Gamma_{\text{anti.}}^J \otimes [L_J, \cdot]$$

multi-index J monot. increasing, $|J|$ odd

[J. Barrett, *J. Math. Phys.* '15], $H_J^* = H_J, L_J^* = -L_J$

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$$D = \sum_J \Gamma_{\text{s.a.}}^J \otimes \{H_J, \cdot\} + \sum_J \Gamma_{\text{anti.}}^J \otimes [L_J, \cdot]$$

multi-index J monot. increasing, $|J|$ odd
[J. Barrett, *J. Math. Phys.* '15], $H_J^* = H_J$, $L_J^* = -L_J$

- Examples:

$$\begin{aligned} - D_{(1,1)} &= \gamma^1 \otimes [L, \cdot] + \gamma^2 \otimes \{H, \cdot\} \\ - D_{(0,4)} &= \sum_{\mu} \gamma^\mu \otimes [L_\mu, \cdot] + \gamma^{\hat{\mu}} \otimes \{H_{\hat{\mu}}, \cdot\} \end{aligned}$$

- [J. Barrett, L. Glaser, *J. Phys. A* 2016]

$$\begin{aligned} \{H, \cdot\} &\mapsto H \otimes 1_N + 1_N \otimes H^T \\ [L, \cdot] &\mapsto L \otimes 1_N - 1_N \otimes L^T \end{aligned}$$

so we will get double traces from

$$\text{Tr}_H = \text{Tr}_{\mathbb{S}} \otimes \text{Tr}_{M_N(\mathbb{C})} = \text{Tr}_{\mathbb{S}} \otimes \text{Tr}_N^{\otimes 2}$$

Notation: $\text{Tr}_V X$ is the trace on operators $X : V \rightarrow V$, $\text{Tr}_V 1 = \dim V$. So $\text{Tr}_N 1 = N$ but $\text{Tr}_{M_N(\mathbb{C})}(1) = N^2$.

Spectral Action: $\text{Tr}_H D^{2m}$ [C.P. 1912.13288] Chord diags. $d = 2$ geometries, $\eta = \text{diag}(\mathbf{e}_1, \mathbf{e}_2)$ go to more CD's \triangleright

$$\text{Tr}_H(D^6) = 2N \sum_{\mu} \overline{\text{Tr}_{\mathbb{S}}(\gamma^{\mu_1} \dots \gamma^{\mu_6})} \times$$

$$+(-1)^2 \eta^{\mu_1 \mu_5} \eta^{\mu_2 \mu_4} \eta^{\mu_3 \mu_6} + (-1)^1 \eta^{\mu_1 \mu_5} \eta^{\mu_2 \mu_6} \eta^{\mu_3 \mu_4} + (-1)^0 \eta^{\mu_6 \mu_1} \eta^{\mu_2 \mu_3} \eta^{\mu_4 \mu_5} + (-1)^1 \eta^{\mu_6 \mu_1} \eta^{\mu_2 \mu_4} \eta^{\mu_3 \mu_5} + (-1)^0 \eta^{\mu_6 \mu_1} \eta^{\mu_2 \mu_5} \eta^{\mu_3 \mu_4}$$

$$+(-1)^1 \eta^{\mu_1 \mu_3} \eta^{\mu_2 \mu_6} \eta^{\mu_4 \mu_5} + (-1)^0 \eta^{\mu_1 \mu_4} \eta^{\mu_2 \mu_3} \eta^{\mu_5 \mu_6} + (-1)^3 \eta^{\mu_1 \mu_4} \eta^{\mu_2 \mu_5} \eta^{\mu_3 \mu_6} + (-1)^2 \eta^{\mu_1 \mu_4} \eta^{\mu_2 \mu_6} \eta^{\mu_3 \mu_5} + (-1)^1 \eta^{\mu_1 \mu_5} \eta^{\mu_2 \mu_3} \eta^{\mu_4 \mu_6}$$

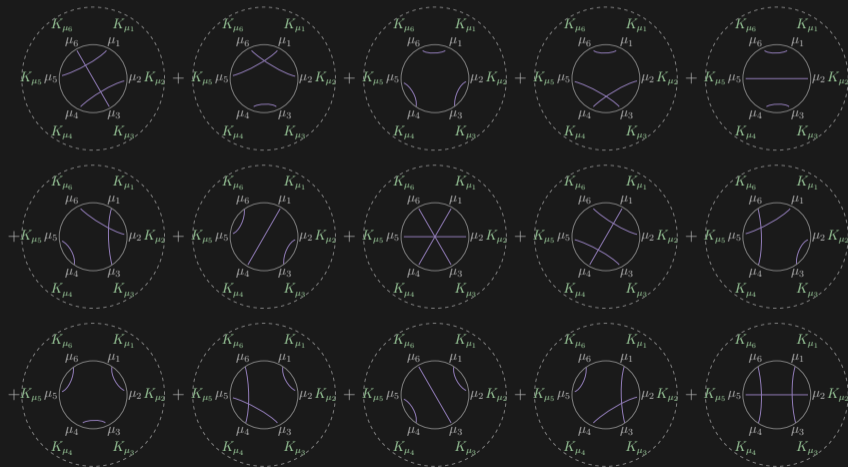
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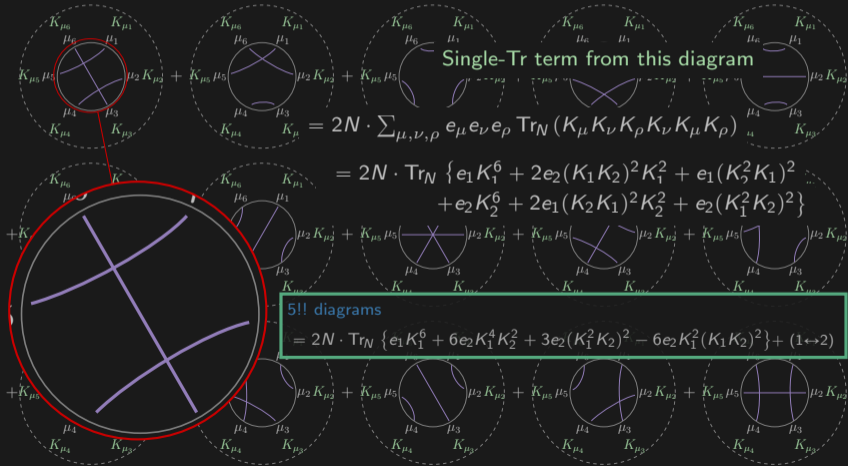
$$\text{Tr}_H(D^6) = 2N \sum_{\mu} \overbrace{\text{Tr}_S(\gamma^{\mu_1} \dots \gamma^{\mu_6})}^{\text{solid circ.}} \times$$

$$\begin{aligned}
 & \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} \\
 & + (-1)^2 \eta^{\mu_1 \mu_5} \eta^{\mu_2 \mu_4} \eta^{\mu_3 \mu_6} + (-1)^1 \eta^{\mu_1 \mu_5} \eta^{\mu_2 \mu_6} \eta^{\mu_3 \mu_4} + (-1)^0 \eta^{\mu_6 \mu_1} \eta^{\mu_2 \mu_3} \eta^{\mu_4 \mu_5} + (-1)^1 \eta^{\mu_6 \mu_1} \eta^{\mu_2 \mu_4} \eta^{\mu_3 \mu_5} + (-1)^0 \eta^{\mu_6 \mu_1} \eta^{\mu_2 \mu_5} \eta^{\mu_3 \mu_4} \\
 & + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} \\
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 \end{aligned}$$

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Multimatrix models with multitraces & ribbon graphs

- The chord-diagram description holds in general dimension and signature [CP '19]

$$\begin{aligned} \mathcal{Z} &= \int_{\text{DIRAC}} e^{-\text{Tr}_H f(D)} dD \quad (\hbar = 1) \\ &= \int_{M_{p,q}} e^{-N \text{Tr}_N P - \text{Tr}_N^{\otimes 2} (Q_{(1)} \otimes Q_{(2)})} d\mathbb{X}_{\text{LEB}} \end{aligned}$$

- $\mathbb{X} \in M_{p,q}$ = products of $\mathfrak{su}(N)$ and \mathcal{H}_N
- $d\mathbb{X}_{\text{LEB}}$ is the Lebesgue measure on $M_{p,q}$
- $P, Q_{(i)}$ in $\mathbb{C}\langle k \rangle = \mathbb{C}\langle X_1, \dots, X_k \rangle = \mathbb{C}\langle \mathbb{X} \rangle$ are certain noncommutative polynomials
- $\mathcal{Z}_{\text{FORMAL}}$ leads to colored ribbon graphs

$$\bar{g}_1 \text{Tr}_N (ABBBAB) \leftrightarrow \text{Diagram 1}$$

$$\bar{g}_2 \text{Tr}_N^{\otimes 2} (AABABA \otimes AA) \leftrightarrow \text{Diagram 2}$$



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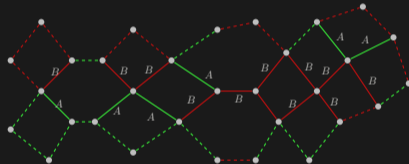
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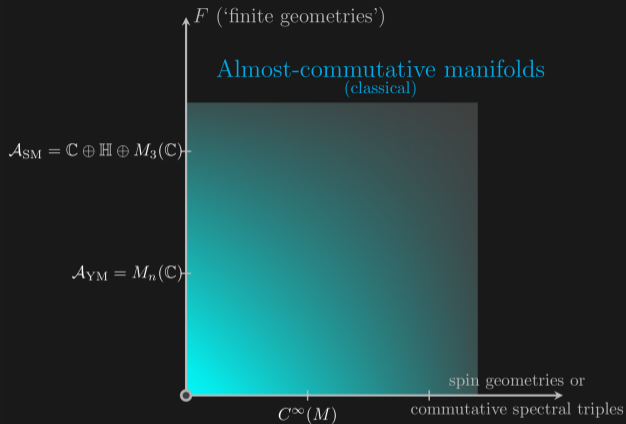
- Multitrace: 'touching interactions' [Klebanov, *PRD* '95], wormholes [Ambjørn-Jurkiewicz-Loll-Vernizzi, *JHEP* '01], 'stuffed maps' [G. Borot *Ann. Inst. Henri Poincaré Comb. Phys. Interact.* '14], AdS/CFT [Witten, *hep-th/0112258*]
- Ribbon graphs: Enumeration of maps [Brezin, Itzykson, Parisi, Zuber, *CMP* '78]. The ones we get are 'face-worded'



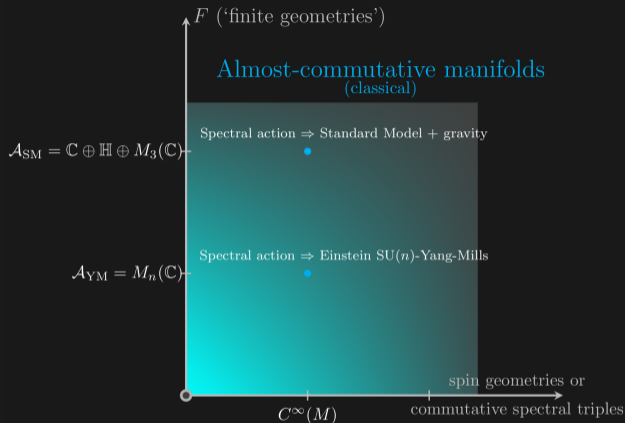
& intersection numbers of ψ -classes [Kontsevich, *CMP*, '92]

$$\begin{aligned} & \sum_{a_1 + \dots + a_n = \dim_{\mathbb{C}} \bar{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdot \psi_2^{a_2} \cdots \psi_n^{a_n} \prod_{j=1}^n \frac{(2a_j - 1)!!}{s_j^{2a_j+1}} \\ &= \sum_{G \text{ trivalent of type } (g,n)} \frac{2^{2g-2+n}}{\#\text{Aut}(G)} \prod_{e \in G} \frac{1}{s_{L(e)} + s_{R(e)}} \end{aligned}$$

III. YANG-MILLS-HIGGS MATRIX THEORY



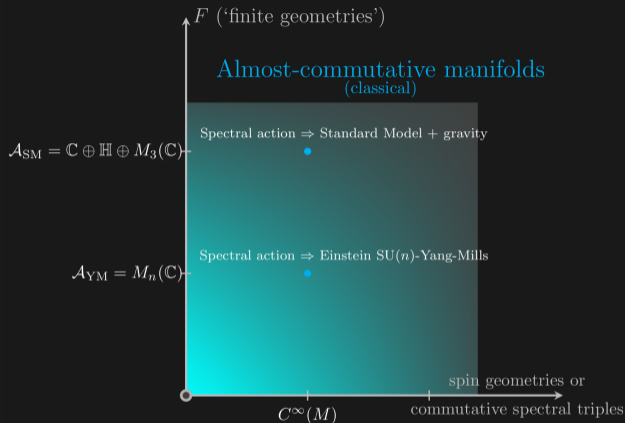
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$$\mathcal{Z}_{\text{AC}} \stackrel{?}{=} \int_{\text{DIRAC}} e^{-\frac{1}{\hbar} \text{Tr} f(D)} dD$$

(hard for almost-commutative manifolds)

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(hard for almost-commutative manifolds)

DEFINITION [CP 2105.01025] We define a *gauge matrix spectral triple* $G_f \times F$ as the spectral triple product of a fuzzy geometry G_f with a finite geometry $F = (A_F, H_F, D_F)$, $\dim A_F < \infty$.

LEMMA-DEFINITION [CP 2105.01025] Consider a gauge matrix spectral triple $G_f \times F$ with

$$F = (M_n(\mathbb{C}), M_n(\mathbb{C}), D_F)$$

and G_f Riemannian ($d = 4$) fuzzy geometry on the algebra $M_N(\mathbb{C})$, whose **fluctuated** Dirac op. is

$$D_\omega = \sum_{\mu=0}^3 \overbrace{\gamma^\mu \otimes (\ell_\mu + a_\mu) + \gamma^{\hat{\mu}} \otimes (x_\mu + s_\mu)}^{D_{\text{gauge}}} + \overbrace{\gamma \otimes \Phi}^{D_{\text{Higgs}}},$$

The **field strength** is given by

$$\mathcal{F}_{\mu\nu} := [\underbrace{\ell_\mu + a_\mu}_{d_\mu}, \ell_\nu + a_\nu] =: [\mathbf{F}_{\mu\nu}, \cdot]$$

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THEOREM For a Yang–Mills–Higgs matrix spectral triple on a 4-dimensional flat ($x = 0 = s$) Riemannian ($p = 0$) fuzzy base, the Spectral Action for a real polynomial $f(x) = \frac{1}{2} \sum_{i=1}^4 a_i x^i$ reads

$$\frac{1}{4} \text{Tr}_H f(D) = S_{\text{YM}}^f + S_{\text{H}}^f + S_{\text{g-H}}^f + S_\vartheta^f,$$

where each sector is defined as follows:

$$S_{\text{YM}}^f(\ell, a) := -\frac{a_4}{4} \text{Tr}_{M_{N \otimes n}^{\mathbb{C}}} (\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}),$$

$$S_{\text{g-H}}^f(\ell, a, \Phi) := -a_4 \text{Tr}_{M_{N \otimes n}^{\mathbb{C}}} (d_\mu \Phi d^\mu \Phi),$$

$$S_{\text{H}}^f(\Phi) := \text{Tr}_{M_{N \otimes n}^{\mathbb{C}}} f_e(\Phi),$$

$$S_\vartheta^f(\ell, a) := \text{Tr}_{M_{N \otimes n}^{\mathbb{C}}} f_e(\vartheta^{1/2}). \quad \vartheta = d_\mu d^\mu$$

Moreover, one obtains positivity for each of the following functionals, independently:

$$S_\vartheta^f, S_{\text{YM}}^f, S_{\text{H}}^f \geq 0, \quad \text{if } a_4 \geq 0.$$

MEANING

RANDOM MATRIX CASE, FLAT $d = 4$ RIEM.

SMOOTH OPERATOR

Tr = TRACE OF OPS. $M_N \otimes M_n \rightarrow M_N \otimes M_n$

Derivation

$$\ell_\mu = [L_\mu \otimes 1_n, \cdot]$$

 ∂_i

Gauge potential

$$a_\mu = [A_\mu, \cdot]$$

 \mathbb{A}_i

Higgs field

 Φ h

Covariant Derivative

$$d_\mu = \ell_\mu + a_\mu$$

$$\mathbb{D}_i = \partial_i + \mathbb{A}_i$$

Field strength

$$[d_\mu, d_\nu] = \overbrace{[\ell_\mu, \ell_\nu]}^{\neq 0} + [\ell_\mu, a_\nu] - [\ell_\nu, a_\mu] + [a_\mu, a_\nu]$$

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Higgs potential

$$\text{Tr}(f_2 \Phi^2 + f_4 \Phi^4)$$

$$\int_M (f_2 |h|^2 + f_4 |h|^4) \text{vol}$$

Gauge-Higgs coupling

$$- \text{Tr}(d_\mu \Phi d^\mu \Phi)$$

$$- \int_M |\mathbb{D}_i h|^2 \text{vol}$$

Yang-Mills action

$$-\frac{1}{4} \text{Tr}(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu})$$

$$-\frac{1}{4} \int_M \text{Tr}_{\text{su}(n)}(\mathbb{F}_{ij} \mathbb{F}^{ij}) \text{vol}$$

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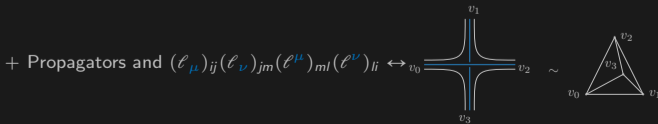
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[Ferrari-Rivasseau-Valette, *CMP*, 2019] is probably useful here

Motivation from '2D-Quantum Gravity'

discrete surfaces	\leftrightarrow	matrix integrals $\mathcal{Z}(\lambda)$ <small>[B. Eynard, <i>Counting Surfaces</i> '16]</small>
smooth surface	\leftrightarrow	$\langle \text{area} \rangle$ finite & infinitesimal mesh a $\langle \text{area} \rangle_g \sim \frac{a^2(2-2g)}{\lambda/\lambda_c - 1}$
all topologies	\leftrightarrow	$\mathcal{Z}(\lambda) = \sum_g N^{2-2g} \underbrace{\mathcal{Z}_g(\lambda)}_{(\lambda_c - \lambda)^{(2-2g)/\theta}}$
\uparrow		
double-scaling limit		$N(\lambda_c - \lambda)^{1/\theta} = C$
lin. RG-flow near a fixed point	\leftrightarrow	$\lambda(N) = \lambda_c + (N/C)^{-\theta}$ $\theta = -(\partial\beta/\partial\lambda) _{\lambda_c}$ <small>[Eichhorn-Kosłowski, PRD, '13]</small>

Two approaches

1. Mathematical construction:

[CP 2007.10914, *Ann. Henri Poincaré* 2021]

- prove Wetterich Equation, or FRGE,
“ $\dot{\Gamma} = \frac{1}{2} \text{STr} \{ \dot{R}_N / [\Gamma^{(2)} + R_N] \}$ ”
- its proof determines the algebra that governs geometric series in the Hessian of Γ (this expansion is originally from [D. Benedetti, K. Groh, P. F. Machado and F. Saueressig, *JHEP* 2011])
- to determine the scalings of the couplings with Z and N we use [A. Eichhorn, T. Koslowski, *Phys. Rev. D* '13], but the proof of the FRGE dictates an algebra not reported there
- fixed-point solution to β -equations for a sextic truncation (48 running operators)

- for the unique real solution g^* leading to a single relevant direction (positive e.v. of $-(\partial\beta_i/\partial g_j)_{i,j}|_{g^*}$) yields an R_N -dependent

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2. Pragmatic approach: [CP 2111.02858]

- write down Wetterich Equation (*Generatio Sponanea*)
- assume an expansion of its rhs in unitary-invariant operators (\neq exact RG)
- impose the one-loop structure
- determine from it the ‘algebra of functional renormalization’; it is unique and the one reported in [CP 2007.10914]

From *Holography for Tensor Models*, vTJC June, 2020

$$F(x, y) = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \dots$$

$$F(\alpha, \alpha) = 2\alpha + 3\alpha^2 + 4\alpha^3 + 6\alpha^4 + 8\alpha^5 + 14\alpha^6 + 20\alpha^7 + 36\alpha^8 + \dots \\ + 1342184\alpha^{25} + 2581428\alpha^{26} + 4971068\alpha^{27} + \dots$$

$$F_1(\alpha) = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \dots$$

“the two-matrix problem is a lot more complicated, and up to now, nobody has found a way to package all different invariants into a nice (collective) field” DE MELLO KOCH

From Holography for Tensor Models, vTJC June, 2020

$$F(x, y) = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \dots$$

$$F(\alpha, \alpha) = 2\alpha + 3\alpha^2 + 4\alpha^3 + 6\alpha^4 + 8\alpha^5 + 14\alpha^6 + 20\alpha^7 + 36\alpha^8 + \dots$$

$$+ 1342184\alpha^{25} + 2581428\alpha^{26} + 4971068\alpha^{27} + \dots$$

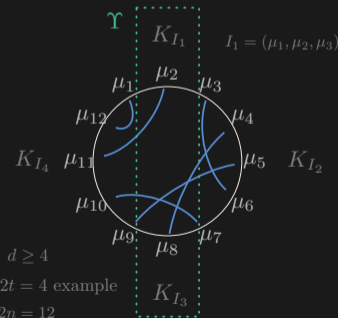
$$F_1(\alpha) = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \dots$$

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Double-traces makes our invariants for 'free' two-matrix models grow even faster [CP '19]: in dimension d

$$\frac{1}{\dim \mathbb{S}} \text{Tr}(D^{2t}) = \sum_{l_1, \dots, l_t \in \Lambda_d^-} \left\{ \sum_{\substack{\chi \in \text{CD}_{2n} \\ 2n = \sum_i |l_i|}} \chi^{l_1 \dots l_t} \right.$$

$$\left. \times \left(\sum_{\Upsilon \in \mathcal{P}_{2t}} \text{sgn}(l_\Upsilon) \times \text{Tr}_N(K_{l_{\Upsilon c}}) \times \text{Tr}_N[(K^T)_{l_\Upsilon}] \right) \right\}$$



go to examples of complexity of double-traces >

Handwaving Functional Renormalization for k -matrix models (w/multitrace-measures)

Quantum theories 'flow' with energy, here in RG-time $t = \log N$, $1 \ll N < \mathcal{N}$. E.g. for $k = 2$ and with bare action

$$S[A, B] = \mathcal{N} \operatorname{Tr}_{\mathcal{N}} \left\{ \frac{1}{2} A^2 + \frac{1}{2} B^2 + g_{A^4} \frac{1}{4} A^4 + g_{B^4} \frac{1}{4} B^4 + \frac{1}{2} g_{ABAB} ABAB \right\}$$

radiative corrections 'generate' *effective vertices*. For instance  generates $\mathcal{N} \operatorname{Tr}_{\mathcal{N}} (ABBA)$.

$$\Gamma_N[A, B] = \operatorname{Tr}_N \left\{ \underbrace{\frac{Z_A}{2} A^2 + \frac{Z_B}{2} B^2 + \bar{g}_{A^4} \frac{1}{4} A^4 + \bar{g}_{B^4} \frac{1}{4} B^4 + \frac{1}{2} \bar{g}_{ABAB} ABAB}_{\text{operators from the bare action (but with 'running couplings')}} + \underbrace{\frac{1}{2} \bar{g}_{ABBA} ABBA + \frac{1}{2} \bar{g}_{A|A} \operatorname{Tr}_N (A) \times A + \dots}_{\text{radiative corrections}} \right\}$$

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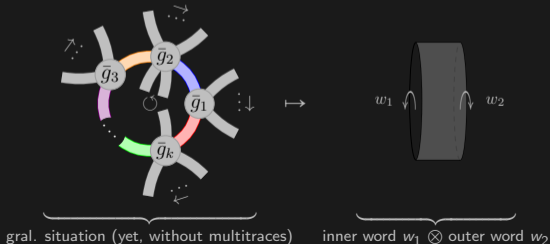
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We are interested in *one-loop graphs*. These are graphs G whose 1-dim skeleton $G^\circ = G / \{\text{interaction vertices collapsed to points}\}$ has $b_1(G^\circ) = 1$. The *effective vertex* O_G^{eff} of such Feynman graph is formed by reading off each word w_i traveling around all ribbon edges (propagators) by both sides:

$$O_G^{\text{eff}} = \underbrace{\text{Tr}_N(w_1) \times \text{Tr}_N(w_2) \times \dots \times \text{Tr}_N(w_s)}_{\text{from vertices contracted with propagators}} \times \underbrace{\text{Tr}_N(U_1) \times \text{Tr}_N(U_2) \dots \times \text{Tr}_N(U_r)}_{\text{from vertices uncontracted with propagators}}$$



- *nc-derivative* $\partial_A : \mathbb{C}\langle k \rangle \rightarrow \mathbb{C}\langle k \rangle^{\otimes 2}$ sums over
‘replacements of A by \otimes ’ [Rota-Sagan-Stein+Voiculescu]:

$$\partial_A(PAAR) = P \otimes AR + PA \otimes R, \text{ but}$$

$$\partial_A(ALGEBRA) = 1 \otimes LGEBRA + ALGEBR \otimes 1$$

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$$\partial_B \partial_A \left(\begin{array}{c} A \\ B \quad \otimes \quad A \\ B \quad \otimes \quad B \\ B \quad \otimes \quad A \end{array} \right) \quad \text{go to examples of nc-Hessians } \nabla$$

$$= 1_N \otimes \left(\begin{array}{c} A \\ B \quad \otimes \quad A \\ B \quad \otimes \quad B \\ B \quad \otimes \quad A \end{array} + \begin{array}{c} A \\ B \quad \otimes \quad A \\ B \quad \otimes \quad B \\ B \quad \otimes \quad A \end{array} + \begin{array}{c} A \\ B \quad \otimes \quad A \\ B \quad \otimes \quad B \\ B \quad \otimes \quad A \end{array} \right)$$

$$+ \left(\begin{array}{c} A \\ B \quad \otimes \quad A \\ B \quad \otimes \quad B \\ B \quad \otimes \quad A \end{array} + \begin{array}{c} A \\ B \quad \otimes \quad A \\ B \quad \otimes \quad B \\ B \quad \otimes \quad A \end{array} + \begin{array}{c} A \\ B \quad \otimes \quad A \\ B \quad \otimes \quad B \\ B \quad \otimes \quad A \end{array} \right) \otimes 1_N + \dots$$

in ellipsis $\sum_{\text{cuts}} \left(\begin{array}{c} A \\ B \quad \otimes \quad A \\ B \quad \otimes \quad B \\ B \quad \otimes \quad A \end{array} \right) \rightarrow BAA \otimes ABB$

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- products of traces \Rightarrow extend by \boxtimes , $\mathcal{A}_k = (\mathbb{C}_{\langle k \rangle} \otimes \mathbb{C}_{\langle k \rangle}) \oplus (\mathbb{C}_{\langle k \rangle} \boxtimes \mathbb{C}_{\langle k \rangle})$



$$\text{Hess}_{a,b}(\text{Tr} P \cdot \text{Tr} Q)$$

$$= \text{Tr} P \cdot \text{Hess}_{a,b}[\text{Tr} Q] + (\partial_{X_a} \text{Tr} P) \boxtimes (\partial_{X_b} \text{Tr} Q) + (P \leftrightarrow Q)$$

- Wetterich Eq. governs the functional RG

$$t = \log N$$

$$\partial_t \Gamma_N[\mathbb{X}] = \frac{1}{2} \text{STr} \left\{ \frac{\partial_t R_N}{\text{Hess} \Gamma_N[\mathbb{X}] + R_N} \right\}$$

$$\stackrel{\text{assume}}{=} \sum_{k=0}^{\infty} \bar{h}_k(N, \eta_1, \dots, \eta_n)$$

$$\times \frac{1}{2} (-1)^k \text{STr} \left\{ (\text{Hess} \Gamma_N^{\text{INT}}[\mathbb{X}])^{*k} \right\}$$

regulator-independent part

- $\text{STr} = \text{Tr}_k \otimes \text{Tr}_{\mathcal{A}_k}$. Tadpoles

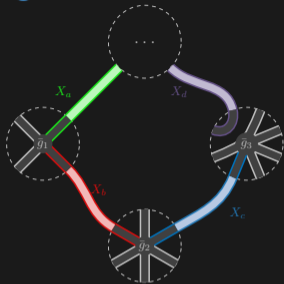


imply

$$\text{Tr}_{\mathcal{A}_k}(P \otimes Q) = \text{Tr}_N P \cdot \text{Tr}_N Q, \text{Tr}_{\mathcal{A}_k}(P \boxtimes Q) = \text{Tr}_N(PQ)$$

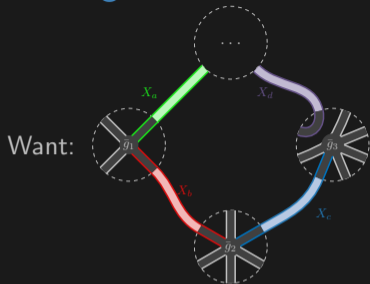
Finding \star

Want:



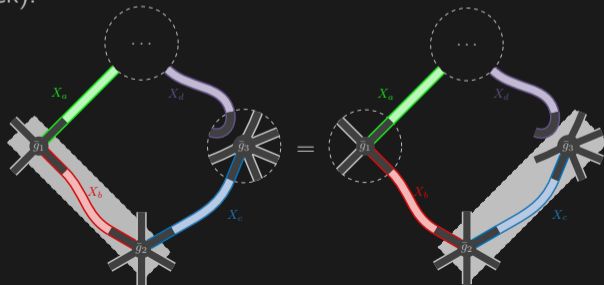
$$\subset \text{Hess}_{a,b} O_1 \star \text{Hess}_{b,c} O_2 \star \text{Hess}_{c,d} O_3 \star \dots \star \text{Hess}_{*,a} O_\ell$$

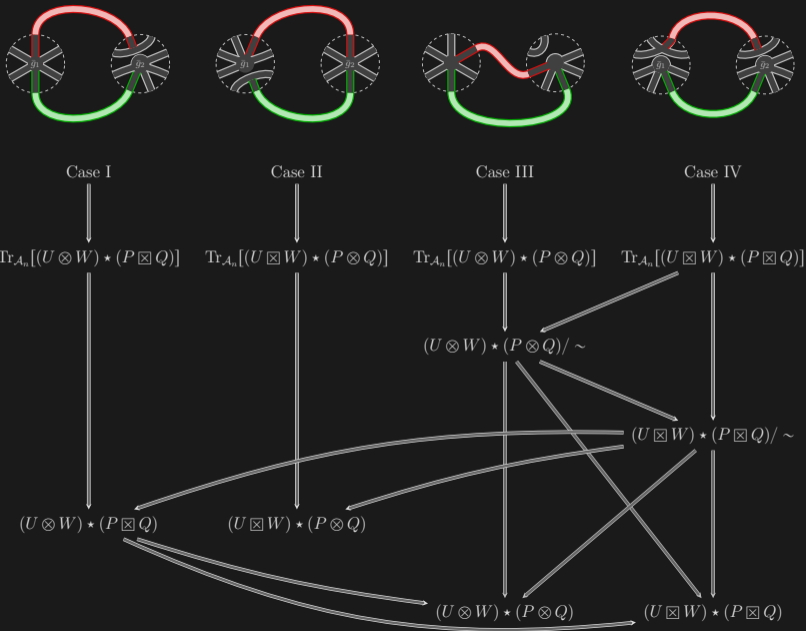
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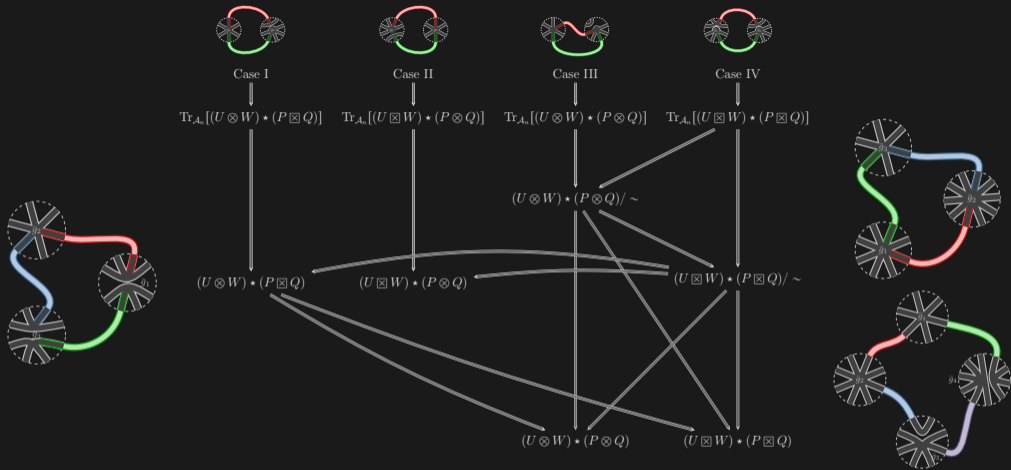


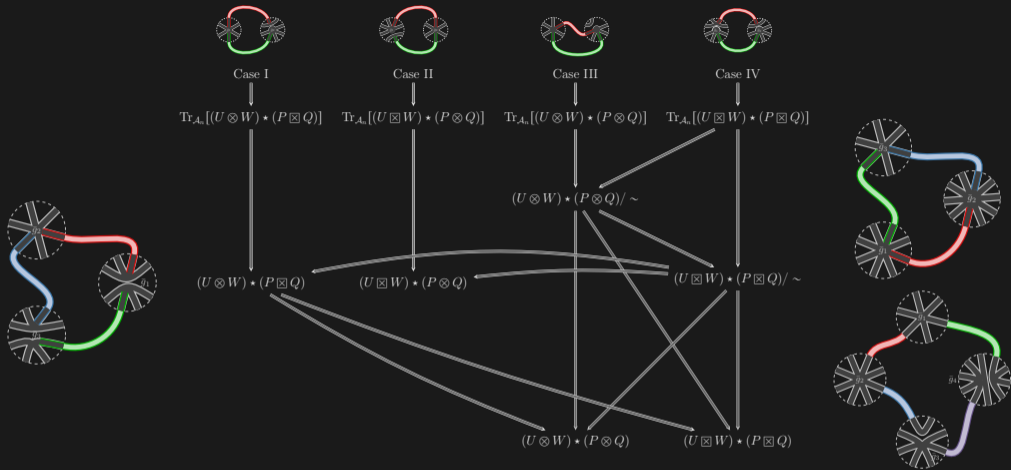
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Associativity (trivial check):









THM. [CP 2111.02858] If the RG-flow is computable in terms of $U(N)$ -invariants, the algebra of Functional Renormalization is $M_k(\mathcal{A}_{N,k}, \star)$ where $\mathcal{A}_{N,k} = (\mathbb{C}_{\langle k \rangle}^{(N)} \otimes \mathbb{C}_{\langle k \rangle}^{(N)}) \oplus (\mathbb{C}_{\langle k \rangle}^{(N)} \boxtimes \mathbb{C}_{\langle k \rangle}^{(N)})$ whose product in homogeneous elements reads:

$$\begin{aligned}
 (U \otimes W) \star (P \otimes Q) &= PU \otimes WQ, \\
 (U \boxtimes W) \star (P \otimes Q) &= U \boxtimes PWQ, \\
 (U \otimes W) \star (P \boxtimes Q) &= WPU \boxtimes Q, \\
 (U \boxtimes W) \star (P \boxtimes Q) &= \text{Tr}(WP)U \boxtimes Q.
 \end{aligned}$$

Example: Hermitian 'free' 3-matrix model

Consider two operators $O_1 = \frac{\bar{g}_1}{2} [\text{Tr}_N(\frac{A^2}{2})]^2$ and $O_2 = \bar{g}_2 \text{Tr}_N(ABC)$. We compute $g_1 g_2^2$ -coefficients:

$$\text{Hess}_{I,J} O_1 = \delta_I^J \delta_I^A \bar{g}_1 \left\{ \underbrace{\text{Tr}_N(A^2/2) \cdot [1_N \otimes 1_N]}_{\text{filled ribbon}} + \underbrace{A \boxtimes A}_{\text{white ribbon}} \right\},$$

where a 'filled ribbon' means contracted in the one-loop graph, and 'white ribbon' uncontracted.

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$$\text{Hess } O_2 = \bar{g}_2 \begin{bmatrix} 0 & C \otimes 1_N & B \otimes 1_N \\ 1_N \otimes C & 0 & A \otimes 1_N \\ 1_N \otimes B & 1_N \otimes A & 0 \end{bmatrix} \Rightarrow [\text{Hess } O_2]^{*2} = \bar{g}_2^2 \begin{bmatrix} \overbrace{C \otimes C} + \overbrace{B \otimes B} & B \otimes A & C \otimes A \\ A \otimes B & A \otimes A + C \otimes C & C \otimes B \\ A \otimes C & B \otimes C & B \otimes B + A \otimes A \end{bmatrix}.$$

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Extracting coefficients

$$[\bar{g}_1 \bar{g}_2] \text{STr} \{ \text{Hess } O_1 \star [\text{Hess } O_2]^{\star 2} \} = \text{Tr}_N (A^2/2) \times [(\text{Tr}_N C)^2 + (\text{Tr}_N B)^2] + \text{Tr}_N (ACAC + ABAB),$$

which are effective vertices of the four one-loop graphs that can be formed with the contractions of (the filled ribbon half-edges of) any of $\left\{ \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right\}$, $\left\{ \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right\}$ with any of $\left\{ \begin{array}{c} \text{---} \\ \times \\ \text{---} \end{array} \right\}$, $\left\{ \begin{array}{c} \text{---} \\ \times \\ \text{---} \end{array} \right\}$. This is a toy example in [CP '21]; in [CP '20] 48 such operators run \Rightarrow (48 less friendly Hessians)³. [go to nc-Hessians examples >](#)

CONCLUSION

- spectral triple \equiv spin without commutativity of the ‘algebra of functions’
- spin $M \times \{\text{finite spectral triple}\} \equiv$ almost-commutative
(reproduces classical Standard Model, but hard to quantize)
- *fuzzy* or *matrix* geometry \approx finite spectral triple + $\mathbb{C}\ell$ -action
($\mathcal{Z}_{\text{FUZZY}} = \int_{\text{DIRAC}} e^{-\frac{1}{\hbar} \text{Tr}_H f(D)} dD$ is a multimatrix model with multitraces)
- $G_\ell \times F =$ fuzzy \times finite = gauge matrix spectra triple
(is $\text{PU}(n)$ -Yang-Mills-Higgs-like if F is over $M_n(\mathbb{C})$;
partition function is a k -matrix model, k large)
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thank you!

Classical Dirac operators (assume d even)

- M (spacetime) will be a closed, Riemannian manifold
- if M is spin, there is a vector bundle \mathbb{S} with fibers satisfying $\text{End}(\mathbb{S}_x) \cong \mathbb{C}\ell(d)$ ($x \in M$). The sections $\Gamma(\mathbb{S})$ are spinors
- the Levi-Civita connection ∇^{LC} can be also lifted to the *spin connection* $\nabla^s : \Gamma(\mathbb{S}) \rightarrow \Omega^1(M) \otimes \Gamma(\mathbb{S})$

$$\nabla^s c(\omega)\psi = c(\nabla^{\text{LC}}\omega)\psi + c(\omega)\nabla^s\psi$$
$$\psi \in \Gamma(\mathbb{S}), \omega \in \Omega^1(M)$$

being c Clifford multiplication, basically $c(dx^\mu) = \gamma^\mu$


- on the space of square integrable spinors $L^2(M, \mathbb{S})$ there is an (ess.) self-adjoint operator, the *Dirac operator*,

$$D_M = -ic \circ \nabla^s \stackrel{\text{loc.}}{=} -i \sum_{\mu=1}^d \gamma^\mu (\partial_\mu + \omega_\mu)$$

and by Leibniz rule

$$[D_M, a] = -ic(da) \quad a \in C^\infty(M)$$

which is bounded

back to 'spectral triples' 

Matrix or Fuzzy Geometries

DEFINITION ("condensed" from [J. Barrett, *J. Math. Phys.* 2015]).

A *fuzzy geometry* of *signature* $(p, q) \in \mathbb{Z}_{\geq 0}^2$ is given by

- a simple matrix algebra \mathcal{A} – we take always $\mathcal{A} = M_N(\mathbb{C})$
- a Hermitian $\mathcal{Cl}(p, q)$ -module \mathbb{S} with a *chirality* γ . That is a linear map $\gamma : \mathbb{S} \rightarrow \mathbb{S}$ satisfying $\gamma^* = \gamma$ and $\gamma^2 = 1$
- a Hilbert space $\mathcal{H} = \mathbb{S} \otimes M_N(\mathbb{C})$ with inner product $\langle v \otimes R, w \otimes S \rangle = (v, w) \text{Tr}_N(R^* S)$ for each $R, S \in M_N(\mathbb{C})$, being (\cdot, \cdot) the inner product of \mathbb{S}
- a left- \mathcal{A} representation $\rho(a)(v \otimes R) = v \otimes (aR)$ on \mathcal{H} , $a \in \mathcal{A}$ and $v \otimes R \in \mathcal{H}$

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- three signs $\epsilon, \epsilon', \epsilon'' \in \{-1, +1\}$ determined through $s := q - p$ by the following table:

$s \equiv q - p \pmod{8}$	0	1	2	3	4	5	6	7
ϵ	+	+	-	-	-	-	+	+
ϵ'	+	-	+	+	+	-	+	+
ϵ''	+	+	-	+	+	+	-	+

- a real structure $J = C \otimes *$, where $*$ is complex conjugation and C is an anti-unitarity on \mathbb{S} satisfying $C^2 = \epsilon$ and $C\gamma^\mu = \epsilon'\gamma^\mu C$ for all the gamma matrices $\mu = 1, \dots, p + q$.

- a self-adjoint operator D on \mathcal{H} satisfying the *order-one condition*

$$[[D, \rho(a)], J\rho(b)J^{-1}] = 0 \quad \text{for all } a, b \in \mathcal{A}$$

- a chirality $\Gamma = \gamma \otimes 1_{\mathcal{A}}$ for \mathcal{H} , where γ is the chirality of \mathbb{S} . The signs above impose:

$$J^2 = \epsilon, \quad JD = \epsilon'DJ, \quad J\Gamma = \epsilon''\Gamma J.$$

Two-matrix model from $\text{Tr}_H D^6$, $\eta = \text{diag}(e_1, e_2)$, $K_i^* = e_i K_i$ [CP '19]

$$\begin{aligned} \mathcal{S}_6[K_1, K_2] = 2 \cdot \text{Tr}_N \{ & e_1 K_1^6 + 6e_2 K_1^4 K_2^2 - 6e_2 K_1^2 (K_1 K_2)^2 + 3e_2 (K_1^2 K_2)^2 \\ & + e_2 K_2^6 + 6e_1 K_2^4 K_1^2 - 6e_1 K_2^2 (K_2 K_1)^2 + 3e_1 (K_2^2 K_1)^2 \} \end{aligned}$$

Two-matrix model from $\text{Tr}_H D^6$, $\eta = \text{diag}(e_1, e_2)$, $K_i^* = e_i K_i$ [CP '19]

$$\mathcal{S}_6[K_1, K_2] = 2 \cdot \text{Tr}_N \left\{ e_1 K_1^6 + 6e_2 K_1^4 K_2^2 - 6e_2 K_1^2 (K_1 K_2)^2 + 3e_2 (K_1^2 K_2)^2 \right. \\ \left. + e_2 K_2^6 + 6e_1 K_2^4 K_1^2 - 6e_1 K_2^2 (K_2 K_1)^2 + 3e_1 (K_2^2 K_1)^2 \right\}$$

and the double-trace part is

$$\mathcal{B}_6[K_1, K_2] = 6 \text{Tr}_N(K_1) \left\{ 2 \text{Tr}_N(K_1^5) + 2 \text{Tr}_N(K_1 K_2^4) + 6e_1 e_2 \text{Tr}_N(K_1^3 K_2^2) - 2e_1 e_2 \text{Tr}_N(K_1^2 K_2 K_1 K_2) \right\} \\ + 6 \text{Tr}_N(K_2) \left\{ 2 \text{Tr}_N(K_2^5) + 2 \text{Tr}_N(K_2 K_1^4) + 6e_1 e_2 \text{Tr}_N(K_2^3 K_1^2) - 2e_1 e_2 \text{Tr}_N(K_2^2 K_1 K_2 K_1) \right\} \\ + 48 \text{Tr}_N(K_1 K_2) \cdot [e_1 \text{Tr}_N(K_1^3 K_2) + e_2 \text{Tr}_N(K_2^3 K_1)] \\ + 6 \text{Tr}_N(K_1^2) \cdot \left\{ e_2 [8 \text{Tr}_N(K_1^2 K_2^2) - 2 \text{Tr}_N(K_2 K_1 K_2 K_1)] + e_1 [5 \text{Tr}_N(K_1^4) + \text{Tr}_N(K_2^4)] \right\} \\ + 6 \text{Tr}_N(K_2^2) \cdot \left\{ e_1 [8 \text{Tr}_N(K_1^2 K_2^2) - 2 \text{Tr}_N(K_1 K_2 K_1 K_2)] + e_2 [5 \text{Tr}_N(K_2^4) + \text{Tr}_N(K_1^4)] \right\} \\ + 4(5[\text{Tr}_N(K_1^3)]^2 + 6e_1 e_2 \text{Tr}_N(K_1 K_2^2) \text{Tr}_N(K_1^3) + 9[\text{Tr}_N(K_1^2 K_2)]^2 \\ + 5[\text{Tr}_N(K_2^3)]^2 + 6e_1 e_2 \text{Tr}_N(K_1^2 K_2) \text{Tr}_N(K_2^3) + 9[\text{Tr}_N(K_1 K_2^2)]^2).$$

Sketch of the Standard Model derivation from NCG

One starts with the $M \times_{s.t.} F$ and $\mathcal{A}_{LR} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})$

- $F = (\mathcal{A}_{LR}, \mathcal{M}_F^{\#\text{generations}}, D_F)$, \mathcal{M}_F an \mathcal{A}_{LR} -module
- \mathcal{M}_F has to be of the form $\mathcal{M}_F = \mathcal{E} \otimes \mathcal{E}^\circ$, with

$$\mathcal{E} = (2_L \otimes 1^\circ) \oplus (2_R \otimes 1^\circ) \oplus (2_L \otimes 3^\circ) \oplus (2_L \otimes 3^\circ), \quad \dim_{\mathbb{C}} \mathcal{E} = 16$$

- Thus the $\mathcal{H}_F \cong \mathbb{C}^{32 \times 3}$. The 96×96 matrix D_F can have off-diagonal elements only for the maximal subalgebra

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

- Lie group part of $SU(\mathcal{A}_F) = U(1) \times SU(2) \times SU(3)$

Sketch of the Standard Model derivation from NCG

With $Q : \mathbb{C} \hookrightarrow \mathbb{H}$, $Q_\lambda = \text{diag}(\lambda, \bar{\lambda})$ and $Q_\lambda |\pm\rangle = \pm\lambda |\pm\rangle$,

• Weak hypercharge:

Y	ν	e	u	d
	$ +\rangle \otimes 1^\circ$	$ -\rangle \otimes 1^\circ$	$ +\rangle \otimes 3^\circ$	$ -\rangle \otimes 3^\circ$
L	-1	-1	$+1/3$	$+1/3$
R	0	-2	$+4/3$	$-2/3$

- $SU(2)$ -adjoint action is 2 on \mathcal{H}_L or trivial in the \mathcal{H}_R sector
- $SU(3)$ -adjoint action is the color action on \mathcal{H}_q and trivial on \mathcal{H}_ℓ

$$\text{Lie}(SU(\mathcal{A}_F)) = U(1)_Y \times SU(2)_L \times SU(3)_{\text{color}}$$

- All D_F such that $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ is a spectral triple are

$$D_F(\Upsilon_R, \Upsilon_\nu, \Upsilon_e, \Upsilon_u, \Upsilon_d)$$

the moduli of such Dirac operators has dimension $31 = \text{num. Yukawa couplings in } \nu\text{MSM}$.

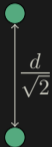
• **Example 1: (Finite spectral triples)** [Exercise from W. van Suijlekom's book]

- $\mathcal{A} = \mathbb{C}^3$

- $\mathcal{H} = \mathbb{C}^3 \hookrightarrow \mathcal{A}$, in defining representation

- $D = \begin{pmatrix} 0 & 1/d & 0 \\ 1/d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (0 \neq d \in \mathbb{R})$

looks like

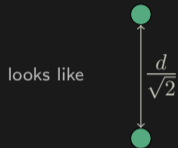


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for $d_{ij} = \sup_{a \in \mathcal{A}} \{|a(i) - a(j)| : \|[D, a]\| \leq 1\}$

$= \sup_{a \in \mathcal{A}} \{|a(i) - a(j)| : |a(1) - a(2)|^2 \leq d^2\} \quad \forall i, j = 1, 2, 3$

• **Example 2: (Finite spectral triples)**

- $\mathcal{A} = M_3(\mathbb{C})$

- $\mathcal{H} = \mathbb{C}^3$

- $D = 0$

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• **Example 2: (Finite spectral triples)**

- $\mathcal{A} = M_3(\mathbb{C}) = \mathbb{C}[\mathcal{G}_\sim]$

- $\mathcal{H} = \mathbb{C}^3$

3 = $\{1, 2, 3\} / \sim$

- $D = 0$

Chord Diagrams of 10 points (1/4)

(appear in $D^{10}|_{d=2}, D^4|_{d=4}, D^2|_{d=6}$)



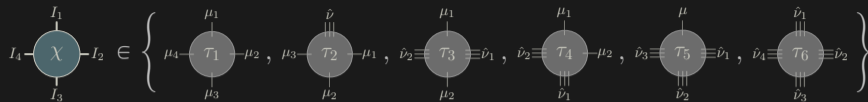
Chord Diagrams of 10 points (2/4)



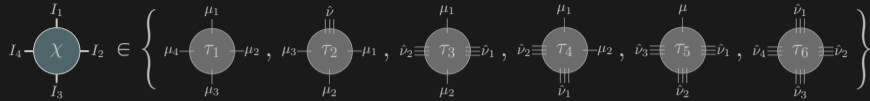
Chord Diagrams of 10 points (4/4)



Non-vanishing ones... $\frac{1}{4} \text{Tr} D^4 = NS_4^{\text{Riemann}} + \sum_i \text{Tr}_N^{\otimes 2}(A_i \otimes B_i)$



Non-vanishing ones... $\frac{1}{4} \text{Tr} D^4 = N S_4^{\text{Riemann}} + \sum_i \text{Tr}_N \otimes^2 (A_i \otimes B_i)$

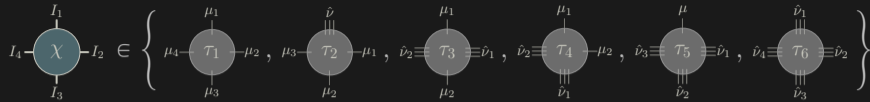


$$\begin{aligned}
 S_4^{\text{Riemann}} = & \text{Tr}_N \left\{ 2 \sum_{\mu} (L_{\mu}^4 + H_{\mu}^4) + 4 \sum_{\mu < \nu} (2L_{\mu}^2 L_{\nu}^2 + 2H_{\mu}^2 H_{\nu}^2 - L_{\mu} L_{\nu} L_{\mu} L_{\nu} - H_{\mu} H_{\nu} H_{\mu} H_{\nu}) \right. \\
 & - \sum_{\mu \neq \nu} [2(L_{\mu} H_{\nu})^2 + 4L_{\mu}^2 H_{\nu}^2] + \sum_{\mu} [2(L_{\mu} H_{\mu})^2 - 4L_{\mu}^2 H_{\mu}^2] \\
 & + 8 \left[H_1 (L_2 [L_3, L_4] + L_3 [L_4, L_2] + L_4 [L_2, L_3]) \right. \\
 & \quad + H_2 (L_1 [L_3, L_4] + L_3 [L_4, L_1] + L_4 [L_1, L_3]) \\
 & \quad + H_3 (L_1 [L_2, L_4] + L_2 [L_4, L_1] + L_4 [L_1, L_2]) \\
 & \quad \left. + H_4 (L_1 [L_2, L_3] + L_2 [L_3, L_1] + L_3 [L_1, L_2]) \right] + 8 [H \leftrightarrow L] \left. \right\}
 \end{aligned}$$

[C.P. '19]

L_{μ}, H_{μ} are random matrices!

Non-vanishing ones... $\frac{1}{4} \text{Tr} D^4 = N S_4^{\text{Riemann}} + \sum_i \text{Tr}_N \otimes^2 (A_i \otimes B_i)$



$$\begin{aligned}
 S_4^{\text{Riemann}} = & \text{Tr}_N \left\{ 2 \sum_{\mu} (L_{\mu}^4 + H_{\mu}^4) + 4 \sum_{\mu < \nu} (2L_{\mu}^2 L_{\nu}^2 + 2H_{\mu}^2 H_{\nu}^2 - L_{\mu} L_{\nu} L_{\mu} L_{\nu} - H_{\mu} H_{\nu} H_{\mu} H_{\nu}) \right. \\
 & - \sum_{\mu \neq \nu} [2(L_{\mu} H_{\nu})^2 + 4L_{\mu}^2 H_{\nu}^2] + \sum_{\mu} [2(L_{\mu} H_{\mu})^2 - 4L_{\mu}^2 H_{\mu}^2] \\
 & + 8 \left[H_1 (L_2 [L_3, L_4] + L_3 [L_4, L_2] + L_4 [L_2, L_3]) \right. \\
 & \quad + H_2 (L_1 [L_3, L_4] + L_3 [L_4, L_1] + L_4 [L_1, L_3]) \\
 & \quad + H_3 (L_1 [L_2, L_4] + L_2 [L_4, L_1] + L_4 [L_1, L_2]) \\
 & \quad \left. + H_4 (L_1 [L_2, L_3] + L_2 [L_3, L_1] + L_3 [L_1, L_2]) \right] + 8 [H \leftrightarrow L] \left. \right\}
 \end{aligned}$$

[C.P. '19]

L_{μ}, H_{μ} are random matrices!

- Analogy $[L_{\mu}, \cdot] \rightarrow \partial_{\mu}$ $\{H_{\mu}, \cdot\} \rightarrow \omega_{\mu}$ [J. Barrett, L. Glaser, *J. Phys. A* 2016]
- Obtained for any signature, also the A_i, B_i noncommutative-polynomials [C.P. '19]
- Lorentzian signature [L. Glaser, *J. Phys. A* '17]

OPERATOR ITS NONCOMMUTATIVE HESSIAN

$$\text{Tr}(A) \text{Tr}(A^3) \quad 3 \cdot \begin{pmatrix} \text{Tr} A \cdot (A \otimes 1 + 1 \otimes A) & 0 \\ +1 \boxtimes A^2 + A^2 \boxtimes 1 & \\ 0 & 0 \end{pmatrix}$$

$$\text{Tr}(ABAB) \quad 2 \cdot \begin{pmatrix} B \otimes B & (1 \otimes BA + AB \otimes 1) \\ (1 \otimes AB + BA \otimes 1) & A \otimes A \end{pmatrix}$$

$$\text{Tr} A \text{Tr}(AAABB) \quad \left(\begin{array}{l} (\text{Tr}(A)1 \otimes (ABB) + \text{Tr}(A)1 \otimes (BBA) + \text{Tr}(A)(ABB) \otimes 1 + \text{Tr}(A)(BBA) \otimes 1 + 1 \boxtimes (AABB) + 1 \boxtimes (ABBA) + 1 \boxtimes (BBAA) + (AABB) \boxtimes 1 + (ABBA) \boxtimes 1 + (BBAA) \boxtimes 1 + \text{Tr}(A)A \otimes B^2 + \text{Tr}(A)B^2 \otimes A) \quad \text{Tr}(A)(1 \otimes (AAB) + (BAA) \otimes 1 + A \otimes AB + B \otimes A^2 + A^2 \otimes B + BA \otimes A) + 1 \boxtimes (A^3 B) + 1 \boxtimes (BA^3) \\ \text{Tr}(A)(1 \otimes (BAA) + (AAB) \otimes 1 + A \otimes BA + B \otimes A^2 + A^2 \otimes B + AB \otimes A) + (A^3 B) \boxtimes 1 + (BA^3) \boxtimes 1 \quad \text{Tr}(A)(1 \otimes (A^3) + (A^3) \otimes 1) \end{array} \right)$$

Table: Some Hessians order operators. Here $\text{Tr} = \text{Tr}_N$.

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Table: Some Hessians order operators. Here $\text{Tr} = \text{Tr}_N$.

β -functions of 'free' two-matrix models, signature $\eta = \text{diag}(e_1, e_2)$

$$2h_1(a_4 + c_{22} + 2d_{2|02} + 6d_{2|2}) = \eta_a$$

$$2h_1(b_4 + c_{22} + 6d_{02|02} + 2d_{2|02}) = \eta_b$$

$$-h_1[e_a(a_4 - c_{1111}) + 2d_{1|12} + 6d_{1|3}] + d_{1|1}(\eta + 1) = \beta(d_{1|1})$$

$$-h_1[e_b(b_4 - c_{1111}) + 6d_{01|03} + 2d_{01|21}] + d_{01|01}(\eta + 1) = \beta(d_{01|01})$$

The next block encompasses the connected quartic couplings:

$$h_2(4a_4^2 + 4c_{22}^2) + a_4(2\eta + 1)$$

$$-h_1(24a_6e_a + 4c_{42}e_b + 4d_{02|4}e_b + 4d_{2|4}e_a) = \beta(a_4)$$

$$h_2(4b_4^2 + 4c_{22}^2) + b_4(2\eta + 1)$$

$$-h_1(24b_6e_b + 4c_{24}e_a + 4d_{02|04}e_b + 4d_{2|04}e_a) = \beta(b_4)$$

$$-h_1(2e_a c_{1212} + e_b 2c_{2121} + 3e_a c_{24} + 3e_b c_{42} + e_a d_{02|22} + e_b d_{2|22})$$

$$+ h_2(2a_4 c_{22} + 2b_4 c_{22} + 2e_a e_b c_{1111}^2 + 2e_a e_b c_{22}^2) + c_{22}(2\eta + 1) = \beta(c_{22})$$

$$8e_a e_b c_{1111} c_{22} h_2 + c_{1111}(2\eta + 1)$$

$$+ h_1(4e_a c_{1311} + 4e_b c_{3111} + 2e_a d_{02|1111} + 2e_b d_{2|1111}) = \beta(c_{1111})$$

$$2h_2(6a_4a_6 + e_a e_b c_{22}c_{42}) + a_6(3\eta + 2) = \beta(a_6)$$

$$2h_2(6b_4b_6 + e_a e_b c_{22}c_{24}) + b_6(3\eta + 2) = \beta(b_6)$$

$$4h_2\{a_4c_{3111} + e_a e_b [c_{22}(c_{1311} + 2c_{3111}) - c_{1111}(2c_{2121} + c_{42})]\} + c_{3111}(3\eta + 2) = \beta(c_{3111})$$

$$2h_2[2a_4c_{2121} + e_a e_b (-2c_{1111}c_{3111} + 4c_{2121}c_{22} + c_{22}c_{24})] + c_{2121}(3\eta + 2) = \beta(c_{2121})$$

$$2h_2[a_4c_{24} + 3b_4c_{24} + 2e_a e_b (c_{22}(3b_6 + c_{2121} + c_{24} + c_{42}) - c_{1111}c_{1311})] + c_{24}(3\eta + 2) = \beta(c_{24})$$

$$4h_2\{b_4c_{1311} + e_a e_b [c_{22}(2c_{1311} + c_{3111}) - c_{1111}(2c_{1212} + c_{24})]\} + c_{1311}(3\eta + 2) = \beta(c_{1311})$$

$$2h_2[2b_4c_{1212} + e_a e_b (c_{22}(4c_{1212} + c_{42}) - 2c_{1111}c_{1311})] + c_{1212}(3\eta + 2) = \beta(c_{1212})$$

$$2h_2[3a_4c_{42} + 2e_a e_b (3a_6c_{22} - c_{1111}c_{3111} + c_{1212}c_{22} + c_{22}c_{24} + c_{22}c_{42}) + b_4c_{42}] + c_{42}(3\eta + 2) = \beta(c_{42})$$